SERRIN'S OVERDETERMINED PROBLEM IN ROUGH DOMAINS

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Abstract. The classical Serrin's overdetermined theorem states that a C^2 bounded domain, which admits a function with constant Laplacian that satisfies both constant Dirichlet and Neumann boundary conditions, must necessarily be a ball. While extensions of this theorem to non-smooth domains have been explored since the 1990s, the applicability of Serrin's theorem to Lipschitz domains remained unresolved. This paper answers this open question affirmatively. Actually, our approach shows that the result holds for domains that are sets of finite perimeter with a uniform upper bound on the density, and it also allows for slit discontinuities.

1. Introduction

Given $\Omega \subset \mathbb{R}^n$ a bounded domain, Serrin's overdetermined problem aims to understand how overdetermined problems for PDEs within Ω influence the geometry of the domain. In its simplest formulation, whenever $\partial\Omega$ is sufficiently smooth, one investigates the following problem:

$$\Delta u = -1 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \quad \partial_\nu u = \mathbf{c} > 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here, $\partial_\nu u$ represents the inward normal derivative of u on $\partial\Omega$.

If Ω is not smooth but its boundary has at least finite $(n-1)$-dimensional Hausdorff measure, then (1.1) is understood in the following weak distributional sense:

$$u \in W_0^{1,2}(\Omega) \qquad \text{and} \qquad \int_\Omega \nabla u \cdot \nabla \varphi \, dx = -\mathbf{c} \int_{\partial\Omega} \varphi \, d\mathscr{H}^{n-1} + \int_\Omega \varphi \, dx \quad \forall \varphi \in C^1(\mathbb{R}^n), \tag{1.2}$$

where $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Given that the Dirichlet problem already yields a unique (weak) solution, the addition of the Neumann boundary condition makes the problem overdetermined. Consequently, (1.1) may not have a solution in general, which implies that the choice of domain Ω cannot be arbitrary.

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1.1. **Serrin's Theorem.** In his seminal theorem, assuming that $\partial\Omega$ is of class C^2 (so that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$), Serrin proved the following celebrated result:

Theorem 1.1 ([37])**.** *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain. Then* (1.1) *admits a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ if and only if, up to a translation, Ω is a ball of radius $R = R(n, \mathbf{c}) > 0$ and u takes the form*

$$u(x) = \frac{R^2 - |x|^2}{2n}. \tag{1.3}$$

This revelation marked the beginning of a burgeoning and fertile area of mathematics, in which the interplay of analysis and geometry gave rise to many applications that span the most diverse areas of mathematics and the natural sciences. Remarkably, the genesis of this field can be traced back to a particular result that intriguingly arose from two questions in mathematical physics: one concerning the torsion of a straight solid rod, and the other concerning the tangential stress of a fluid on the walls of a rectilinear pipe; this was the original motivation of Serrin as stated in [37].

Serrin's proof builds on and refines the original concept introduced by Alexandrov [2, 3], known today as the "moving plane method". Subsequently, Weinberger [42] presented an alternative proof of Theorem 1.1. Inspired by Weinberger's approach, researchers have explored alternative methods to establish this result, as evidenced in [7, 11, 33]

1.2. **Generalizations.** Subsequently, Garofalo and Lewis demonstrated a similar result in [23] for the p-Laplacian. Then, these results were extended to operators in divergence form of p-Laplacian type, and even to certain special cases of ∞-Laplacian, in [18, 20, 8, 15]. Moreover, equations involving fully nonlinear operators of non-divergence form, such as k-Hessian equations [7], and problems in space forms [31, 26, 16, 13, 35, 14, 19, 21, 22], have garnered significant attention. We also record some recent results in [1] about a version of Serrin's problem on planar ring domains, where the solutions are not necessarily radially symmetric, except when adding further conditions on the number of critical points.

Given the extensive body of literature surrounding Serrin's original problem, we can only provide a glimpse of the breadth of results here. We suggest that interested readers look at the surveys [36, 32] and the references therein for a comprehensive overview.

On a separate note, the proof in [11], relying on Alexandrov's theorem, initially unveiled a connection between the results of Alexandrov and Serrin. Subsequently, a deeper linkage has been explored by [12, 28, 29]. We also recommend the insightful survey [27] which comprehensively investigates these findings. This connection has also appeared in overdetermined elliptic problems within unbounded domains, initially conjectured by Berestycki, Caffarelli, and Nirenberg [6] for balls and cylinders, and eventually disproved in [38]. Subsequently, a multitude of counterexamples have been constructed based on unbounded constant mean curvature surfaces. Given the focus of our paper on bounded domains, we refer to the survey [39] for further elucidation on related results.

1.3. **Serrin's Theorem for more singular domains.** In 1992, Vogel [41] proved that if Ω is a C^1 domain for which a solution to (1.1) exists, then Ω is actually C^2 and therefore Theorem 1.1 holds. To be precise, Vogel assumed that

$$u(x) \to 0 \quad \text{and} \quad |\nabla u|(x) \to \mathbf{c} \ \text{ uniformly as } x \to \partial\Omega,$$

and then applied the regularity theory of free boundary problems of Alt-Caffarelli type. We note that his assumption is stronger than just assuming the validity of (1.2) (see also Remark 1.5 below).

Later, Berestycki posed the following question:
Suppose Ω is C^2 throughout except for a potential corner, and u represents a strong solution to (1.1) *everywhere except at said corner. Does Serrin's Theorem remain applicable in this scenario?*
This problem was solved in [34] using an adapted moving plane method, strategically circumventing the exceptional point. Subsequently, interest arose regarding the extension of Serrin's theorem to more general domains. In particular, in [24, Question 7.1] it was asked:
Does Theorem 1.1 hold if Ω is merely Lipschitz and u solves (1.2)*?*

1.4. **Main result.** In this paper, we give a positive answer to the above question and we actually prove the validity of Theorem 1.1 to a much wider class of domains. To state our result we first observe that, since $u \in W_0^{1,2}(\Omega)$, we can extend u to zero outside of Ω and rewrite

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi \, dx = -\int_{\mathbb{R}^n} \Delta u \, \varphi \, dx,$$

where Δu denotes the distributional of u on $\mathbb{R}^n$. Hence (1.2) is equivalent to asking

$$u \in W^{1,2}(\mathbb{R}^n), \qquad u = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega, \qquad \Delta u = \mathbf{c}\mathscr{H}^{n-1}|_{\partial\Omega} - \mathbf{1}_\Omega \, dx,$$

where the last equality should be intended in the sense of distribution.

Note that the formula above does not require Ω to be open but could be any Borel set, provided that we have a good notion of boundary that allows one to perform integration by parts. This naturally leads to the notion of sets of finite perimeter, where $\partial\Omega$ should be replaced by the so-called "reduced boundary" $\partial^*\Omega$. Moreover, we need a version of connectedness for sets of finite perimeter, called indecomposability. We refer to Section 2 below for more details.

Remark 1.2. We choose to work with sets of finite perimeters because the proof of Serrin's theorem would not be significantly easier if we assumed Ω to be a Lipschitz domain; the main concepts introduced in this paper would still be necessary. For readers who are not concerned with this level of generality, we suggest reading our paper with the assumption that Ω is a Lipschitz domain, so that $\partial^*\Omega$ corresponds to the set of points where the boundary is differentiable (which is true $\mathscr{H}^{n-1}$-a.e. by Rademacher's theorem).

We can now state our main theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded indecomposable set of finite perimeter satisfying*

$$\mathscr{H}^{n-1}(B_r(x) \cap \partial^*\Omega) \le A r^{n-1} \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \partial^*\Omega \text{ and } r \in (0,1), \tag{1.4}$$

for some constant $A > 0$. Then Ω admits a solution $u \in W^{1,2}(\mathbb{R}^n)$ to

$$u = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega, \qquad \Delta u = \mathbf{c}\mathscr{H}^{n-1}|_{\partial^*\Omega} - \mathbf{1}_\Omega \, dx, \tag{1.5}$$

if and only if, up to a translation, Ω is a ball of radius $R = R(n, \mathbf{c})$ and u is given by (1.3).

Remark 1.4. Theorem 1.3 implies, in particular, the validity of Serrin's theorem for any domain Ω whose boundary $\partial\Omega$ satisfies (1.4). Assumption (1.4) is notable mild and encompasses many geometries, including Lipschitz domains, and it also allows for the presence of countably many cusps. Consequently, Theorem 1.3 not only addresses [24, Question 7.1] but also includes all previously established results.

Remark 1.5. Our theorem closely aligns with the result in [17] regarding the validity of Alexandrov's theorem for sets of finite perimeter. However, despite the similarity, the two problems are distinct. In particular, while our theorem is non-trivial even when Ω is a Lipschitz domain, the validity of Alexandrov's theorem for Lipschitz domains follows directly from elliptic regularity theory (which immediately implies that such domains must be smooth).

To elucidate this point, it is important to note that Serrin's problem is related to the one-phase Bernoulli problem (we refer to the recent monograph [40] for more details). Consequently, one might consider leveraging its regularity theory to demonstrate that solutions to Serrin's problem in Lipschitz domains are smooth. Unfortunately, this regularity theory relies either on a minimality property or a viscosity-type approach, and it is unclear to us whether it can be applied in our context. Therefore, our proof utilizes techniques from geometric measure theory, and it would not be simpler for Lipschitz domains.

Remark 1.6. One may wonder whether Theorem 1.1 holds also in the case of slit domains, for instance, a slit ball. These domains cannot be treated in the framework of sets of finite perimeters, since the latter are defined up to sets of measure zero. Moreover, in the presence of slits, one must properly define in which sense (1.1) is satisfied. As we shall see in Section 4, our method can be adapted to prove Serrin's theorem even in this setting.

The paper is structured as follows: In the next section, we collect some preliminary results that will be used in Section 3 to prove Theorem 1.3. Then, Section 4 is devoted to the proof of Serrin's Theorem in slit domains, see Theorem 4.1 below.

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2. Preliminary results

In this section, we establish certain regularity results for u and Ω and prove a volume identity that will be used in the proof of our main theorem. Before that, we recall the definition of sets of finite perimeter and their main properties.

A measurable set $\Omega \subseteq \mathbb{R}^n$ is a set of *finite perimeter* if the distributional gradient of its characteristics function $\mathbf{1}_\Omega$ is a $\mathbb{R}^n$-valued Radon measure $D\mathbf{1}_\Omega$ with finite total variation, i.e., $|D\mathbf{1}_\Omega|(\mathbb{R}^n) < \infty$. It follows from the Lebesgue-Besicovitch theorem on differentiation of measures that for $|D\mathbf{1}|$-a.e. x, it holds

$$\lim_{r\to 0^+} \frac{D\mathbf{1}_\Omega(x + rB^n)}{|D\mathbf{1}_\Omega|(x + rB^n)} = \nu_x \quad \text{and} \quad |\nu_x| = 1. \tag{2.1}$$

The set of points x such that (2.1) holds is called the *reduced boundary of* Ω and denoted by $\partial^*\Omega$. Also, at points of the reduced boundary, ν_x is the *measure-theoretic inner unit normal* to Ω at x. According to De Giorgi Rectifiability Theorem, the reduced boundary is a $(n-1)$-rectifiable set. Also, up to changing Ω in a set of measure zero, one can assume that $\overline{\partial^*\Omega} = \partial\Omega$. Finally, a set of finite perimeter E is said *indecomposable* if for every $F \subset E$ having finite perimeter and such that

$$\mathscr{H}^{n-1}(\partial^* E) = \mathscr{H}^{n-1}(\partial^* F) + \mathscr{H}^{n-1}(\partial^*(E \setminus F)),$$

one has either $|F| = 0$ or $|E \setminus F| = 0$. We refer the interested reader to [5] and [25, Sections 12 and 15] for more details on sets of finite perimeter.

In the next lemma, $\mathring{\Omega}$ denotes the (topological) interior part of Ω.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set of finite perimeter satisfying* (1.4)*, and let $u \in W^{1,2}(\mathbb{R}^n)$ satisfy* (1.5)*. Then:*

(1) u is L-Lipschitz continuous, with $L = L(n, \mathbf{c}, A)$.
(2) u is nonnegative, $\mathring{\Omega} = \{u > 0\}$, and $u \in C^\infty(\mathring{\Omega})$.
(3) $\Omega = \mathring{\Omega}$ up to a set of measure zero. In particular, without loss of generality, we can assume Ω to be open.
(4) At every point $x \in \partial^*\Omega$ it holds

$$\frac{u(x+rz)}{r} \to \mathbf{c}\big(\nu_x \cdot z\big)_+ \qquad \text{as } r \to 0,$$

where ν_x denotes the measure-theoretic inner unit normal at x.

Proof. Equation (1.5) implies that Δu is a Radon measure. Also, it follows from (1.4) that, given $x \in \partial^*\Omega$,

$$\Delta u(B_r(x)) \le \big(\Delta u + \mathbf{1}_\Omega\big)(B_r(x)) \le A\mathbf{c}r^{n-1} \quad \forall\, r \in (0,1).$$

Since $u = 0$ a.e. outside Ω, the classical identity

$$\frac{d}{dr} \fint_{\partial B_r(x)} u \, d\mathscr{H}^{n-1} = \frac{\Delta u(B_r(x))}{n|B_1|r^{n-1}} \tag{2.2}$$

(see for instance the proof of [40, Lemma 3.10]) combined with the bound above implies that

$$\fint_{\partial B_r(x)} u \, d\mathscr{H}^{n-1} \le C(n, A, \mathbf{c})r \qquad \text{for } x \in \partial^*\Omega,\ r \in (0,1),$$

therefore

$$\fint_{B_r(x)} u(y)\, dy \le C(n, A, \mathbf{c})r \qquad \text{for } x \in \partial^*\Omega,\ r \in (0,1).$$

Since the map $x \mapsto \fint_{B_r(x)} u(y)\, dy$ is continuous for $r > 0$ fixed and $\overline{\partial^*\Omega} = \partial\Omega$, we deduce that

$$\fint_{B_r(x)} u(y)\, dy \le C(n, A, \mathbf{c})r \qquad \text{for } x \in \partial\Omega,\ r \in (0,1).$$

Recalling that $\Delta u = -1$ inside $\mathring{\Omega}$, interior regularity estimates imply that u is uniformly Lipschitz continuous function inside $\mathring{\Omega}$, see for instance the proof of [40, Lemma 3.5]. Therefore, since u vanishes on $\partial\Omega$ and $u = 0$ a.e. outside Ω, u is globally Lipschitz, which proves (1).

Since $\Delta u = -1 < 0$ inside $\mathring{\Omega}$, the strong maximum principle implies that $\{u > 0\}$ inside $\mathring{\Omega}$. Since u vanishes outside $\mathring{\Omega}$, this shows that $\{u > 0\} = \mathring{\Omega}$. The smoothness of u inside $\mathring{\Omega}$ follows immediately from the equation $\Delta u = -1$. This proves (2).

Note that, since $u \geq 0$, the distributional Laplacian of u (which we know to be a Radon measure) is non-negative in the set $\{u = 0\} = \mathbb{R}^n \setminus \mathring{\Omega}$. Also, inside $\mathring{\Omega} = \{u > 0\}$, $\Delta u = -1$. This implies that

$$\Delta u = \mu_+ - \mathbf{1}_{\mathring{\Omega}}\, dx,$$

for some nonnegative measure μ_+. Comparing this equation with (1.5), we conclude that $\mu_+ = \mathbf{c}\mathscr{H}^{n-1}|_{\partial^*\Omega}$ and $\mathbf{1}_\Omega\, dx = \mathbf{1}_{\mathring{\Omega}}\, dx$. This last equality implies that Ω and $\mathring{\Omega}$ coincide a.e., proving (3).

Finally, we prove (4). Given $x \in \partial^*\Omega$, we consider the sequence of Lipschitz functions

$$v_r(z) = \frac{u(x + rz)}{r}, \qquad \Delta v_r = \mathbf{c}\mathscr{H}^{n-1}|_{\partial^*\Omega_{x,r}} - r\mathbf{1}_{\Omega_{x,r}} dx,$$

where $\Omega_{x,r} = \frac{\Omega - x}{r} = \{z \in \mathbb{R}^n : x + rz \in \Omega\}$. Then, since x belongs to $\partial^*\Omega$, $\Omega_{x,r} \to H_x$, where H_x is a half space. In addition, for any converging subsequence $r_i \to 0$, the Lipschitz functions v_{r_i} converge to a nonnegative Lipschitz function v_0 satisfying

$$\Delta v_0 = 0 \quad \text{in } H_x, \qquad v_0 = 0 \quad \text{in } \mathbb{R}^n \setminus H_x.$$

Then, Liouville Theorem implies that $v_0(z) = \mathbf{a}(\nu_x \cdot z)_+$ for some $\mathbf{a} \geq 0$.

On the other hand, using again $x \in \partial^*\Omega$, it follows that $\mathscr{H}^{n-1}|_{\partial^*\Omega_{x,r}} \rightharpoonup \mathscr{H}^{n-1}|_{\partial H_x}$. Hence $\Delta v_0 = \mathbf{c}\mathscr{H}^{n-1}|_{\partial H_x}$. Combining these two facts, we conclude that $\mathbf{a} = \mathbf{c}$. This shows that $v_{r_i} \to \mathbf{c}(\nu_x \cdot z)_+$ for any converging subsequence r_i, so the entire sequence v_r converges to $\mathbf{c}(\nu_x \cdot z)_+$, as desired.

□

We now demonstrate the following volume identity that will play a crucial role in our proof. In the classical setting [42], this identity is proved via a Pohozaev's approach. Unfortunately, this approach requires too much regularity on the solution u, so a new proof is needed.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set of finite perimeter satisfying* (1.4)*, and let $u \in W^{1,2}(\mathbb{R}^n)$ satisfy* (1.5)*. Then*

$$(n+2)\int_\Omega u\, dx = \mathbf{c}^2 n|\Omega|. \tag{2.3}$$

Proof. Thanks to Lemma 2.1(3), we can assume that Ω is open. Given $\epsilon > 0$, consider

$$\varphi_\epsilon(x) = \frac{u((1+\epsilon)x) - u((1-\epsilon)x)}{2\epsilon} - 2u(x). \tag{2.4}$$

Note that φ_ϵ is Lipschitz continuous for $\epsilon > 0$ fixed, and that $|\varphi_\epsilon| \leq C = C(L, \Omega)$ for all $\epsilon > 0$ (by the Lipschitz continuity of u). Therefore, testing (1.5) against φ_ϵ we get

$$\mathbf{c}\int_{\partial^*\Omega} \varphi_\epsilon\, d\mathscr{H}^{n-1} - \int_\Omega \varphi_\epsilon\, dx = -\int_\Omega \nabla u \cdot \nabla\varphi_\epsilon\, dx = \int_\Omega u\Delta\varphi_\epsilon\, dx. \tag{2.5}$$

We first want to compute the Laplacian of φ_ϵ. To this aim, we define

$$\mathcal{N}_\epsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) \le C_0\epsilon\}$$

with $C_0 = \operatorname{diam}(\Omega)$, so that

$$x, (1+\epsilon)x, (1-\epsilon)x \in \Omega \qquad \text{for all } x \in \Omega \setminus \mathcal{N}_\epsilon.$$

Hence, noticing that $\frac{(1+\epsilon)^2-(1-\epsilon)^2}{2\epsilon} - 2 = 0$, it follows from (1.5) that

$$\Delta\varphi_\epsilon(x) = \mathbf{c}\frac{(1+\epsilon)^2\mathscr{H}^{n-1}|_{(1+\epsilon)^{-1}\partial^*\Omega} - (1-\epsilon)^2\mathscr{H}^{n-1}|_{(1-\epsilon)^{-1}\partial^*\Omega}}{2\epsilon} - 2\mathbf{c}\mathscr{H}^{n-1}|_{\partial^*\Omega} + O\left(\frac{1}{\epsilon}dx|_{\mathcal{N}_\epsilon}\right).$$

Therefore, by a change of variables,

$$\begin{aligned}\int_\Omega u\Delta\varphi_\epsilon\,dx &= \frac{\mathbf{c}}{2\epsilon}\left[\int_{(1+\epsilon)^{-1}\partial^*\Omega}(1+\epsilon)^2u\,d\mathscr{H}^{n-1} - \int_{(1-\epsilon)^{-1}\partial^*\Omega}(1-\epsilon)^2u\,d\mathscr{H}^{n-1}\right] + \int_{\mathcal{N}_\epsilon} u\,O\left(\frac{1}{\epsilon}\right)dx\\ &= \mathbf{c}\left[\int_{\partial^*\Omega}\frac{(1+\epsilon)^{3-n}u((1+\epsilon)^{-1}x) - (1-\epsilon)^{3-n}u((1-\epsilon)^{-1}x)}{2\epsilon}\,d\mathscr{H}^{n-1}(x)\right] + \int_{\mathcal{N}_\epsilon} u\,O\left(\frac{1}{\epsilon}\right)dx.\end{aligned} \tag{2.6}$$

Hence, if we define

$$\psi_\epsilon(x) = -\frac{(1+\epsilon)^{3-n}u((1+\epsilon)^{-1}x) - (1-\epsilon)^{3-n}u((1-\epsilon)^{-1}x)}{2\epsilon} + \varphi_\epsilon(x), \tag{2.7}$$

it follows from (2.5) that

$$\mathbf{c}\int_{\partial^*\Omega}\psi_\epsilon\,d\mathscr{H}^{n-1} - \int_\Omega \varphi_\epsilon\,dx = \int_{\mathcal{N}_\epsilon} u\,O\left(\frac{1}{\epsilon}\right)dx. \tag{2.8}$$

Note now that, applying Lemma 2.1(4), for $x \in \partial^*\Omega$ we have

$$\psi_\epsilon(x) \to -\mathbf{c}\big((\nu_x\cdot x)_+ - (\nu_x\cdot x)_-\big) = -\mathbf{c}\nu_x\cdot x.$$

Thus, by dominated convergence (recall that $|\varphi_\epsilon| \le C$)

$$\mathbf{c}\int_{\partial^*\Omega}\psi_\epsilon\,d\mathscr{H}^{n-1} \to -\mathbf{c}^2\int_{\partial^*\Omega}\nu_x\cdot x\,d\mathscr{H}^{n-1} \qquad \text{as } \epsilon \to 0.$$

Also, since $\varphi_\epsilon(x) \to \nabla u(x)\cdot x - 2u(x)$ inside Ω (recall that, without loss of generality, we can assume that Ω is open), by dominated convergence and an integration by parts we have

$$\int_\Omega \varphi_\epsilon\,dx \to \int_\Omega \big(\nabla u\cdot x - 2u\big)\,dx = -(n+2)\int_\Omega u\,dx \qquad \text{as } \epsilon \to 0.$$

Finally, since u is Lipschitz and vanished on $\partial\Omega$ we deduce that $u = O(\epsilon)$ inside $\mathcal{N}_\epsilon$, therefore

$$\int_{\mathcal{N}_\epsilon} u\,O\left(\frac{1}{\epsilon}\right)dx = O(|\mathcal{N}_\epsilon|) \to 0 \qquad \text{as } \epsilon \to 0$$

(the convergence $|\mathcal{N}_\epsilon| \to 0$ is a simple consequence of dominated convergence, since $\mathcal{N}_\epsilon \to \emptyset$ as $\epsilon \to 0$). Combining all together, we proved that

$$-\mathbf{c}^2 \int_{\partial^*\Omega} \nu_x \cdot x \, d\mathscr{H}^{n-1} = (n+2) \int_\Omega u \, dx.$$

Finally, by the divergence theorem for sets of finite perimeter (recall that ν_x is the inner normal),

$$-\int_{\partial^*\Omega} \nu_x \cdot x \, d\mathscr{H}^{n-1} = \int_\Omega \operatorname{div}(x) \, dx = n|\Omega|,$$

which concludes the proof. □

3. Proof of Theorem 1.3

Our proof of Theorem 1.3 relies on Proposition 3.3 below, stating that $|\nabla u| \le \mathbf{c}$ in Ω. Proving this fact is nontrivial for two reasons:
(i) in a rough domain as in our setting, the standard maximum principle for $|\nabla u|$ is not available;
(ii) in our situation, at least formally, Lemma 2.1(4) tells us that $|\nabla u| = \mathbf{c}$ on the reduced boundary, but we do not have any information at points of $\partial\Omega \setminus \partial^*\Omega$.
To circumvent these difficulties and to prove Proposition 3.3, we shall carefully exploit the properties of the Green function of Ω. As we shall see, to prove Lemma 3.2 we will crucially use the fact that a solution to Serrin's problem exists in Ω.

Remark 3.1. A Green function approach to prove a maximum principle on $|\nabla u|$ has already been used, in a similar context, in [4, Theorem 6.3]. There, however, the authors assume that solutions are non-degenerate, which is something that we do not have in our context. For this reason, our proofs are completely different.

Recall that, due to Lemma 2.1(3), we can assume that Ω is open. To define the Green function, we consider an increasing sequence Ω_k of smooth sets contained inside Ω such that $\Omega_k \to \Omega$ as $k \to \infty$. Then, given $x \in \Omega$, we note that $x \in \Omega_k$ for k sufficiently large, so we can define $G_{x,k}$ as the solution of

$$\begin{cases} \Delta G_{x,k} = -\delta_x, & \text{in } \Omega_k \\ G_{x,k} = 0 & \text{in } \mathbb{R}^n \setminus \Omega_k \end{cases}.$$

Noticing that $G_{x,k} \le G_{x,j}$ for $k < j$ (by the maximum principle), we can define

$$G_x := \lim_{k\to\infty} G_{x,k}.$$

Lemma 3.2. *Let Ω be an open bounded set satisfying the assumptions in Theorem 1.3, and let a solution u of* (1.5) *exist. Fix $x \in \Omega$, and let G_x be the Green function constructed above. Then:*

(1) G_x is Lipschitz continuous near $\partial\Omega$, with the Lipschitz constant depending only on n, A, $\mathbf{c}$, Ω, and x. Moreover, there exists a bounded measurable function $\alpha : \partial^*\Omega \to [0,\infty)$ such that

$$\Delta G_x = \alpha \, \mathscr{H}^{n-1}|_{\partial^*\Omega} - \delta_x.$$

(2) At every point $y \in \partial^*\Omega$ it holds

$$\frac{G_x(y+rz)}{r} \to \mathbf{a}_y\big(\nu_y \cdot z\big)_+ \qquad \text{as } r \to 0,$$

where ν_y denotes the measure-theoretic inner unit normal at y and $\mathbf{a}_y = \alpha(y)$.

Proof. Recall that G_x is the monotone limit of $G_{x,k}$. As $u > 0$ in Ω, we can choose $\rho = \rho(x,\Omega) > 0$ small and $M = M(x,\Omega) > 0$ large enough (independent of k), so that

$$G_{x,k} < Mu \quad \text{ on } \; \partial B_\rho(x)$$

for all $k \gg 1$. Also, as $u > 0$ on $\partial\Omega_k$ we have $Mu > G_{x,k}$ on $\partial\Omega_k$, therefore

$$G_{x,k} < Mu \quad \text{ in } \; \Omega_k \setminus B_\rho(x)$$

by the maximum principle (recall that $\Delta u = -1 < 0 = \Delta G_{x,k}$ inside $\Omega_k \setminus B_\rho(x)$). Letting $k \to \infty$, this gives

$$G_x \leq Mu \quad \text{ in } \; \Omega \setminus B_\rho(x). \tag{3.1}$$

Recalling that u vanishes on $\partial\Omega$ and it is Lipschitz, this proves that G_x grows at most linearly at every boundary point. So, by interior regularity estimates, we conclude that G_x is Lipschitz continuous in a neighborhood of $\partial\Omega$

Recalling that $G_{x,k} = 0$ outside Ω_k and that $\Delta G_{x,k} + \delta_x \geq 0$, we see that $G_x = 0$ outside Ω and that $\Delta G_x + \delta_x \geq 0$ as a distribution. In addition, $\Delta G_x = 0$ inside $\Omega \setminus \{x\}$.

Now, given $y \in \mathbb{R}^n \setminus \Omega$ and $r > 0$ small, it follows from (2.2), (3.1), and (1.5) that

$$\int_0^r \frac{\Delta G_x(B_s(y))}{s^{n-1}}\, ds \leq M \int_0^r \frac{\Delta u(B_s(y))}{s^{n-1}}\, ds = M\mathbf{c} \int_0^r \frac{\mathscr{H}^{n-1}(\partial^*\Omega \cap B_s(y))}{s^{n-1}}\, ds.$$

Hence, by the rectifiability of the reduced boundary,

$$\liminf_{s\to 0} \frac{\Delta G_x(B_s(y))}{s^{n-1}} \leq M\mathbf{c} \lim_{s\to 0} \frac{\mathscr{H}^{n-1}(\partial^*\Omega \cap B_s(y))}{s^{n-1}} = \begin{cases} M\mathbf{c}\omega_{n-1} & \text{for } \mathscr{H}^{n-1}\text{-a.e. } y \in \partial^*\Omega \\ 0 & \text{for } \mathscr{H}^{n-1}\text{-a.e. } y \notin \partial^*\Omega, \end{cases}$$

where ω_{n-1} denotes the volume of the $(n-1)$-dimensional ball. By [30, Theorem 6.11], this implies that

$$\Delta G_x|_{\mathbb{R}^n\setminus\Omega} \leq C(M,\mathbf{c},n)\mathscr{P}^{n-1}|_{\partial^*\Omega},$$

where $\mathscr{P}^{n-1}$ denotes the $(n-1)$-dimensional packing measure (see [30, Chapter 5.10]). Since $\partial^*\Omega$ is rectifiable it follows that $\mathscr{P}^{n-1}|_{\partial^*\Omega} = \mathscr{H}^{n-1}|_{\partial^*\Omega}$.

Thus, combining all together, we proved that

$$0 \leq \Delta G_x + \delta_x \leq C(M,\mathbf{c},n)\mathscr{H}^{n-1}|_{\partial^*\Omega}.$$

By the Radon-Nikodym Theorem, this implies the existence of a measurable function $\alpha : \partial^*\Omega \to [0, C(M,n)]$ such that

$$\Delta G_x = \alpha\mathscr{H}^{n-1}|_{\partial^*\Omega} - \delta_x.$$

This concludes the proof of (1).

The proof of (2) is similar to that of Lemma 2.1(4). Indeed, given $y \in \partial^*\Omega$, it follows by (1) that the sequence of functions

$$F_r(z) = \frac{G_x(y+rz)}{r}$$

are uniformly Lipschitz and harmonic inside $U_{y,r} = \frac{U-y}{r}$, where $U = \Omega \setminus B_\rho(x)$. Also, since $y \in \partial^*\Omega$,

$$U_{y,r} \to H_y \quad \text{as} \quad r \to 0,$$

and, up to a subsequence, F_r converges to a nonnegative Lipschitz function F_0 that is harmonic in H_y and vanishes outside H_y. Then Liouville Theorem yields the existence of a constant $\mathbf{a}_y \geq 0$ so that

$$F_0 = \mathbf{a}_y(\nu_y \cdot z)_+.$$

On the other hand, using again that $y \in \partial^*\Omega$, it follows that $\mathscr{H}^{n-1}|_{\partial^*\Omega_{y,r}} \rightharpoonup \mathscr{H}^{n-1}|_{\partial H_y}$. Hence $\Delta F_0 = \alpha(y)\mathscr{H}^{n-1}|_{\partial H_y}$. Combining these two consequences, we conclude that $\alpha(y) = \mathbf{a}_y$. This shows that $F_{r_k} \to \mathbf{a}_y(\nu_y \cdot z)_+$ for any converging subsequence r_k, so the entire sequence F_r converges to $\mathbf{a}_y(\nu_y \cdot z)_+$, as desired. □

We can now prove a maximum principle for $|\nabla u|$.

Proposition 3.3. *Let Ω be an open bounded set satisfying the assumptions in Theorem 1.3, and let u solve* (1.5)*. Then*

$$\sup_\Omega |\nabla u| \leq \mathbf{c}.$$

Proof. For $\epsilon > 0$ and $e \in \mathbb{S}^{n-1}$, let

$$\phi_\epsilon(y) = \frac{G_x(y+\epsilon e) - G_x(y-\epsilon e)}{2\epsilon},$$

and note that ϕ_ϵ is uniformly bounded and Lipschitz continuous near $\partial\Omega$. Thus, testing (1.5) against ϕ_ϵ we obtain

$$\mathbf{c}\int_{\partial^*\Omega} \phi_\epsilon \, d\mathscr{H}^{n-1} - \int_\Omega \phi_\epsilon \, dy = \int_\Omega u\Delta\phi_\epsilon \, dy. \tag{3.2}$$

Recall that, by Lemma 3.2(2), we have

$$\Delta\phi_\epsilon(y) = \frac{\alpha\,\mathscr{H}^{n-1}|_{\partial^*\Omega-\epsilon e} - \alpha\,\mathscr{H}^{n-1}|_{\partial^*\Omega+\epsilon e}}{2\epsilon} - \frac{\delta_{x-\epsilon e} - \delta_{x+\epsilon e}}{2\epsilon}.$$

Therefore, by a change of variable and Lemma 2.1(4) we get

$$\begin{aligned}
\int_\Omega u\Delta\phi_\epsilon \, dx &= \frac{1}{2\epsilon}\left[\int_{\partial^*\Omega-\epsilon e} \alpha(y+\epsilon e)u(y)\, d\mathscr{H}^{n-1} - \int_{\partial^*\Omega+\epsilon e} \alpha(y-\epsilon e)u(y)\, d\mathscr{H}^{n-1}\right] \\
&\qquad - \left\langle u, \, \frac{\delta_{x-\epsilon e} - \delta_{x+\epsilon e}}{2\epsilon}\right\rangle \\
&= -\int_{\partial^*\Omega} \frac{u(y+\epsilon e) - u(y-\epsilon e)}{2\epsilon}\alpha(y)\, d\mathscr{H}^{n-1} + \left\langle \frac{u(y+\epsilon e) - u(y-\epsilon e)}{2\epsilon}, \, \delta_x\right\rangle \\
&= -\frac{\mathbf{c}}{2}\int_{\partial^*\Omega} \alpha\nu_y \cdot e \, d\mathscr{H}^{n-1} + \partial_e u(x) + o(1).
\end{aligned}$$

Concerning the left-hand side of (3.2), we note that $\phi_\epsilon \to DG \cdot e$ inside Ω. Thus, by dominated convergence we get

$$\int_\Omega \phi_\epsilon \, dy = \int_\Omega \partial_e G_x \, dy + o(1).$$

In addition, Lemma 3.2 tells that, for $\mathscr{H}^{n-1}$-almost every $y \in \partial^*\Omega$, we have

$$\phi_\epsilon(y) = \frac{G_x(y+\epsilon e) - G_x(y-\epsilon e)}{2\epsilon} \to \frac{\mathbf{a}_y}{2}\big((\nu_y \cdot e)_+ - (\nu_y \cdot e)_-\big) = \frac{\alpha(y)}{2}\nu_y \cdot e.$$

Thus, the left-hand side of (3.2) equals to

$$\frac{\mathbf{c}}{2}\int_{\partial^*\Omega} \alpha\nu_y \cdot e\, d\mathscr{H}^{n-1} - \int_\Omega \partial_e G_x\, dy + o(1).$$

As a result, by letting $\epsilon \to 0$ in (3.2) we eventually

$$\partial_e u(x) = \mathbf{c}\int_{\partial^*\Omega} \alpha\nu_y \cdot e\, d\mathscr{H}^{n-1} - \int_\Omega \partial_e G_x\, dy. \tag{3.3}$$

Note now that, by the divergence theorem in sets for finite perimeter (recall that ν_y is the inner unit normal),

$$\int_\Omega \partial_e G_x\, dy = -\int_{\partial^*\Omega} G_x \nu_y \cdot e\, d\mathscr{H}^{n-1} = 0. \tag{3.4}$$

Also, thanks to Lemma 3.2,

$$0 = \int_{\mathbb{R}^n} \Delta G_x\, dy = -1 + \int_{\partial^*\Omega} \alpha\, d\mathscr{H}^{n-1}. \tag{3.5}$$

Combining (3.3), (3.4), and (3.5), we get

$$|\partial_e u(x)| \le \mathbf{c}\int_{\partial^*\Omega} \alpha\, d\mathscr{H}^{n-1} = \mathbf{c}.$$

Since x and e are arbitrary, the result follows. □

Proof of Theorem 1.3. Thanks to the previous results, we can basically repeat Weinberger's argument [42] with minor modifications.

More precisely, recalling that we can assume Ω to be open (see Lemma 2.1(3)), define

$$P = |\nabla u|^2 + \frac{2}{n}u \qquad \text{inside } \Omega.$$

Since $\Delta u = -1$ inside Ω, it follows that

$$\Delta P = 2|D^2u|^2 - \frac{2}{n}\Delta u = 2\left|D^2u + \frac{1}{n}\mathrm{Id}\right|^2 \ge 0. \tag{3.6}$$

In particular, as u is smooth in $\{u > \eta\}$, Proposition 3.3 and the weak maximum principle imply that

$$\max_{\{u\ge\eta\}} P = \max_{\partial\{u\ge\eta\}} P \le \mathbf{c}^2 + \frac{2}{n}\eta \qquad \forall\, \eta > 0,$$

so letting $\eta \to 0$ we deduce that

$$P \le \mathbf{c}^2 \qquad \text{in } \Omega. \tag{3.7}$$

Using again that $\Delta u = -1$ inside Ω, thanks to (2.3) and (3.7) we have

$$\mathbf{c}^2|\Omega| = \frac{n+2}{n}\int_\Omega u\, dx = \int_\Omega u\left(-\Delta u + \frac{2}{n}\right) dx = \int_\Omega \left(|\nabla u|^2 + \frac{2}{n}u\right) dx = \int_\Omega P\, dx \le \mathbf{c}^2|\Omega|.$$

The equation above implies that $P = \mathbf{c}^2$ inside Ω. In particular $\Delta P = 0$, so it follows from (3.6) that

$$D^2 u = -\frac{1}{n}\mathrm{Id} \qquad \text{in } \Omega.$$

As Ω is indecomposable, this implies that Ω is a ball and that, up to a translation, u is given by (1.3). □

4. Extension to slit domains

In the previous section we proved the validity of Serrin's Theorem in the setting of sets of finite perimeter. However, as already mentioned in Remark 1.6, this formalism does not allow one to treat sets with slit discontinuities, say a slit ball. In fact, from a measure-theoretic point of view, a slit ball is equivalent to a ball.

To extend our result to slit domains, we need to find a suitable reformulation of (1.1). To this end, we note that in the proof of Theorem 1.3, the Neumann condition on u was crucially used to prove the blow-up result in Lemma 2.1(4). In the case of slit domains, the assumption that $\Delta u = 2\mathbf{c}\mathscr{H}^{n-1}$ on the slit does not ensure that on both sides of the slit the function u behaves like a linear function with slope $\mathbf{c}$ (the slopes on the two sides may be different). As we will see, to prove Lemma 2.2 in the case of a slit domain, the equality of the slopes from the two sides of the slit is required. In addition, we need to guarantee that u vanishes on the slit in a suitable weak sense. Both conditions are included in the assumption (4.3) below.

The following generalization of the theorem 1.3 holds.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a indecomposable set of finite perimeter, let $u \in W^{1,2}(\mathbb{R}^n)$ satisfy*

$$u = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega, \qquad \Delta u = \mathbf{c}\mathscr{H}^{n-1}|_{\partial^*\Omega} + 2\mathbf{c}\mathscr{H}^{n-1}|_{\Sigma} - \mathbf{1}_\Omega \, dx \tag{4.1}$$

for some $(n-1)$-rectifiable set $\Sigma \subset \mathring{\Omega}$, and assume that

$$\mathscr{H}^{n-1}\big(B_r(x) \cap (\partial^*\Omega \cup \Sigma)\big) \le A r^{n-1} \qquad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in \partial^*\Omega \cup \Sigma \text{ and } r \in (0,1). \tag{4.2}$$

Also, suppose that at $\mathscr{H}^{n-1}$-a.e. $x \in \Sigma$ it holds

$$\frac{u(x+rz)}{r} \to \mathbf{c}|\nu_x \cdot z| \qquad \text{as } r \to 0, \tag{4.3}$$

where ν_x denotes the measure-theoretic unit normal at x. Then, up to a translation, Ω is a ball of radius $R = R(n, \mathbf{c}) > 0$ and u is given by (1.3).

To prove this theorem, we first note that the following generalization of Lemma 2.1 holds with essentially the same proof.

Lemma 4.2. *Let Ω and u satisfy the assumption of Theorem 4.1. Then:*

(1) u is L-Lipschitz continuous, with $L = L(n, \mathbf{c}, A)$.
(2) u is nonnegative, $\{u > 0\} = \mathring{\Omega} \setminus \overline{\Sigma}$, and $u \in C^\infty(\mathring{\Omega} \setminus \overline{\Sigma})$.
(3) $\Omega = \mathring{\Omega} \setminus \overline{\Sigma}$ up to a set of measure zero.

(4) At every point $x \in \partial^*\Omega$ it holds

$$\frac{u(x+rz)}{r} \to \mathbf{c}\big(\nu_x \cdot z\big)_+ \qquad \text{as } r \to 0,$$

where ν_x denotes the measure-theoretic inner unit normal at x.

We then show that also Lemma 2.2 holds.

Lemma 4.3. *Let Ω and u satisfy the assumptions in Theorem 4.1. Then*

$$(n+2)\int_\Omega u\,dx = \mathbf{c}^2 n|\Omega|.$$

Proof. As in the proof of Lemma 2.2 we consider the function φ_ϵ and ψ_ϵ defined as in (2.4) and (2.7). Testing (4.1) against φ_ϵ we get

$$\mathbf{c}\int_{\partial^*\Omega} \varphi_\epsilon\, d\mathscr{H}^{n-1} + 2\mathbf{c}\int_\Sigma \varphi_\epsilon\, d\mathscr{H}^{n-1} - \int_\Omega \varphi_\epsilon\, dx = \int_\Omega u\Delta\varphi_\epsilon\, dx. \tag{4.4}$$

We only need to treat the second and last term above, since the others are treated as in the proof of Lemma 2.2.

In this case it holds

$$\begin{aligned}\Delta\varphi_\epsilon(x) = {} & \mathbf{c}\frac{(1+\epsilon)^2\mathscr{H}^{n-1}|_{(1+\epsilon)^{-1}\partial^*\Omega} - (1-\epsilon)^2\mathscr{H}^{n-1}|_{(1-\epsilon)^{-1}\partial^*\Omega}}{2\epsilon}\\ & + 2\mathbf{c}\frac{(1+\epsilon)^2\mathscr{H}^{n-1}|_{(1+\epsilon)^{-1}\Sigma} - (1-\epsilon)^2\mathscr{H}^{n-1}|_{(1-\epsilon)^{-1}\Sigma}}{2\epsilon}\\ & - 2\mathbf{c}\big(\mathscr{H}^{n-1}|_{\partial^*\Omega} + 2\mathscr{H}^{n-1}|_\Sigma\big) + O\left(\frac{1}{\epsilon}dx|_{\mathcal{N}_\epsilon}\right),\end{aligned}$$

therefore (cp. (2.6))

$$\begin{aligned}\int_\Omega u\Delta\varphi_\epsilon\, dx = {} & \mathbf{c}\left[\int_{\partial^*\Omega} \frac{(1+\epsilon)^{3-n}u((1+\epsilon)^{-1}x) - (1-\epsilon)^{3-n}u((1-\epsilon)^{-1}x)}{2\epsilon}\, d\mathscr{H}^{n-1}(x)\right]\\ & + 2\mathbf{c}\left[\int_\Sigma \frac{(1+\epsilon)^{3-n}u((1+\epsilon)^{-1}x) - (1-\epsilon)^{3-n}u((1-\epsilon)^{-1}x)}{2\epsilon}\, d\mathscr{H}^{n-1}(x)\right] + \int_{\mathcal{N}_\epsilon} u\, O\left(\frac{1}{\epsilon}\right) dx.\end{aligned}$$

Hence, recalling (4.4) we deduce that (cp. (2.8))

$$\mathbf{c}\int_{\Omega^*} \psi_\epsilon\, d\mathscr{H}^{n-1} + 2\mathbf{c}\int_\Sigma \psi_\epsilon\, d\mathscr{H}^{n-1} - \int_\Omega \varphi_\epsilon\, dx = \int_{\mathcal{N}_\epsilon} u\, O\left(\frac{1}{\epsilon}\right) dx. \tag{4.5}$$

Note now that applying Lemma 4.2(4), for $\mathscr{H}^{n-1}$-almost every $x \in \partial^*\Omega$ we have

$$\psi_\epsilon(x) \to -\mathbf{c}\big((\nu_x\cdot x)_+ - (\nu_x\cdot x)_-\big) = -\mathbf{c}\nu_x\cdot x,$$

while (4.3) implies that, for $\mathscr{H}^{n-1}$-almost every $x \in \Sigma$,

$$\psi_\epsilon(x) \to \mathbf{c}\big((\nu_x\cdot x)_+ - (\nu_x\cdot x)_-\big) + \mathbf{c}\big((\nu_x\cdot x)_- - (\nu_x\cdot x)_+\big) = 0.$$

Thus, by dominated convergence,

$$\mathbf{c}\int_{\Omega^*} \psi_\epsilon\, d\mathscr{H}^{n-1} + 2\mathbf{c}\int_\Sigma \psi_\epsilon\, d\mathscr{H}^{n-1} \to -\mathbf{c}^2\int_{\partial^*\Omega} \nu_x\cdot x\, d\mathscr{H}^{n-1} \qquad \text{as } \epsilon \to 0.$$

Combining this fact with (4.5), we conclude as in the proof of Lemma 2.2. □

The next step is to prove a maximum principle for $|\nabla u|$. In this case, given a point $x \in \mathring{\Omega} \setminus \overline{\Sigma}$, we define the Green function G_x by considering an increasing sequence Ω_k of smooth sets contained inside $\mathring{\Omega} \setminus \overline{\Sigma}$ such that $\Omega_k \to \mathring{\Omega} \setminus \overline{\Sigma}$ as $k \to \infty$. In this way, the following analog of Lemma 3.2 holds.

Lemma 4.4. *Let G_x be the Green function constructed above. Then:*

(1) G_x is Lipschitz continuous near $\partial\Omega \cup \overline{\Sigma}$, with the Lipschitz constant depending only on n, A, $\mathbf{c}$, Ω, Σ, and x. Moreover, there exist bounded measurable functions $\alpha : \partial^*\Omega \to [0,\infty)$ and $\beta : \Sigma \to [0,\infty)$ such that

$$\Delta G_x = \alpha\, \mathscr{H}^{n-1}|_{\partial^*\Omega} + \beta\, \mathscr{H}^{n-1}|_{\Sigma} - \delta_x.$$

(2) At every point $y \in \partial^*\Omega$ it holds

$$\frac{G_x(y+rz)}{r} \to \mathbf{a}_y\big(\nu_y \cdot z\big)_+ \qquad \text{as } r \to 0,$$

while at every point $y \in \Sigma$ it holds

$$\frac{G_x(y+rz)}{r} \to \mathbf{a}_y^+\big(\nu_y \cdot z\big)_+ + \mathbf{a}_y^-\big(\nu_y \cdot z\big)_- \qquad \text{as } r \to 0,$$

where ν_y denotes the measure-theoretic inner unit normal at y, $\mathbf{a}_y = \alpha(y)$, $\mathbf{a}_y^\pm \geq 0$, and $\mathbf{a}_y^+ + \mathbf{a}_y^- = \beta(y)$.

We can now prove the maximum principle for $|\nabla u|$.

Proposition 4.5. *Let Ω and u satisfy the assumptions in Theorem 4.1. Then*

$$\sup_{\mathring{\Omega}\setminus\overline{\Sigma}} |\nabla u| \leq \mathbf{c}.$$

Proof. As in the proof of Proposition 3.3, we test (4.1) against $\phi_\epsilon(y) = \frac{G_x(y+\epsilon e)-G_x(y-\epsilon e)}{2\epsilon}$ to get

$$\mathbf{c}\int_{\partial^*\Omega} \phi_\epsilon\, d\mathscr{H}^{n-1} + 2\mathbf{c}\int_{\Sigma} \phi_\epsilon\, d\mathscr{H}^{n-1} - \int_\Omega \phi_\epsilon\, dy = \int_\Omega u\Delta\phi_\epsilon\, dy. \tag{4.6}$$

Recalling Lemma 4.4(1) we have

$$\Delta\phi_\epsilon(y) = \frac{\alpha\, \mathscr{H}^{n-1}|_{\partial^*\Omega-\epsilon e} - \alpha\, \mathscr{H}^{n-1}|_{\partial^*\Omega+\epsilon e}}{2\epsilon} + \frac{\beta\, \mathscr{H}^{n-1}|_{\Sigma-\epsilon e} - \beta\, \mathscr{H}^{n-1}|_{\Sigma+\epsilon e}}{2\epsilon} - \frac{\delta_{x-\epsilon e} - \delta_{x+\epsilon e}}{2\epsilon}.$$

Hence, by a change of variable, Lemma 4.2(4), and (4.3), we now get

$$\begin{aligned}\int_\Omega u\Delta\phi_\epsilon\,dx &= \frac{1}{2\epsilon}\left[\int_{\partial^*\Omega-\epsilon e}\alpha(y+\epsilon e)u(y)\,d\mathscr{H}^{n-1} - \int_{\partial^*\Omega+\epsilon e}\alpha(y-\epsilon e)u(y)\,d\mathscr{H}^{n-1}\right]\\ &\quad + \frac{1}{2\epsilon}\left[\int_{\Sigma-\epsilon e}\beta(y+\epsilon e)u(y)\,d\mathscr{H}^{n-1} - \int_{\Sigma+\epsilon e}\beta(y-\epsilon e)u(y)\,d\mathscr{H}^{n-1}\right]\\ &\qquad - \left\langle u,\, \frac{\delta_{x-\epsilon e}-\delta_{x+\epsilon e}}{2\epsilon}\right\rangle\\ &= -\left[\int_{\partial^*\Omega}\frac{u(y+\epsilon e)-u(y-\epsilon e)}{2\epsilon}\alpha(y)\,d\mathscr{H}^{n-1}\right]\\ &\quad - \left[\int_\Sigma \frac{u(y+\epsilon e)-u(y-\epsilon e)}{2\epsilon}\beta(y)\,d\mathscr{H}^{n-1}\right] + \left\langle \frac{u(x+\epsilon e)-u(x-\epsilon e)}{2\epsilon},\, \delta_x\right\rangle\\ &= -\frac{\mathbf{c}}{2}\int_{\partial^*\Omega}\alpha\nu_y\cdot e\,d\mathscr{H}^{n-1} + \partial_e u(x) + o(1)\end{aligned}$$

For the left-hand side of (4.6), we have

$$\int_\Omega \phi_\epsilon\,dy = \int_\Omega \partial_e G_x\,dy + o(1).$$

Moreover, Lemma 4.4(2) implies that, for $\mathscr{H}^{n-1}$-almost every $y\in\partial^*\Omega$ we have

$$\phi_\epsilon(y) = \frac{G_x(y+\epsilon e)-G_x(y-\epsilon e)}{2\epsilon} \to \frac{\mathbf{a}_y}{2}\big((\nu_y\cdot e)_+ - (\nu_y\cdot e)_-\big) = \frac{\alpha(y)}{2}\nu_y\cdot e,$$

while for $\mathscr{H}^{n-1}$-almost every $y\in\Sigma$ we have

$$\phi_\epsilon(y)\to\frac{\mathbf{a}_y^+}{2}\big((\nu_y\cdot e)_+ - (\nu_y\cdot e)_-\big) + \frac{\mathbf{a}_y^-}{2}\big((\nu_y\cdot e)_- - (\nu_y\cdot e)_+\big) = \frac{\mathbf{a}_y^+ - \mathbf{a}_y^-}{2}(\nu_y\cdot e).$$

Thus the left-hand side of (4.6) equals to

$$\frac{\mathbf{c}}{2}\int_{\partial^*\Omega}\alpha\nu_y\cdot e\,d\mathscr{H}^{n-1} + \mathbf{c}\int_\Sigma(\mathbf{a}_y^+ - \mathbf{a}_y^-)\nu_y\cdot e\,d\mathscr{H}^{n-1} - \int_\Omega\partial_e G_x\,dy + o(1).$$

As a result, by letting $\epsilon\to 0$ in (4.6) we eventually arrive at

$$\partial_e u(x) = \mathbf{c}\int_{\partial^*\Omega}\alpha\nu_y\cdot e\,d\mathscr{H}^{n-1} + \mathbf{c}\int_\Sigma(\mathbf{a}_y^+ - \mathbf{a}_y^-)\nu_y\cdot e\,d\mathscr{H}^{n-1} - \int_\Omega\partial_e G_x\,dy.$$

As before, the divergence theorem implies

$$\int_\Omega \partial_e G_x\,dy = -\int_{\partial\Omega}G_x\nu_y\cdot e\,d\mathscr{H}^{n-1} = 0,$$

while Lemma 3.2 yields

$$\begin{aligned}0 = \int_{\mathbb{R}^n}\Delta G_x\,dy = -1 + \int_{\partial^*\Omega}\alpha\,d\mathscr{H}^{n-1} &+ \int_\Sigma\beta\,d\mathscr{H}^{n-1}\\ &\geq -1 + \int_{\partial^*\Omega}\alpha\,d\mathscr{H}^{n-1} + \int_\Sigma|\mathbf{a}_y^+ - \mathbf{a}_y^-|\,d\mathscr{H}^{n-1}.\end{aligned}$$

where we used that $|\mathbf{a}_y^+ - \mathbf{a}_y^-| \le \mathbf{a}_y^+ + \mathbf{a}_y^- = \beta(y)$. This proves that $|\partial_e u(x)| \le \mathbf{c}$, and we conclude by the arbitrariness of x and e. □

Thanks to these preliminary results, we can now repeat the argument in the proof of Theorem 1.3 (with the only difference that P is now defined inside $\mathring{\Omega} \setminus \overline{\Sigma}$) to conclude the validity of Theorem 4.1.

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