

Sharp Quantitative Stability for the Prékopa-Leindler and Borell-Brascamp-Lieb Inequalities

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Abstract

The Borell-Brascamp-Lieb inequality is a classical extension of the Prékopa-Leindler inequality, which in turn is a functional counterpart of the Brunn-Minkowski inequality. The stability of these inequalities has received significant attention in recent years. Despite substantial progress in the geometric setting, a sharp quantitative stability result for the Prékopa-Leindler inequality has remained elusive, even in the special case of log-concave functions. In this work, we provide a unified and definitive stability framework for these foundational inequalities. By establishing the optimal quantitative stability for the Borell-Brascamp-Lieb inequality in full generality, we resolve the conjectured sharp stability for the Prékopa-Leindler inequality as a particular case. Our approach builds on the recent sharp stability results for the Brunn-Minkowski inequality obtained by the authors in [FvHT23, FvHT24].

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1 Introduction

The stability of geometric inequalities has received a lot of attention over recent years. This study goes back to the isoperimetric inequality stating that among all bodies of a given volume, the Euclidean ball minimizes the surface, a classical result attributed to Queen Dido of Carthage. In the landmark result [FMP08] (resp. [FMP10a]), the authors settled the quantitative stability of that inequality (resp. its anisotropic version) by showing that bodies with nearly minimal surface area must be close to a ball (resp. a Wulff shape) in a sharp quantitative sense. However, the isoperimetric is merely the base of a hierarchy of geometric inequalities (see e.g. [Gar02]).

1.1 The Brunn-Minkowski inequality

The Brunn-Minkowski inequality is a fundamental result on which much of convex geometry is built [Sch13]. It asserts that for sets $A, B \subset \mathbb{R}^n$ of equal volume and a parameter $\lambda \in (0, 1)$, we have

$$|\lambda A + (1 - \lambda)B| \geq |A|,$$

with equality if and only if A and B are the same convex body less a measure zero set, up to translation. Note that, up to exchanging the roles of A and B , one can always assume that $\lambda \in (0, 1/2]$.

The stability of the Brunn-Minkowski inequality has a rich history with a series of results obtained in the last years [Ruz91, Ruz97, Ruz06, FMP09, FMP10b, Chr12b, Chr12a, EK14, FJ15, Fig15, FJ17, BJ17, CM17, FJ21, vHST22, vHK23, vHST23a, vHST23b, vHK24], see the introduction of [FvHT23] for a description of these contributions. Recently, the current authors managed to conclude this line of inquiry [FvHT23, FvHT24] showing the following result.¹

Theorem 1.1. *Let $|A| = |B|$ and $\lambda \in (0, 1/2]$, and assume that $|\lambda A + (1 - \lambda)B| \leq (1 + \delta)|A|$ for some $\delta > 0$ sufficiently small in terms of n and λ . Then there exists a convex set K so that, up to a translation,*

$$K \supset A \cup B, \quad |K \setminus A| + |K \setminus B| = O_{n,\lambda}(\sqrt{\delta})|A|.$$

In other words, the theorem above states that A and B are $\sqrt{\delta}$ -close to the same convex set. Moreover, if we do not insist on the same convex set, then a stronger linear stability holds (see [FvHT23]):

$$|\text{co}(A) \setminus A| + |\text{co}(B) \setminus B| = O_{n,\lambda}(\delta)|A|,$$

where $\text{co}(A)$ is the convex hull of A , namely, the smallest convex set containing A . Both the square root and the linear dependence are optimal, in the sense that the powers $1/2$ and 1 cannot be replaced by anything bigger.

¹Here and in the sequel, given $m, a > 0$ we shall write $O_m(a)$ (resp. $\Omega_m(a)$) to denote a quantity that is bounded from above (resp. from below) by $C_m a$, where the constant $C_m > 0$ may depend on m .

1.2 The Prékopa-Leindler inequality

The Brunn-Minkowski inequality has a functional extension called the Prékopa-Leindler inequality. Given integrable functions $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and a parameter $\lambda \in (0, 1/2]$, assume that $\int f dx = \int g dx$ and that, for all $x, y \in \mathbb{R}^n$, we have

$$h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$$

(here f, g, h mirror $A, B, \lambda A + (1 - \lambda)B$ in the Brunn-Minkowski inequality). Then, the Prékopa-Leindler inequality states that $\int h dx \geq \int f dx$, with equality if and only if f and g are (up to a translation) the same *log-concave* function almost everywhere.²

When restricted to indicator functions of sets, the Prékopa-Leindler inequality directly implies the Brunn-Minkowski inequality. In the reverse direction, the Prékopa-Leindler inequality can be deduced from the Brunn-Minkowski inequality, by associating to a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ the set $A_f \subset \mathbb{R}^{n+k}$ defined by

$$A_f := \bigcup_{x \in \mathbb{R}^n} \{x\} \times \underbrace{\left[0, f(x)^{1/k}\right] \times \dots \times \left[0, f(x)^{1/k}\right]}_{k \text{ times}},$$

so that $|A_f| = \int f dx$. Applying Brunn-Minkowski to the sets A_f and A_g , and noticing that $|\lambda A_f + (1 - \lambda)A_g| \downarrow \int h dx$ as $k \rightarrow \infty$,³ the Prékopa-Leindler inequality follows.

Despite the interest in the stability of Prékopa-Leindler [BF14, BB10, BB11, BD21, BFR23], sharp stability results have been harder to obtain. Indeed, all previous results provide quantitative estimates but with non-sharp exponents. Furthermore, except for the very special case where one considers one-dimensional log-concave functions [FR24], even for special classes of functions (like log-concave functions in \mathbb{R}^n or one-dimensional arbitrary functions), no optimal quantitative results are known.

As for the Brunn-Minkowski inequality, it was believed that a sharp $\sqrt{\delta}$ -stability should persist in the functional context. More precisely, let f, g, h , and $\lambda \in (0, 1/2]$ as before. If $\int h dx \leq (1 + \delta) \int f dx$ for some $\delta \geq 0$, then there should exist a log-concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ so that, up to a translation,

$$\int (|\ell - f| + |g - \ell|) dx = O_{\lambda, n}(\sqrt{\delta}) \int f dx.$$

Despite being a very natural statement, one major challenge here is to identify the “correct” log-concave function ℓ . Indeed, even when $f = g$, ℓ *cannot* be chosen as the log-concave hull of f (see Remark 1.11 below). In this paper, we settle this line of investigation by proving the bound above as a special case of our main theorem.

1.3 The Borell-Brascamb-Lieb inequality

Although the Prékopa-Leindler condition $h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$ seems natural from the perspective of the previously mentioned proof using limits of sets in \mathbb{R}^n , it is, in fact, stronger than what is needed, as noticed by Borell [Bor75] and independently by Brascamb and Lieb [BL76]. To state their result, consider the following definition.

Definition 1.2. *Given parameters $\lambda \in [0, 1]$ and $p \in \mathbb{R}$, and real numbers $x, y \geq 0$, let*

$$M_{\lambda, p}(x, y) = \begin{cases} 0 & \text{if } xy = 0 \\ (\lambda x^p + (1 - \lambda)y^p)^{1/p} & \text{if } p \neq 0 \text{ and } xy \neq 0. \\ x^\lambda y^{1-\lambda} & \text{if } p = 0 \text{ and } xy \neq 0 \end{cases}$$

For $p = 0$, this definition corresponds to the Prékopa-Leindler condition, as $h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} = M_{\lambda, 0}(f(x), g(y))$. Also, by the monotonicity of means, we have $M_{\lambda, p}(x, y) \leq M_{\lambda, q}(x, y)$ for $p < q$, so the condition $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$ gets weaker as p decreases.

The following definition is also important:

Definition 1.3. *A function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is p -concave if for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, we have that $\ell(\lambda x + (1 - \lambda)y) \geq M_{p, \lambda}(\ell(x), \ell(y))$.*

²Recall that a function f is log-concave if for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$.

³More precisely, assuming that f, g are bounded and with bounded support, then $|\lambda A_f + (1 - \lambda)A_g| \downarrow \int h dx$ as $k \rightarrow \infty$ by dominated convergence. This implies the Prékopa-Leindler inequality in this case, and the general case follows by approximation.

In these terms, 0-concave is the same as log-concave. Perhaps more intuitively, a function f is p -concave for $p > 0$ (resp. $p < 0$) if its support is convex and f^p is concave (resp. convex) inside its support. The Borell-Brascamb-Lieb inequality asserts that we can decrease the parameter p in the Prékopa-Leindler condition all the way down to $p = -1/n$.

Theorem 1.4 (Borell-Brascamb-Lieb Inequality). *Let $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in [-1/n, \infty)$, and $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_+$ integrable functions. Assume that $\int f dx = \int g dx$ and*

$$h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int h dx \geq \int f dx.$$

In addition, if $p > -1/n$, then equality holds if and only if f and g are, up to a translation, the same p -concave function a.e.

Remark 1.5. *The family of equality cases for $p = -1/n$ is richer than the one for $p > -1/n$, e.g., equality holds for the triple*

$$f = \mathbf{1}_{[0,1]^n}, \quad g = t^{-n} \mathbf{1}_{[0,t]^n}, \quad \text{and} \quad h = M_{\lambda,-1/n}(1, t^{-n}) \mathbf{1}_{[0,\lambda+(1-\lambda)t]^n} = (\lambda + (1 - \lambda)t)^{-n} \mathbf{1}_{[0,\lambda+(1-\lambda)t]^n}.$$

Characterizing the equality cases for $p = -1/n$ is a non-trivial task done by Dubuc [Dub77].

Although the equality case for the Borell-Brascamb-Lieb inequality is well understood (see [BK18] for a recent proof that works also on curved spaces), very little is known at the level of stability. In particular, the papers [GS17, RS17] deal with the case $p > 0$, which is weaker than Prékopa-Leindler.

1.4 Main results

The main result of this paper is a sharp stability result for the Borell-Brascamb-Lieb inequality for $p > -1/n$.

Theorem 1.6. *Let $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, and $p \in (-1/n, \infty)$. Let $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

- $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx$,
- $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$ for all $x, y \in \mathbb{R}^n$, and
- $\int_{\mathbb{R}^n} h dx = (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $\delta \geq 0$.

Then there exists a p -concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that, up to a translation⁴,

$$\int_{\mathbb{R}^n} (|f - \ell| + |g - \ell|) dx = O_{n,\lambda,p}(\sqrt{\delta}) \int_{\mathbb{R}^n} f dx.$$

In the special case $p = 0$, we obtain the optimal quantitative stability of the Prékopa-Leindler inequality, resolving the main conjecture in this direction and improving the results from [BF14, BB10, BB11, BD21, BFR23, FR24].

Corollary 1.7. *Let $n \in \mathbb{N}$ and $\lambda \in (0, 1/2]$. Let $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

- $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx$,
- $h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$ for all $x, y \in \mathbb{R}^n$, and
- $\int_{\mathbb{R}^n} h dx = (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $\delta \geq 0$.

Then there exists a log-concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that, up to a translation,

$$\int_{\mathbb{R}^n} (|f - \ell| + |g - \ell|) dx = O_{n,\lambda}(\sqrt{\delta}) \int_{\mathbb{R}^n} f dx.$$

⁴That is, there exists a $v \in \mathbb{R}^n$, so that $\int_{\mathbb{R}^n} (|f(x) - \ell(x)| + |g(x - v) - \ell(x)|) dx = O_{n,\lambda,p}(\sqrt{\delta}) \int_{\mathbb{R}^n} f(x) dx$.

The assumption that $\int f dx = \int g dx$ in these theorems and all previous discussions is a matter of normalization, and the general case can be obtained by rescaling (which simply changes the value of λ).

While the sharp stability of the Brunn-Minkowski inequality (Theorem 1.1) serves as a fundamental tool throughout the proofs, the functional nature of these inequalities demands a highly refined analysis and the introduction of several new ideas. More precisely, as detailed in Sections 3 and 8 below, our results rely critically on the stability of the Brunn-Minkowski inequality, yet achieving these results requires a completely novel approach.

Remark 1.8. *As mentioned before, the square root dependence is optimal, as can be seen by considering the one-dimensional example*

$$f = \frac{1}{1 + \sqrt{\delta}} \mathbf{1}_{[0, 1 + \sqrt{\delta}]}, \quad g = (1 + \sqrt{\delta}) \mathbf{1}_{[0, \frac{1}{1 + \sqrt{\delta}}]}, \quad \text{and} \quad h = \mathbf{1}_{\left[0, \frac{1 + \sqrt{\delta} + \frac{1}{1 + \sqrt{\delta}}}{2}\right]}.$$

Since the family of equality cases for $p = -1/n$ is richer than the class of $(-1/n)$ -concave functions (see Remark 1.5), Theorem 1.6 does not hold for $p = -1/n$. In particular, the constant multiplying $\sqrt{\delta}$ must blow up as $p \rightarrow -1/n^+$.

Regarding the dependence on λ , based on corresponding results for the Brunn-Minkowski inequality (see, in particular, [FMP09, BJ17, FMM18]), it is natural to expect that the expression $O_{n, \lambda, p}(\sqrt{\delta})$ can be replaced by $O_{n, p}(\sqrt{\delta/\lambda})$. However, proving this bound would require nontrivial work beyond the scope of the current paper, which already introduces several intricate tools and ideas. Therefore, we leave this investigation for future work.

1.5 Proof of Theorem 1.6

We will prove Theorem 1.6 in two steps. First, as stated in the following theorem, we prove that the two functions are close to each other (this is the equivalent of finding an upper bound on $|A \triangle B|$ in the context of the Brunn-Minkowski inequality).

Theorem 1.9. *Let $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, and $p \in (-1/n, \infty)$. Let $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

- $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx$,
- $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$ for all $x, y \in \mathbb{R}^n$, and
- $\int_{\mathbb{R}^n} h dx = (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $\delta \geq 0$.

Then, up to a translation,

$$\int_{\mathbb{R}^n} |f - g| dx = O_{n, \lambda, p}(\sqrt{\delta}) \int_{\mathbb{R}^n} f dx.$$

Second, we prove the following linear stability result in the case where $f = g$, showing that f must be linearly close to a p -concave function.

Theorem 1.10. *Let $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, and $p \in (-1/n, \infty)$. Let $f, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that*

- $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), f(y))$ for all $x, y \in \mathbb{R}^n$, and
- $\int_{\mathbb{R}^n} h dx = (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $\delta \geq 0$.

Then there exists a p -concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\int_{\mathbb{R}^n} |f - \ell| dx = O_{n, \lambda, p}(\delta) \int_{\mathbb{R}^n} f dx.$$

Remark 1.11. *This second result can be seen as a functional extension of the linear results in [Fig15, FJ17, vHST23a], which were proved in the run-up to the recently established stability of the Brunn-Minkowski inequality. More precisely, as shown in [vHST23a], given $A \subset \mathbb{R}^n$ with $|\frac{A+A}{2} \setminus A|$ small, we have*

$$|\text{co}(A) \setminus A| = O_n \left(\left| \frac{A+A}{2} \setminus A \right| \right).$$

However, contrary to the case of sets (where $\text{co}(A)$ contains A), in Theorem 1.10 we cannot additionally demand that $\ell \geq f$ everywhere. This can be seen by considering $f = \mathbf{1}_{[0,1]} + \delta^{100/\lambda} \mathbf{1}_{[v,v+1]}$ for some large number $v \in \mathbb{R}$. One can note that, for any p -concave function ℓ with $\ell \geq f$ a.e., $\int \ell \rightarrow \infty$ as $v \rightarrow \infty$. On the other hand, if we let $h = \mathbf{1}_{[0,1]} + \delta^{100(1-\lambda)/\lambda} \mathbf{1}_{[(1-\lambda)v, (1-\lambda)v+1]} + \delta^{100} \mathbf{1}_{[\lambda v, \lambda v+1]} + \delta^{100/\lambda} \mathbf{1}_{[v, v+1]}$, then

$$h(\lambda x + (1-\lambda)y) \geq M_{\lambda,0}(f(x), f(y)) \geq M_{\lambda,p}(f(x), f(y)) \quad \forall p \leq 0$$

and $\int h dx \leq (1+\delta) \int f dx$, so our theorem applies.

We now show how Theorems 1.9 and 1.10 directly imply our main result.

Proof of Theorem 1.6. By Theorem 1.9 we have that, up to a translation, $\int |f - g| dx \leq O_{n,\lambda,p}(\sqrt{\delta}) \int f dx$. Thus, if we define $k := \min\{f, g\}$, it follows that

$$\int k dx \geq \left(1 - O_{n,\lambda,p}(\sqrt{\delta})\right) \int f dx \quad \text{and} \quad h(\lambda x + (1-\lambda)y) \geq M_{\lambda,p}(k(x), k(y)).$$

Hence, applying Theorem 1.10 to k and h , we find a p -concave function ℓ such that $\int |\ell - k| dx = O_{n,\lambda,p}(\sqrt{\delta}) \int k dx$. Combining these two results yield

$$\int (|f - \ell| + |g - \ell|) dx \leq \int (|f - k| + |k - \ell| + |g - k| + |k - \ell|) dx = O_{n,\lambda,p}(\sqrt{\delta}) \int f dx,$$

as desired. \square

Because of this discussion, the main goal of this paper will be to prove Theorems 1.9 and 1.10.

1.6 Structure of the paper

After introducing some notation, Sections 3-7 are devoted to the proof of Theorem 1.9. More precisely, after describing the structure of its proof, we first prove the result in one dimension, then in two dimensions, and finally in the general case. The final Sections 8-9 are devoted to the proof of Theorem 1.10. To better navigate through the paper, we invite the reader to look at the table of contents on the first page.

2 Notation

Here and in the following, $p \in (-1/n, \infty)$ and $\lambda \in (0, 1/2]$.

Definition 2.1. For a function $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$, we denote by $\text{co}_p(f): \mathbb{R}^n \rightarrow \mathbb{R}_+$ the minimal p -concave function such that $\text{co}_p(f) \geq f$ everywhere in \mathbb{R}^n .

Definition 2.2. Given functions $f, g: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ define their (λ, p) -sup-convolution $M_{\lambda,p}^*(f, g)$ by

$$M_{\lambda,p}^*(f, g): \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}, \quad z \mapsto \sup_{x,y:\lambda x+(1-\lambda)y=z} M_{\lambda,p}(f(x), g(y)).$$

Definition 2.3. A convex set $C \subset \mathbb{R}^n$ is called a cone if there exists a hyperplane H not containing the origin and a bounded convex set $P \subset H$ such that

$$C = \bigsqcup_{t \geq 0} tP.$$

Note that a cone always has its vertex at the origin, which we will denote by o .

Definition 2.4. Let S_n denote a regular simplex in \mathbb{R}^n . Denote by F_n^0, \dots, F_n^n the faces of S_n . Assume the distance from o to F_n^0, \dots, F_n^n is 1. Denote H_n^0, \dots, H_n^n the supporting hyperplanes. Finally, construct the half-spaces $H_n^{i,-}, H_n^{i,+}$, where the former contains o .

We shall assume that the simplex S_n and the basis e_1, \dots, e_n of \mathbb{R}^n are such that $e_1 \perp H_n^1$.

Definition 2.5. Construct the cones $C_n^i = \bigsqcup_{t \geq 0} tF_n^i$ and let $\mathfrak{C}_n = \{C_n^0, \dots, C_n^n\}$.

Note that \mathfrak{C}_n forms an essential partition of \mathbb{R}^n .

3 Theorem 1.9: structure of the proof

The proof of Theorem 1.9 consists of three parts. More precisely, after collecting in the next section a series of preliminary results, we first prove the 1-dimensional version. Then we use this result to prove the 2-dimensional case, and finally we show that the 2-dimensional result implies the n -dimensional case.

3.1 Overview of the 1-dimensional proof

The initial reduction for this part of the proof is included in Section 4. The specific reduction is then in Section 5.

1. We define level sets $F_s := \{x \in \mathbb{R} : f(x) > s\}$ (and G_s and H_s analogously).
2. Then F_0 is the support of f and $\int_{\mathbb{R}} f(x) dx = \int_0^\infty |F_s| ds$ (analogously for g and h).
3. We use the inclusion $\lambda F_s + (1 - \lambda)G_t \subset H_u$ for $M_{\lambda,p}(s, t) = u$ with properly chosen s and t .
4. Using this lower bound on $|H_u|$, we find that typically $|F_s| \approx |G_t|$ and $\lambda|F_s| + (1 - \lambda)|G_t| \approx |H_u|$ for these s and t .
5. Combining this information with the stability of the 1-dimensional Brunn-Minkowski inequality, we deduce that $|\text{co}(F_s) \setminus F_s|$ and $|\text{co}(G_t) \setminus G_t|$ are typically small.
6. Up to changing f and g little, we can assume that all the level sets are nearly intervals.
7. Considering the transport map T from f to g and using the bound $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$, we find that typically $\frac{dT}{dx}(x) = \frac{f(x)}{g(T(x))}$ is close to 1.
8. In particular, changing f and g only little, we find new functions f_1 and g_1 so that $\frac{dT}{dx}$ is almost everywhere close to one.
9. Since all of these steps may have caused gaps in the support of f_1 and g_1 , we next push the support of these functions together to remove the gaps to create f_2 and g_2 .
10. Using that the level sets were near intervals, we show that f_1 and g_1 differ little from f_2 and g_2 .
11. Next, we fill in the level sets so that they become intervals, to find function f_3 and g_3 changing little from f_2 and g_2 .
12. Finally we define the auxiliary function $G(x) := g_3(T_2(x))$, where T_2 is the transport map pushing f_2 onto g_2 , and we control the differences $|f_3 - G|$ and $|g_3 - G|$.

3.2 Overview of the 2-dimensional proof

The initial reduction for this part of the proof is included in Section 4. The specific reduction is then in Section 6. First we establish the result on well-behaved tubes. More precisely, assume first that f and g are defined on parallel tubes that are infinite in one direction, and that f and g are large near the base. Then the result follows as described in the next six steps.

1. Let $f : [0, r] \times [0, \infty) \rightarrow [0, 1]$ and $g : [0, r'] \times [0, \infty) \rightarrow [0, 1]$, and assume that for all $x \in [0, r] \times [0, 1]$ we have $f(x) \geq 0.1$, and similarly for $x \in [0, r'] \times [0, 1]$ we have $g(x) \geq 0.1$.
2. Consider the map $T : [0, r] \rightarrow [0, r']$ so that $T(0) = 0$ and $\frac{dT}{dx}(x) = \frac{\int f(x, y) dy}{\int g(T(x), y) dy}$.
3. Using the 1-dimensional Borell-Brascamb-Lieb inequality, we find that

$$\int h(\lambda x + (1 - \lambda)T(x), y) dy \geq M_{\lambda, q} \left(\int f(x, y) dy, \int g(T(x), y) dy \right),$$

for some $q > -1$ depending on $p > -1/2$. Integrating over x , this implies

$$\int h(x, y) dx dy \geq \int f(x, y) dx dy.$$

- Using the sharp stability in the one-dimensional case twice (in both of the two inequalities above), we find that for all $x \in [0, r]$ there is some translation v_x so that

$$\int |f(x, y) - g(T(x), y - v_x)| dy$$

is small. Moreover, $\frac{dT}{dx}$ is typically close to 1, and thus $T(x)$ is close to x (recall that $T(0) = 0$).

- Since the functions are “big” (say $f, g \geq 0.1$) close to the base $x = 0$, we find that v_x must be small for all x .
- Finally, we use that the level sets of f and g are nearly convex, so that shifting the functions by the small translations $T(x) - x$ and v_x doesn't affect the symmetric difference much, so that indeed $\int |f - g| dx$ is small.

The next step is to use the previous result for suitable restrictions of f and g to tubes. To this aim, we need to make sure that the corresponding tubes are close together. We can do this well enough to find small symmetric difference in the region where f and g are not smaller than say 0.0001.

- After affinely transforming the domain, we find that in a disk D of constant radius around the origin both f and g are mostly at least 0.1.
- We can partition \mathbb{R}^2 into 3 cones C at the origin so that, inside each of them, $\int_C f dx = \int_C g dx \geq 0.1 \int f dx$.
- Consider a tube (i.e., an affine image of $[0, 1] \times [0, \infty)$) whose base (i.e., the image of $[0, 1] \times \{0\}$) is contained in $C \cap \frac{1}{2}D$.
- There is a corresponding parallel tube R' so that $\int_R f dx = \int_{R'} g dx$ and $\int_{\lambda R + (1-\lambda)R'} h dx \leq (1 + O(\delta)) \int_R f dx$. Moreover, R and R' are close together, so that $\int_{R \Delta R'} (f + g) dx$ is small.
- Using the above result in tubes, we find that $\int_R |f - g| dx$ is small.
- As we can choose the width of the tube R on the scale of D , we can cover $100D$ with say 1000 different tubes R , so that $\int_{100D} |f - g| dx$ is small.

Finally, we want to repeatedly trim the functions f and g so that the different instances of $100D$ cover everything.

- Consider level set $F_t := \{x \in \mathbb{R}^2 : f(x) > t\}$ (and the analogous G_t and H_t). By a transport approach leveraging the stability of the Brunn-Minkowski inequality [FvHT23] we find that these level sets are nearly convex and of comparable size and shape.
- We additionally find that they cannot change size too rapidly, so that $F_{0.1t} \subset 100F_t$.
- One consequence is that not too much mass can be in low level sets, i.e., $\int_{\mathbb{R}^2 \setminus F_{\delta^{100}}} f dx$ is small, so we can restrict our attention to $f \cdot \mathbf{1}_{F_{\delta^{100}}}$.
- Consider the truncated functions $f^i := \min\{f, 2^{-i}\}$ (and corresponding g^i) for $i = 1, \dots, \log_2(\delta^{100})$.
- By the previous result controlling the symmetric difference for not too small f and g , we find that there exist translates v_i so that $\int_{F_{0.001 \cdot 2^{-i}}} |f_i(x) - g_i(x + v_i)| dx$ is small for all i .
- Since there is a serious overlap between these integrals for consecutive i 's, we find that the difference between v_i and v_{i+1} must be small.
- By a careful analysis of the interaction between the different i 's we find that, in fact, $\sum_i \int_{F_{0.001 \cdot 2^{-i}}} |f_i(x) - g_i(x)| dx$ is small.
- This sum provides an upper bound for $\int |f - g| dx$ so that we can conclude.

3.3 Overview of the n -dimensional proof

The initial reduction for this part of the proof is included in Section 4. The specific reduction is then in Section 7. For the reduction from n dimensions to 2 dimensions, we use the following steps.

1. We first use a transport approach through the level sets, leveraging the sharp stability of the Brunn-Minkowski inequality [FvHT23], to show that the level sets $F_t := \{x \in \mathbb{R}^n : f(x) > t\}$ (and the analogous G_t and H_t) are nearly convex and of similar size.
2. After affinely transforming the domain, we find that in a ball around the origin both f and g are mostly at least 0.1 (cf. Proposition 4.1).
3. We then partition \mathbb{R}^n into $n + 1$ reasonable looking cones K at the origin, so that $\int_K f dx = \int_K g dx \geq \Omega_n (\int f dx)$ (cf. Proposition 7.1).
4. Since f and g are ≥ 0.1 on most of a ball around the origin, if we prove the result in each of these cones, the translation in each of the cones can be taken to be zero.
5. Using a technical result from [FvHT23], we can further partition \mathbb{R}^n into smaller (very very small) cones (maintaining the previous properties) with the additional property that in all but one direction the cone is so narrow that the function is nearly constant on fibers in those directions (cf. Proposition 7.2).
6. Given a small cone C provided by the previous step, let $C^{z,w}$ be those $(n - 2)$ -dimensional fibers so that $\max_{x,y \in C^{z,w}} |f(x) - f(y)| + |g(x) - g(y)|$ is small (here the smallness can be chosen as small as desired).
7. If we let $F(z, w) = |C^{z,w}| \cdot \min_{x \in C^{z,w}} f(x)$ (and $G(z, w), H(z, w)$ analogously), then

$$\int_{\mathbb{R}^2} F(z, w) dzdw \approx \int_C f dx \text{ and } \int_{\mathbb{R}^2} |F(z, w) - G(z, w)| dzdw \approx \int_C |f - g| dx$$

(cf. Proposition 7.11).

8. Since the cone is convex (so $(z, w) \mapsto |C^{z,w}|$ is $1/(n - 2)$ -concave), we can use Hölder's inequality to find the following: For every $p > -1/n$ there is a $q > -1/2$ so that if $M_{\lambda,p}(f(x), g(y)) \geq h(\lambda x + (1 - \lambda)y)$ for some $x \in C^{z,w}$ and $y \in C^{z',w'}$, then $M_{\lambda,q}(F(z, w), G(z', w')) \geq H(\lambda z + (1 - \lambda)z', \lambda w + (1 - \lambda)w')$ (cf. Theorem 6.1).
9. We finally apply the two-dimensional theorem to the functions F, G, H to prove the result in each of the cones, and thus in general.

4 Preliminary results for the proof of Theorem 1.9

In this section we show how Theorem 1.9 is implied by the following proposition. Most of the weight of that reduction is in Lemma 4.6 and its more easily applicable Corollary 4.7.

Proposition 4.1. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $r \in (0, \infty)$ there exists $d = d_{n,\lambda,p,r}$ such that the following holds. Let $f, g, h: \mathbb{R}^n \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx = 1$,
- for every $i \in [0, n]$, $\int_{C_n^i} f dx = \int_{C_n^i} g dx$,
- $|\{x \in rS_n : f(x) < \beta \text{ or } g(x) < \beta\}| = O_{n,\lambda,p,r}(\delta)$ and
- for all $x, y \in \mathbb{R}^n$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$.

If $\int_{\mathbb{R}^n} h dx = 1 + \delta$ for some $0 \leq \delta \leq d$, then $\int_{\mathbb{R}^n} |f - g| dx = O_{n,\lambda,p,r,\beta}(\sqrt{\delta})$.

First, we show that we may assume that f, g, h are continuous.

Lemma 4.2. *Given $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ integrable functions with $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx$ and $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$ for all $x, y \in \mathbb{R}^n$, then for any $\eta > 0$ there exist functions $f', g', h': \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ continuous functions with $h'(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f'(x), g'(y))$ and $\int_{\mathbb{R}^n} (|f - f'| + |g - g'| + |h - h'|) dx < \eta$.*

Next we show that f, g, h take values in $[0, 1]$ and have bounded domains.

Lemma 4.3. *Let $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be continuous functions such that*

- $\int_{\mathbb{R}^n} f \, dx = \int_{\mathbb{R}^n} g \, dx = 1$,
- for all $x, y \in \mathbb{R}^n$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$, and
- $\int_{\mathbb{R}^n} h \, dx = 1 + \delta$ for some $0 \leq \delta \leq d$.

Then there exist continuous functions $f', g', h': \mathbb{R}^n \rightarrow [0, 1]$ and an affine transformation $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the following properties.

- the support of f', g' , and h' is bounded,
- $\int_{\mathbb{R}^n} f' \, dx = \int_{\mathbb{R}^n} g' \, dx = 1$,
- for all $x, y \in \mathbb{R}^n$ we have $h'(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f'(x), g'(y))$,
- $\int_{\mathbb{R}^n} (|f(x) - \det(A)^{-1} f'(Ax)| + |g(x) - \det(A)^{-1} g'(Ax)|) \, dx \leq \delta^{100}$.

We can find translates of the cones from Definition 2.5 which evenly partition f and g .

Lemma 4.4. *Let $f, g: \mathbb{R}^n \rightarrow [0, 1]$ be continuous functions with bounded support.*

Then, there exist $a, b \in \mathbb{R}^n$ such that for every $i \in \{0, \dots, n\}$

$$\int_{a+C_n^i} f \, dx = \int_{b+C_n^i} g \, dx = \frac{1}{n+1} \int_{\mathbb{R}^n} f \, dx.$$

Using the stability results for the Brunn-Minkowski inequality from [FvHT23, FvHT24], we derive in Lemma 4.6 that the level sets of f and g must be almost convex and almost the same. Moreover, we find that most of f and g is contained in a level set that is not too low. We use the following fact about the averages $M_{\lambda, p}$.

Lemma 4.5. *For $p \in (-1, 0)$, $\lambda \in (0, 1)$, $t \in \mathbb{R}$ and differentiable map $T: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, we have*

$$\frac{d}{dt} M_{\lambda, p}(t, T(t)) \geq \frac{1}{M_{\lambda, -p}\left(1, \frac{1}{\frac{dT}{dt}}\right)}$$

Lemma 4.6. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $\alpha, \alpha' > 0$ there exists $d = d_{n, \lambda, p, \alpha'} > 0$ and $\beta = \beta_{n, \lambda, p, \alpha'} > 0$ such that the following holds. Let $f, g, h: \mathbb{R}^n \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int_{\mathbb{R}^n} f \, dx = \int_{\mathbb{R}^n} g \, dx = 1$,
- for all $x, y \in \mathbb{R}^n$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$,
- $\int_{\mathbb{R}^n} h \, dx = (1 + \delta)$ for some $\delta \leq d$.

Let $F_t := \{x \in \mathbb{R}^n : f(x) > t\}$, $G_t := \{x \in \mathbb{R}^n : g(x) > t\}$, and $H_t := \{x \in \mathbb{R}^n : h(x) > t\}$. Let $T: [0, 1] \rightarrow [0, 1]$ be the transport map with $\frac{dT}{dt}(t) = \frac{|F_t|}{|G_{T(t)}}$, i.e., so that $\int_0^t |F_s| \, ds = \int_0^{T(t)} |G_s| \, ds$. Define the following subsets of $[0, 1]$ depending on parameter $\alpha > 0$;

$$\begin{aligned} I_1 &:= \left\{ t \in [0, 1] : \frac{dT}{dt}(t) \notin [1 - \alpha, 1 + \alpha] \right\} \\ I_2 &:= \{t \in [0, 1] : |\text{co}(F_t) \setminus F_t| \geq \alpha |F_t| \text{ or } |\text{co}(G_{T(t)}) \setminus G_{T(t)}| \geq \alpha |G_{T(t)}|\} \\ I_3 &:= \{t \in [0, 1] : |\lambda F_t + (1 - \lambda)G_{T(t)}| \geq (1 + \alpha) M_{\lambda, 1/n}(|F_t|, |G_{T(t)}|)\} \\ I_4 &:= \{t \in [0, 1] : \forall x \in \mathbb{R}^n, |(x + F_t) \triangle H_{M_{\lambda, p}(t, T(t))}| \geq \alpha |F_t|\} \\ I_5 &:= \{t \in [0, 1] : \forall x \in \mathbb{R}^n, |(x + \text{co}(F_t)) \triangle \text{co}(G_{T(t)})| \geq \alpha |F_t|\} \end{aligned}$$

and let $I := \bigcup I_i$. Then

$$\int_I |F_t| \, dt = \int_{T(I)} |G_t| \, dt = O_{n, p, \lambda, \alpha}(\delta),$$

$$\int_{[0,1] \setminus I} (|\text{co}(F_t) \setminus F_t| + |\text{co}(G_{T(t)}) \setminus G_{T(t)}|) dt = O_{n,p,\lambda}(\delta).$$

Moreover,

$$\int_{F_{\beta \sup f}} f dx, \int_{G_{\beta \sup g}} g dx \geq 1 - \alpha'.$$

Corollary 4.7. *Given f, g, h as in Lemma 4.6, then for every $\alpha > 0$ we can find $f', g' : \mathbb{R}^n \rightarrow [0, 1]$ so that*

- $f' \leq f$ and $g' \leq g$,
- $\int f' = \int g'$,
- $\int (|f - f'| + |g - g'|) \leq O_{n,\lambda,p,\alpha}(\delta)$, and
- $\int h \leq (1 + O_{n,\lambda,p,\alpha}(\delta)) \int f'$.

Let $F'_t := \{x \in \mathbb{R}^n : f'(x) > t\}$, $G'_t := \{x \in \mathbb{R}^n : g'(x) > t\}$, and $H_t := \{x \in \mathbb{R}^n : h(x) > t\}$, the level sets of f', g', h respectively. Let $T : [0, 1] \rightarrow [0, 1]$ be the transport map with $\frac{d}{dt}T(t) = \frac{|F'_t|}{|G'_{T(t)}|}$, i.e., so that $\int_0^t |F'_s| ds = \int_0^{T(t)} |G'_s| ds$. Then for all $t \in [0, 1]$ we have

- $\frac{d}{dt}T(t) \in [1 - \alpha, 1 + \alpha]$,
- $|\text{co}(F'_t) \setminus F'_t| \leq \alpha |F'_t|$ and $|\text{co}(G'_t) \setminus G'_t| \leq \alpha |G'_t|$,
- $|\lambda F'_t + (1 - \lambda)G'_{T(t)}| \leq (1 + \alpha)M_{\lambda,1/n}(|F'_t|, |G'_{T(t)}|)$,
- $\exists x_t \in \mathbb{R}^n, |(x_t + F'_t) \Delta H'_{M_{\lambda,p}(t, T(t))}| \leq \alpha |F'_t|$, and
- $\exists x_t \in \mathbb{R}^n, |\text{co}(x_t + F'_t) \Delta \text{co}(G'_{T(t)})| \leq \alpha |F'_t|$.
- $\int_{F'_{\beta \sup f}} f dx, \int_{G'_{\beta \sup g}} g dx \geq 1 - \alpha'$.

One example of a quick consequence of Lemma 4.6 that we will use in the proof in two dimensions is the following result for functions which are rarely small. As in Lemma 4.6, β can be thought of as $\Omega_{n,\lambda,p}(1)$.

Lemma 4.8. *Given $f, g, h : \mathbb{R}^n \rightarrow \{0\} \cup [\beta, 1]$ and $\delta > 0$ sufficiently small with respect to n, p , and λ with*

- $h = M_{\lambda,p}^*(f, g)$
- $\int f dx = \int g dx = 1$
- $\int h dx \leq 1 + \delta$

Then (up to translation) $|\text{supp}(f) \Delta \text{supp}(g)| \leq O_{n,\lambda,p,\beta}(\sqrt{\delta})$.

Moreover, if functions $f, g, h : \mathbb{R}^n \rightarrow [0, 1]$ don't just take values in $\{0\} \cup [\beta, 1]$, but we do have $|\{x : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma$, then (up to translation) $|\text{supp}(f) \Delta \text{supp}(g)| \leq O_{n,\lambda,p,\beta}(\sqrt{\delta + \gamma})$.

Finally, we record that for functions as in Corollary 4.7, slight translations don't result in big symmetric difference.

Lemma 4.9. *Let $f : \mathbb{R}^n \rightarrow [0, 1]$ with $\int f dx = 1$ and level sets $F_t := \{x : f(x) > t\}$, so that $\int_0^1 |\text{co}(F_t) \setminus F_t| dt \leq \delta$. If for some $t_0, \eta > 0$, we have $o, v \in \eta \text{co}(F_{t_0})$ and $|\text{co}(F_{t_0})| \leq 2|F_{t_0}|$, then*

$$\int (f(x) - f(x + v)) dx = O_{n,t_0}(\delta + \eta).$$

4.1 Proof of Lemmas

Proof of Lemma 4.3. Without loss of generality, we may assume $\sup f \geq \sup g$. Let $\ell := (\max\{\sup f, \sup g\})^{-1/n}$ and let

$$f_1 : \mathbb{R}^n \rightarrow [0, 1], x \mapsto \ell^n f(\ell^{-1}x), g_1 : \mathbb{R}^n \rightarrow [0, 1], x \mapsto \ell^n g(\ell^{-1}x) \text{ and } h_1 : \mathbb{R}^n \rightarrow [0, 1], x \mapsto \ell^n h(\ell^{-1}x).$$

To reduce to sets with bounded support, note that for sufficiently large $r = r_{f,\delta}$ in terms of f and δ , we have that $\int_{[-r,r]^n} f_1 dx = 1 - \delta^{1000}$. Similarly, we can find $r' = r'_{g_1,\delta}$, so that $\int_{[-r',r']^n} g dx = 1 - \delta^{1000}$. Hence, let $f' := f_1 \cdot \mathbf{1}[-r,r]^n$ and $g' := g_1 \cdot \mathbf{1}[-r',r']^n$. Let $h' = h_1$, so that

$$\int h' dx = \int h_1 dx = \int h dx \leq (1 + \delta) \int f dx \leq \frac{1 + \delta}{1 - \delta^{1000}} \int f' dx \leq (1 + 2\delta) \int f' dx.$$

□

Before we prove Lemma 4.4, recall the Knaster-Kuratowski-Mazurkiewicz lemma [KKM29], which is a consequence of Sperner's lemma.

Lemma 4.10 (KKM Lemma). *Consider a simplex $\text{co}(x_0, \dots, x_n) \subset \mathbb{R}^n$ and closed sets Y_0, \dots, Y_n so that for all subsets $I \subset \{0, \dots, n\}$ we have $\text{co}(x_i : i \in I) \subset \bigcup_{i \in I} Y_i$. Then $\bigcap_{i=0}^n Y_i \neq \emptyset$.*

Proof of Lemma 4.4. We show that a translate a exists for f , the result for b follows analogously. Define the closed sets $Y_i \subset \mathbb{R}^n$ as follows, let $y \in Y_i$ if

$$\int_{C_n^i + y} g dx \geq \frac{1}{n+1} \int_{\mathbb{R}^n} f dx.$$

Note that for a given $y \in \mathbb{R}^n$ we can't have $\int_{C_n^i + y} f dx < \frac{1}{n+1} \int_{\mathbb{R}^n} f dx$ for all $i = 0, \dots, n$ as that would imply $\int_{\mathbb{R}^n + y} f dx < \int_{\mathbb{R}^n} f dx$. Hence, $\bigcup_{i=0}^n Y_i = \mathbb{R}^n$.

To apply the KKM lemma, we use the simplex ℓS_n for a sufficiently large ℓ . Indeed, since g has bounded support, we can choose ℓ so large that $\text{supp}(f) \subset \ell S_n$. For that ℓ we find that if $y \in \ell F_n^i$, then $\int_{C_n^i + y} g dx = 0$ as $C_n^i + y$ falls completely outside of the support of g . Hence, $Y_i \cap \ell F_n^i = \emptyset$, which can be seen to imply the condition in the KKM lemma. We can thus find a point $a \in \bigcap_{i=0}^n Y_i$, i.e., so that

$$\int_{C_n^i + a} f dx \geq \frac{1}{n+1} \int_{\mathbb{R}^n} f dx,$$

for all $i = 0, \dots, n$. This in fact implies

$$\int_{C_n^i + a} f dx = \frac{1}{n+1} \int_{\mathbb{R}^n} f dx,$$

for all $i = 0, \dots, n$, which concludes the lemma. □

Proof of Lemma 4.5. Simply expanding we find

$$\frac{d}{dt} M_{\lambda,p}(t, T(t)) = \frac{d}{dt} (\lambda t^p + (1-\lambda)T(t)^p)^{1/p} = (\lambda t^p + (1-\lambda)T(t)^p)^{1/p-1} \left(\lambda t^{p-1} + (1-\lambda)T(t)^{p-1} \frac{dT}{dt}(t) \right).$$

On the other hand, we have

$$\frac{1}{M_{\lambda,-p}\left(1, \frac{dT}{dt}(t)\right)} = M_{\lambda,p}\left(1, \frac{dT}{dt}(t)\right) = \left(\lambda + (1-\lambda) \left(\frac{dT}{dt}(t) \right)^p \right)^{1/p}.$$

Hence, it suffices to show that

$$\left(\lambda t^{p-1} + (1-\lambda)T(t)^{p-1} \frac{dT}{dt}(t) \right) \left(\lambda + (1-\lambda) \left(\frac{dT}{dt}(t) \right)^p \right)^{-1/p} \geq (\lambda t^p + (1-\lambda)T(t)^p)^{1-1/p},$$

or, taking both sides to the power $1/(1-1/p) = p/(p-1)$

$$\left(\lambda t^{p-1} + (1-\lambda)T(t)^{p-1} \frac{dT}{dt}(t) \right)^{p/(p-1)} \left(\lambda + (1-\lambda) \left(\frac{dT}{dt}(t) \right)^p \right)^{-1/(p-1)} \geq (\lambda t^p + (1-\lambda)T(t)^p).$$

This final inequality follows immediately by Hölders inequality with exponents $(p-1)/p$ and $1-p$ satisfying $\frac{1}{(p-1)/p} + \frac{1}{1-p} = 1$. □

Proof of Lemma 4.6. For notational convenience assume $\sup g \leq \sup f = 1$. Other cases follow analogously. Consider the level sets

$$F_t := \{x : f(x) > t\}, G_t = \{x : g(x) > t\}, H_t = \{x : h(x) > t\},$$

so that e.g. $F_0 = \text{supp}(f)$. By the construction of h , we have inclusion

$$\lambda F_t + (1 - \lambda)G_s \subset H_{M_{\lambda,p}(t,s)}.$$

Consider the transport map $T: [0, 1] \rightarrow [0, 1]$ defined by the differential equation $\frac{dT}{dt}(t) = \frac{|F_t|}{|G_{T(t)}}$, i.e., so that for all $t \in [0, 1]$, we have

$$\int_0^t |F_s| ds = \int_0^{T(t)} |G_s| ds.$$

With all of this machinery set up, we consider the following lower bound on $\int h dx$;

$$\begin{aligned} \int_{\mathbb{R}^n} h dx &= \int_0^1 |H_s| ds \geq \int_0^1 |H_{M_{\lambda,p}(t,T(t))}| dM_{\lambda,p}(t, T(t)) \geq \int_0^1 |\lambda F_t + (1 - \lambda)G_{T(t)}| \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} \\ &\geq \int_0^1 \frac{|\lambda F_t + (1 - \lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt + \int_0^1 M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)}, \end{aligned}$$

where in the second inequality we use Lemma 4.5. Note that by the Brunn-Minkowski inequality, we have $|\lambda F_t + (1 - \lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \geq 0$. Using that $p \geq -1/n$ (and thus $-p \leq 1/n$) combined with $\frac{1}{\frac{dT}{dt}(t)} = \frac{|G_{T(t)}|}{|F_t|}$, we find that

$$\int_0^1 M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} = \int_0^1 |F_t| \frac{M_{\lambda,1/n}\left(1, \frac{|G_{T(t)}|}{|F_t|}\right)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt = \int_0^1 |F_t| dt = 1.$$

In order to show that we typically have that $\frac{dT}{dt}(t)$ is close to 1, note that for $x \in I_1$, we have

$$\frac{M_{\lambda,1/n}(1, x)}{M_{\lambda,-p}(1, x)} \geq 1 + \Omega_{\lambda,n,p,\alpha}(1),$$

so that we find

$$\int_0^1 |F_t| \frac{M_{\lambda,1/n}\left(1, \frac{|G_{T(t)}|}{|F_t|}\right)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \geq 1 + \Omega_{\lambda,n,p,\alpha} \left(\int_{I_1} |F_t| dt \right),$$

and using the bound on $\int_{\mathbb{R}^n} h dx$, this implies that little mass is in the levels in I_1 , in particular,

$$\int_{I_1} |F_t| dt = O_{\lambda,n,p,\alpha}(\delta).$$

Outside of I_1 , we have $\frac{|F_t|}{|G_{T(t)}|} \in [1 - \alpha, 1 + \alpha]$, so that we can use the stability of Brunn-Minkowski [FvHT23] to find

$$|\lambda F_t + (1 - \lambda)G_{T(t)}| \geq M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) + \Omega_{n,\lambda,\alpha}(\min\{|\text{co}(F_t) \setminus F_t|, |F_t|\}).$$

Hence, the previous lower bound on $\int_{\mathbb{R}^n} h dx$ combined with the note that $M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right) \in [1 - \alpha, 1 + \alpha]$ implies

$$\delta \geq \int_{[0,1] \setminus I_1} \frac{|\lambda F_t + (1 - \lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \geq \Omega_{n,\lambda,\alpha} \left(\int_{[0,1] \setminus I_1} \min\{|\text{co}(F_t) \setminus F_t|, |F_t|\} dt \right).$$

Hence, we find

$$\int_{[0,1] \setminus I_1} \min\{|\text{co}(F_t) \setminus F_t|, |F_t|\} dt \leq O_{\lambda,n,p}(\delta).$$

The same bound for G_t follows analogously. Hence, recalling the definition of I_2 this implies

$$\begin{aligned} \int_{I_1 \cup I_2} |F_t| dt &\leq O_{\lambda, n, p, \alpha}(\delta) \\ \int_{[0,1] \setminus (I_1 \cup I_2)} (|\text{co}(F_t) \setminus F_t| + |\text{co}(G_{T(t)}) \setminus G_{T(t)}|) dt &\leq O_{\lambda, n, p, \alpha}(\delta). \end{aligned}$$

Similarly for I_3 , we have

$$\begin{aligned} \delta &\geq \int_{[0,1]} \frac{|\lambda F_t + (1-\lambda)G_{T(t)}| - M_{\lambda, 1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda, -p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \geq \alpha \int_{I_3} \frac{M_{\lambda, 1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda, -p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \\ &\geq \alpha \int_{I_3} |F_t| \frac{M_{\lambda, 1/n}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)}{M_{\lambda, -p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \geq \alpha \int_{I_3} |F_t| dt. \end{aligned}$$

For I_4 , note that I_4 for a given α is contained in $I_1 \cup I_3$ for a different smaller α' . Indeed if F_t and $G_{T(t)}$ are sufficiently similar in size (at most a factor $(1+\alpha')$ away from each other) and $|\lambda F_t + (1-\lambda)G_{T(t)}| \leq (1+\alpha'')M_{\lambda, 1/n}(|F_t|, |G_{T(t)}|)$, then by Theorem 1.1, there exists a set convex set K tightly containing scaled copies of F_t and $G_{T(t)}$ and thus of $\lambda F_t + (1-\lambda)G_{T(t)}$. To be precise, consider $\alpha''' \leq \alpha'$ so that $|(1+\alpha''')F_t| = \left| \frac{G_{T(t)}}{1+\alpha'''} \right|$ (assume that $|F_t| \geq |G_{T(t)}|$, the other case follows analogously). Then (up to translation) there exists a convex set K so that $(1+\alpha''')F_t, \frac{G_{T(t)}}{1+\alpha'''} \subset K$, and $|K| \leq \left(1 + O_{n, \lambda}(\sqrt{\alpha''})\right) |(1+\alpha''')F_t|$. The former also implies that

$$\begin{aligned} \lambda F_t + (1-\lambda)G_{T(t)} &= \frac{\lambda}{(1+\alpha''')} (1+\alpha''')F_t + (1-\lambda)(1+\alpha''') \frac{G_{T(t)}}{(1+\alpha''')} \\ &\subset \frac{\lambda}{(1+\alpha''')} K + (1-\lambda)(1+\alpha''')K \subset (1+\alpha''')K. \end{aligned}$$

Hence,

$$|(\lambda F_t + (1-\lambda)G_{T(t)}) \Delta F_t| \leq |(1+\alpha''')K \setminus (\lambda F_t + (1-\lambda)G_{T(t)})| + |(1+\alpha''')K \setminus F_t| \leq O_{n, \lambda}(\sqrt{\alpha'' + \alpha'''}) |F_t|,$$

so that the bound for I_4 follows from the bound on $I_1 \cup I_3$. The bound for I_5 follows analogously.

For the final conclusion, we will show the result for g , the result for f follows analogously. We first consider the following more modest claim.

Claim 4.11. *There exists a $t_0 = \Omega_{n, \lambda, p}(1)$ so that $\int_{\mathbb{R}^n \setminus G_{t_0}} g(x) dx \leq 1 - \Omega_{n, \lambda, p}(1)$.*

Proof of Claim.

$$\int_{\mathbb{R}^n} h dx \geq \int_{\mathbb{R}^n \setminus (1-\lambda)G_t} h(x) dx \geq \int_{\mathbb{R}^n \setminus (1-\lambda)G_t} M_{\lambda, p}\left(f(o), g\left(\frac{x}{1-\lambda}\right)\right) dx \geq \int_{\mathbb{R}^n \setminus (1-\lambda)G_t} \frac{M_{\lambda, p}(1, t)}{t} g\left(\frac{x}{1-\lambda}\right) dx,$$

where in the last inequality, we use that $\frac{M_{\lambda, p}(t, 1)}{t}$ is decreasing in t . We can rewrite to

$$\int_{\mathbb{R}^n \setminus (1-\lambda)G_t} \frac{M_{\lambda, p}(1, t)}{t} g\left(\frac{x}{1-\lambda}\right) dx = \frac{M_{\lambda, p}(1, t)}{t} (1-\lambda)^n \int_{\mathbb{R}^n \setminus G_t} g(x) dx.$$

Noting that as $t \rightarrow 0$, we have $\frac{M_{\lambda, p}(1, t)}{t} \rightarrow (1-\lambda)^{1/p} = (1-\lambda)^{-n} (1 + \Omega_{n, \lambda, p}(1))$, so that for some $t_0 = \Omega_{n, \lambda, p}(1)$, we have $\frac{M_{\lambda, p}(1, t_0)}{t_0} \geq (1-\lambda)^{-n} (1 + \Omega_{n, \lambda, p}(1))$, so that

$$\int_{\mathbb{R}^n} h(x) dx \geq (1 + \Omega_{n, \lambda, p}(1)) \int_{\mathbb{R}^n \setminus G_{t_0}} g(x) dx.$$

Since $\int h dx < 1 + \delta$, this implies

$$\int_{\mathbb{R}^n \setminus G_{t_0}} g(x) dx \leq \frac{1 + \delta}{1 + \Omega_{n, \lambda, p}(1)} \leq 1 - \Omega_{n, \lambda, p}(1),$$

which concludes the claim. \square

To get from some to most of the mass, we iterate as follows until we find an appropriate function. Given the t_0 from the claim, consider functions $g' = \min\{g, t_0/2\}$, and $f' := \min\{f, T^{-1}(t_0/2)\}$, so that $\int f' dx = \int g' dx$. Also, let $h' = \min\{h, M_{\lambda,p}(T^{-1}(t_0/2), t_0/2)\}$, so that $h' \geq M_{\lambda,p}^*(f', g')$. If $\int g' dx \leq \alpha' \int g dx$, then in particular $\int_{\mathbb{R}^n \setminus G_{t_0/2}} g dx \leq \int g' dx \leq \alpha' \int g dx$, so we can set $t_0/2 = \beta$ and the result follows.

If, on the other hand, we have $\int g' dx > \alpha' \int g dx$, then we will see that in fact $\int h' dx \leq \delta + \int f' dx \leq (1 + \delta/\alpha') \int f' dx$. Indeed, similar to the earlier computation, we find

$$\begin{aligned} \int (h - h') dx &= \int_{M_{\lambda,p}(T^{-1}(t_0/2), t_0/2)}^1 |H_s| ds \geq \int_{M_{\lambda,p}(T^{-1}(t_0/2), t_0/2)}^1 |H_{M_{\lambda,p}(t, T(t))}| dM_{\lambda,p}(t, T(t)) \\ &\geq \int_{T^{-1}(t_0/2)}^1 \frac{|\lambda F_t + (1 - \lambda)G_{T(t)}}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt \geq \int_{T^{-1}(t_0/2)}^1 |F_t| dt = \int (f - f') dx = \int (g - g') dx. \end{aligned} \quad (4.1)$$

Rearranging shows that

$$\int h' dx \leq \left(\int h dx - \int f dx \right) + \int f' dx \leq \delta + \int f' dx \leq (1 + \delta/\alpha') \int f' dx.$$

Finally, we also have removed a significant amount from g ; indeed

$$\int_{\mathbb{R}^n} (g - g') dx \geq \int_{G_{t_0}} (g - g') dx \geq \frac{1}{2} \int_{G_{t_0}} g dx = \Omega_{n,\lambda,p}(1).$$

Hence, when we iterate, we find after at most $\log_{1 - \Omega_{n,\lambda,p}(1)}(\alpha') = O_{n,\lambda,p,\alpha'}(1)$ steps, we find a $\beta = O_{n,\lambda,p,\alpha'}(1)$ so that if we let $g'' = \min\{g, \beta\}$, then $\int g'' dx \leq \alpha' \int g dx$. Hence, we find $\int_{\mathbb{R}^n \setminus G_\beta} g dx \leq \int g'' dx \leq \alpha' \int g dx$. This concludes the proposition. \square

Proof of Lemma 4.8. Consider the level sets

$$F_t := \{x : f(x) > t\}, G_t = \{x : g(x) > t\}, H_t = \{x : h(x) > t\},$$

so that $F_0 = \text{supp}(f)$. As we have seen before, we have inclusion

$$\lambda F_t + (1 - \lambda)G_s \subset H_{M_{\lambda,p}(t,s)}.$$

Consider the transport map $T: [0, 1] \rightarrow [0, 1]$ defined by the differential equation $\frac{dT}{dt}(t) = \frac{|F_t|}{|G_{T(t)}}$, i.e., so that for all $t \in [0, 1]$, we have

$$\int_0^t |F_s| ds = \int_0^{T(t)} |G_s| ds.$$

With all of this machinery set up, we consider the following lower bound on $\int h dx$;

$$\begin{aligned} \int_{\mathbb{R}^n} h dx &= \int_0^1 |H_s| ds \geq \int_0^1 |H_{M_{\lambda,p}(t, T(t))}| dM_{\lambda,p}(t, T(t)) \geq \int_0^1 |\lambda F_t + (1 - \lambda)G_{T(t)}| \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} \\ &\geq \int_0^1 \frac{|\lambda F_t + (1 - \lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt + \int_0^1 M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)}, \end{aligned}$$

where in the second inequality we use Lemma 4.5. Note that by Brunn-Minkowski, $|\lambda F_t + (1 - \lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \geq 0$. Using that $p \geq -1/n$ (and thus $-p \leq 1/n$) combined with $\frac{1}{\frac{dT}{dt}(t)} = \frac{|G_{T(t)}|}{|F_t|}$, we find that

$$\int_0^1 M_{\lambda,1/n}(|F_t|, |G_{T(t)}|) \frac{dt}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} = \int_0^1 |F_t| \frac{M_{\lambda,1/n}\left(1, \frac{|G_{T(t)}|}{|F_t|}\right)}{M_{\lambda,-p}\left(1, \frac{1}{\frac{dT}{dt}(t)}\right)} dt = \int_0^1 |F_t| dt = 1.$$

Distinguish two cases; either $T(\beta) \leq \beta$ or $T(\beta) > \beta$. We assume the former and the latter follows analogously interchanging the roles of f and g . Under that assumption we find $F_t = \text{supp}(f)$, $G_{T(t)} = \text{supp}(g)$, and

$\frac{dT}{dt}(t) = \frac{|supp(f)|}{|supp(g)|}$ for all $t \in [0, \beta]$. By the above computation, we find

$$\begin{aligned} \delta &\geq \int_0^\beta |F_t| \left(\frac{M_{\lambda,1/n} \left(1, \frac{|G_{T(t)}}{|F_t|} \right)}{M_{\lambda,-p} \left(1, \frac{1}{\frac{dT}{dt}(t)} \right)} - 1 \right) dt \\ &\geq \int_0^\beta |F_t| \left(\frac{M_{\lambda,1/n} \left(1, \frac{|supp(g)|}{|supp(f)|} \right)}{M_{\lambda,-p} \left(1, \frac{|supp(g)|}{|supp(f)|} \right)} - 1 \right) dt \geq \beta \left(\frac{M_{\lambda,1/n} \left(1, \frac{|supp(g)|}{|supp(f)|} \right)}{M_{\lambda,-p} \left(1, \frac{|supp(g)|}{|supp(f)|} \right)} - 1 \right). \end{aligned}$$

Hence, using that $p > -1/n$, this implies that

$$\frac{dT}{dt}(t) = \frac{|supp(g)|}{|supp(f)|} = 1 \pm O_{n,\lambda,p,\beta}(\sqrt{\delta}).$$

By the above, we also know that

$$\beta \frac{(|\lambda F_0 + (1-\lambda)G_0| - M_{\lambda,1/n}(|F_0|, |G_0|))}{1 + O_{n,\lambda,p}(\sqrt{\delta})} \leq \int_0^\beta \frac{|\lambda F_t + (1-\lambda)G_{T(t)}| - M_{\lambda,1/n}(|F_t|, |G_{T(t)}|)}{M_{\lambda,-p} \left(1, \frac{1}{\frac{dT}{dt}(t)} \right)} dt \leq \delta,$$

so that

$$|\lambda F_0 + (1-\lambda)G_0| \leq (1 + 2\delta/\beta)M_{\lambda,1/n}(|F_0|, |G_0|).$$

Hence, by Theorem 1.1 (the main result of [FvHT23]) we find that for $\alpha := \left(\frac{|F_0|}{|G_0|} \right)^{1/n} = 1 \pm O_{n,\lambda,p,\beta}(\sqrt{\delta})$, we have

$$\min_{x \in \mathbb{R}^n} |(x + F_0)\Delta \alpha G_0| = O_{n,\lambda,\beta}(\sqrt{\delta}).$$

Since α is so close to 0, we find that

$$\min_{x \in \mathbb{R}^n} |(x+F_0)\Delta G_0| \leq \min_{x \in \mathbb{R}^n} |(x+F_0)\Delta \alpha G_0| + \min_{x \in \mathbb{R}^n} |(x+G_0)\Delta \alpha G_0| = O_{n,\lambda,\beta}(\sqrt{\delta}) + |1 - \alpha^n| \cdot |G_0| = O_{n,\lambda,p,\beta}(\sqrt{\delta}).$$

This concludes the proof of the main part of the Lemma.

We now extend to functions which do take values below β , but only rarely, i.e., with $|\{x : f(x) < \beta \text{ or } g(x) < \beta\}| = O_{n,\lambda,p,\beta}(\delta)$. Simply consider the functions $f' = f \cdot \mathbf{1}\{x : f(x) \geq \beta\}$ and $g' = g \cdot \mathbf{1}\{x : g(x) \geq \beta\}$, so that $\int f' dx, \int g' dx \geq 1 - \gamma$ and thus $\int h dx \leq 1 + \delta \leq (1 + O_{n,\lambda,p,\beta}(\delta + \gamma)) \int f' dx$. Hence, applying the above result for f' and g' gives (up to translation) $|supp(f')\Delta supp(g')| = O_{n,\lambda,p,\beta}(\sqrt{\delta + \gamma})$. Combining with the obvious observation that $supp(f)\Delta supp(f') = \{x : f(x) < \beta\}$, we find $|supp(f)\Delta supp(g)| = O_{n,\lambda,p,\beta}(\sqrt{\delta + \gamma})$. \square

Proof of Lemma 4.9. We first note that it suffices to consider the level sets as follows

$$\int (f(x) - f(x+v)) dx = \int_0^1 |F_t \Delta (F_t + v)| dt \leq \int_0^1 |\text{co}(F_t) \Delta (\text{co}(F_t) + v)| dt + 2\delta.$$

We split this final integral in $t \geq t_0$ and $t < t_0$. First note that for t_0 , we have $|\text{co}(F_{t_0}) \Delta (\text{co}(F_{t_0}) + v)| \leq O_n(\eta) |\text{co}(F_{t_0})|$. For all $t \leq t_0$, we have $\text{co}(F_t) \supset \text{co}(F_{t_0})$, so that again $|\text{co}(F_t) \Delta (\text{co}(F_t) + v)| \leq O_n(\eta) |\text{co}(F_t)|$. Integrating over those t , gives

$$\int_0^{t_0} |\text{co}(F_t) \Delta (\text{co}(F_t) + v)| dt \leq O_n(\eta) \int_0^{t_0} |\text{co}(F_t)| dt \leq O_n(\eta) \int f dx.$$

For $t > t_0$, we find that as $\text{co}(F_t) \subset \text{co}(F_{t_0})$, we have

$$|\text{co}(F_t) \Delta (\text{co}(F_t) + v)| \leq |\text{co}(F_{t_0}) \Delta (\text{co}(F_{t_0}) + v)| \leq O_n(\eta) |\text{co}(F_{t_0})|.$$

Recall that $|\text{co}(F_{t_0})| \leq 2|F_{t_0}| \leq \frac{2}{t_0} \int_{F_{t_0}} f dx \leq \frac{2}{t_0}$, so that

$$\int_{t_0}^1 |\text{co}(F_t) \Delta (\text{co}(F_t) + v)| dt \leq (1 - t_0) |\text{co}(F_{t_0}) \Delta (\text{co}(F_{t_0}) + v)| \leq \frac{2(1 - t_0)}{t_0} O_n(\eta).$$

Combining these upper bounds proves the lemma. \square

4.2 Proof of the reduction

Proof that Proposition 4.1 implies Theorem 1.9. We may assume δ is sufficiently small in terms of n, λ , and p , since otherwise the conclusion is trivially true by virtue of the trivial upper bound

$$\int_{\mathbb{R}^n} |f - g| dx \leq \int_{\mathbb{R}^n} (f + g) dx = 2 \int_{\mathbb{R}^n} f dx.$$

We will repeatedly use this assumption throughout this proof.

Apply Lemma 4.2 with $\eta = \delta^{1000}$ to find continuous f_1, g_1, h_1 . Then apply Lemma 4.3 to find f_2, g_2, h_2 taking values in $[0, 1]$. Then apply Corollary 4.7 with some $\alpha = \alpha' > 0$ small in terms of n, λ, p to make various statements in the following true, and find f_3, g_3, h_3 and $\beta = \Omega_{n, \lambda, p}(1)$ for which it suffices to prove the result and renormalize so that $\sup f_3 = 1$. Consider level sets F^3, G^3, H^3 and transport map $T : [0, 1] \rightarrow [0, \sup g_3]$ as in Corollary 4.7. Note that $\sup g \in [1 - \alpha, 1 + \alpha]$ since $\frac{d}{dt}T(t) \in [1 - \alpha, 1 + \alpha]$ for all $t \in [0, 1]$ and $T(0) = 0$. Assume $\sup g_3 \leq \sup f_3$, the other case follows analogously.

Claim 4.12. *Up to translation, we have $\left| \frac{\lambda}{2} \text{co}(F_\beta^3) \setminus F_\beta^3 \right| = O_{n, \lambda, p}(\delta)$*

Proof of claim. Find $x \in \mathbb{R}^n$, so that $g_3(x) \geq 0.9$. Then (up to translation) $H_{M_{\lambda, p}(\beta, 0.9)} \supset \lambda F_\beta$. Find the t_0 so that $M_{\lambda, p}(t_0, T(t_0)) = M_{\lambda, p}(\beta, 0.9)$ and note that $t_0 - \beta \geq \Omega_{\lambda, p}(1)$. Since $\left| H_{M_{\lambda, p}(t_0, T(t_0))}^3 \Delta F_{t_0}^3 \right| \leq \alpha |F_{t_0}^3|$, and $\left| \text{co}(F_\beta^3) \setminus F_\beta^3 \right| \leq \alpha |F_\beta^3|$, we find that (up to translation)

$$\left| \text{co}(\lambda F_\beta^3) \setminus F_{t_0}^3 \right| \leq \left| \text{co}(\lambda F_\beta^3) \setminus H_{M_{\lambda, p}(t_0, T(t_0))}^3 \right| + \left| H_{M_{\lambda, p}(t_0, T(t_0))}^3 \Delta F_{t_0}^3 \right| \leq O_{\lambda, n}(\alpha) |F_\beta^3|,$$

so that

$$\left| \text{co}(\lambda F_\beta^3 \cap F_{t_0}^3) \right| \geq (1 - O_{\lambda, n}(\alpha)) \left| \text{co}(\lambda F_\beta^3) \right|,$$

so that (up to translation and for sufficiently small α in λ and n)

$$\frac{\lambda}{2} \text{co}(F_\beta^3) \subset \text{co}(\lambda F_\beta^3 \cap F_{t_0}^3) \subset \text{co}(F_{t_0}^3).$$

Hence, we find that $\left| \frac{\lambda}{2} \text{co}(F_\beta^3) \setminus F_\beta^3 \right|$ is a lower bound on $\left| \text{co}(F_t^3) \setminus F_t^3 \right|$ for all $t \in [\beta, t_0]$, so that

$$(t_0 - \beta) \left| \frac{\lambda}{2} \text{co}(F_\beta^3) \setminus F_\beta^3 \right| \leq \int_0^1 \left| \text{co}(F_t^3) \setminus F_t^3 \right| dt \leq O_{n, \lambda, p}(\delta).$$

The claim follows. □

Claim 4.13. $\left| \text{co}(F_\beta^3) \setminus F_{\beta/2}^3 \right| \leq O_{n, \lambda, p}(\delta)$

Proof. For all $t \in (\beta/2, \beta)$, we have $\left| \text{co}(F_t^3) \setminus F_t^3 \right| \geq \left| \text{co}(F_\beta^3) \setminus F_{\beta/2}^3 \right|$. Hence, we have

$$(\beta - \beta/2) \left| \text{co}(F_\beta^3) \setminus F_{\beta/2}^3 \right| \leq \int_0^1 \left| \text{co}(F_t^3) \setminus F_t^3 \right| dt \leq O_{n, \lambda, p}(\delta).$$

Since $\beta = \Omega_{n, \lambda, p}(1)$, the claim follows. □

Analogously we have $\left| \text{co}(G_{T(\beta)}^3) \setminus G_{\beta/2}^3 \right| = O_{n, \lambda, p}(\delta)$. Recalling from Corollary 4.7, we have that (up to translation) we have $\left| \text{co}(F_\beta^3) \Delta \text{co}(G_{T(\beta)}^3) \right| \leq \alpha |F_\beta^3|$. Hence, we can consider an affine transformation so that $R'S_n \subset \text{co}(F_\beta^3)$, $\text{co}(G_{T(\beta)}^3) \subset RS_n$ and $R/R' \leq O_n(1)$ using John's Theorem.

Apply Lemma 4.4 to find translations $a, b \in \mathbb{R}^n$ so that

$$\int_{a+C_i^n} f dx = \int_{b+C_i^n} g dx = \frac{1}{n+1} \int f dx.$$

We'll see that a and b are not too far from the origin. Recall that $\int_{F_\beta^3} f dx \geq 1 - \alpha$, so

$$\int_{(a+C_i^n) \cap F_\beta^3} f dx = \left(\frac{1}{n+1} - \alpha \right) \int f dx \geq \frac{1}{2n}.$$

On the other hand, since $\sup f \leq 1$, we have $\int_{(a+C_i^n) \cap F_\beta^3} f dx \leq |(a+C_i^n) \cap \text{co}(F_\beta^3)|$. Since $\beta = \Omega_{n,\lambda,p}(1)$, we find that $\beta |\text{co}(F_\beta^3)| \leq 2\beta |F_\beta^3| \leq \int f dx \leq 1$ implies $|\text{co}(F_\beta^3)| = O_{n,\lambda,p}(1)$, and

$$|(a+C_i^n) \cap \text{co}(F_\beta^3)| \geq \Omega_{n,\lambda,p}(|\text{co}(F_\beta^3)|),$$

for all $i = 0, \dots, n$. Combined with the above normalization that $R'S_n \subset \text{co}(F_\beta^3) \subset RS_n$, this implies that there is some $r = \Omega_{n,\lambda,p}(1)$ so that $a+rS_n \subset \text{co}(F_\beta^3)$. Analogously $b+rS_n \subset \text{co}(G_{T(\beta)}^3)$. Translating f and g so that $a=b=0$, this implies

$$\left| rS_n \setminus F_{\beta/2}^3 \right| + \left| rS_n \setminus G_{\beta/2}^3 \right| = O_{n,\lambda,p}(\delta).$$

We can now apply Proposition 4.1 to find that $\int_{\mathbb{R}^n} |f_3 - g_3| dx = O_{n,\lambda,p,r,\beta}(\sqrt{\delta}) = O_{n,\lambda,p}(\sqrt{\delta})$ as $r, \beta = \Omega_{n,\lambda,p}(1)$. Pulling this back, we find that for some translates $v, v_1, v_2 \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |f(v+x) - g(x)| dx &\leq \eta + \int_{\mathbb{R}^n} |f_1(v_1+x) - g_1(x)| dx \\ &\leq O_{n,\lambda,p}(\delta) + \int_{\mathbb{R}^n} |f_2(v_2+x) - g_2(x)| dx \\ &\leq O_{n,\lambda,p}(\delta) + \int_{\mathbb{R}^n} |f_3(v_3+x) - g_3(x)| dx = O_{n,\lambda,p}(\sqrt{\delta}). \end{aligned}$$

Thus, Theorem 1.9 follows. \square

5 Proof of Theorem 5.1, i.e., Theorem 1.9 in \mathbb{R}

We aim to prove the following theorem which is the one dimensional instance of Theorem 1.9.

Theorem 5.1. *Given $\lambda \in (0, 1/2]$ and $p \in (-1/2, \infty)$ there exists $d = d_{1,\lambda,p} > 0$ such that the following holds. Let $f, g, h: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ be integrable functions such that $\int_{\mathbb{R}} f dx = \int_{\mathbb{R}} g dx = 1$ and for all $x, y \in \mathbb{R}$ we have $h(\lambda x + (1-\lambda)y) \geq M_{\lambda,p}(f(x), g(y))$. If $\int_{\mathbb{R}} h dx = 1 + \delta$ for some $0 \leq \delta \leq d$, then, up to translation, $\int_{\mathbb{R}^2} |f - g| dx = O_{\lambda,p}(\sqrt{\delta})$.*

Given the information given by Lemma 4.6 it suffices to prove the following proposition.

Proposition 5.2. *For all $\lambda \in (0, 1), p > -1$, there exists a $d = d_{\lambda,p} > 0$, so that the following holds. Let $f, g, h: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ so that for all $x, y \in \mathbb{R}$ we have $h(\lambda x + (1-\lambda)y) \geq M_{\lambda,p}(f(x), g(y))$ and $\int h dx \leq (1+\delta) \int f dx$ with $\delta < d$. Additionally assume that for level sets $F_t := \{x \in \mathbb{R} : f(x) > t\}$ and G_t we have*

$$\int_{t \in \mathbb{R}_{>0}} (|\text{co}(F_t) \setminus F_t| + |\text{co}(G_t) \setminus G_t|) dt \leq \delta.$$

Then there exists some $v \in \mathbb{R}$ so that

$$\int_{\mathbb{R}} |f(x) - g(x+v)| dx = O(\sqrt{\delta}) \int_{\mathbb{R}} f(x) dx$$

Proof of Proposition 5.2. Let $T: F_0 \rightarrow G_0$ be the map pushing f into g , so that $\int_{-\infty}^x f(y) dy = \int_{-\infty}^{T(x)} g(y) dy$, i.e., $\frac{dT}{dx}(x) = \frac{f(x)}{g(T(x))}$ almost everywhere. We use the bound $h(\lambda x + (1-\lambda)T(x)) \geq M_{\lambda,p}(f(x), g(T(x)))$, to find

$$\int h(\lambda x + (1-\lambda)T(x)) d(\lambda x + (1-\lambda)T(x)) \geq \int M_{\lambda,p}(f(x), g(T(x))) \left(\frac{\lambda + (1-\lambda)\frac{dT}{dx}(x)}{2} \right) dx$$

By definition $\frac{dT}{dx}(x) = \frac{f(x)}{g(T(x))}$, so that, we can rewrite

$$\begin{aligned}
& \int h(x) dx \\
& \geq \int \frac{1}{\frac{\lambda}{f(x)} + \frac{1-\lambda}{g(T(x))}} \left(\lambda + (1-\lambda) \frac{dT}{dx}(x) \right) dx + \int \left(M_{\lambda,p}(f(x), g(T(x))) - \frac{1}{\frac{\lambda}{f(x)} + \frac{1-\lambda}{g(T(x))}} \right) \left(\lambda + (1-\lambda) \frac{dT}{dx}(x) \right) dx \\
& \geq \int f(x) dx + \int f(x) \left(M_{\lambda,p} \left(\sqrt{\frac{dT}{dx}(x)}, \frac{1}{\sqrt{\frac{dT}{dx}(x)}}} \right) - \frac{1}{\frac{\lambda}{\sqrt{\frac{dT}{dx}(x)}} + (1-\lambda)\sqrt{\frac{dT}{dx}(x)}} \right) \left(\frac{\lambda}{\sqrt{\frac{dT}{dx}(x)}} + (1-\lambda)\sqrt{\frac{dT}{dx}(x)} \right) dx \\
& \geq \int f(x) dx + \int f(x) \cdot \Omega_{\lambda,p} \left(\min \left\{ 1, \left| 1 - \sqrt{\frac{dT}{dx}(x)} \right|^2 \right\} \right) \cdot \Omega_{\lambda}(1) dx \\
& \geq \int f(x) dx + \Omega_{\lambda,p} \left(\int f(x) \min \left\{ 1, \left| 1 - \sqrt{\frac{dT}{dx}(x)} \right|^2 \right\} dx \right)
\end{aligned}$$

Hence, as $\int h dx \leq (1 + \delta) \int f dx$, we find

$$\int f(x) \min \left\{ 1, \left| 1 - \sqrt{\frac{dT}{dx}(x)} \right|^2 \right\} dx = O_{\lambda,p}(\delta) \int f dx,$$

so that $\frac{dT}{dx}(x)$ is close to 1 most of the time. To formalize that, consider the following set

$$I := (\text{co}(F_0) \setminus F_0) \cup \left\{ x \in \text{co}(F_0) : \frac{dT}{dx}(x) \notin (1/2, 3/2) \right\}.$$

We find that

$$\int_I f(x) dx \leq 4 \int_I f(x) \min \left\{ 1, \left| 1 - \sqrt{\frac{dT}{dx}(x)} \right|^2 \right\} dx = O_{\lambda,p}(\delta),$$

and

$$\int_{T(I)} g(x) dx = \int_I f(x) dx = O_{\lambda,p}(\delta),$$

so little mass of f is in I . Hence, define

$$f_1(x) := \begin{cases} f(x) & \text{if } x \notin I \\ 0 & \text{if } x \in I. \end{cases} \quad \text{and} \quad g_1(x) := \begin{cases} g(x) & \text{if } x \notin T(I) \\ 0 & \text{if } x \in T(I), \end{cases}$$

so that

$$\int (|f - f_1| + |g - g_1|) dx = O_{\lambda,p}(\delta).$$

Defining $F_t^1 := \{x : f_1(x) > t\} \subset F_t$ and $G_t^1 := \{x : g_1(x) > t\} \subset G_t$, we find

$$\begin{aligned}
\int_{t \in \mathbb{R}_{\geq 0}} (|\text{co}(F_t^1) \setminus F_t^1| + |\text{co}(G_t^1) \setminus G_t^1|) dt & \leq \int_{t \in \mathbb{R}_{\geq 0}} (|\text{co}(F_t) \setminus F_t| + |\text{co}(G_t) \setminus G_t|) dt \\
& + \int_{\mathbb{R}} (|f(x) - f_1(x)| + |g(x) - g_1(x)|) dx \\
& = O_{\lambda,p}(\delta).
\end{aligned}$$

We next compress the functions to skip the points in I . To this end consider the increasing surjective map $S_f: \mathbb{R} \rightarrow (\mathbb{R} \setminus I)$ with $\frac{dS_f}{dx} = 1$ almost everywhere and $S_f(x) = x$ for some x with $f_1(x) = \sup f_1$. Similarly define the increasing surjective map $S_g: \mathbb{R} \rightarrow (\mathbb{R} \setminus T(I))$ with $S'_g = 1$ almost everywhere and $S_g(y) = y$ for some y with $g_1(y) = \sup g_1$.

Using this map, define

$$f_2(x) := f_1(S_f(x)) \text{ and } g_2(x) := g_1(S_g(x)),$$

so that

$$\int f_2 dx = \int g_2 dx = \int f_1 dx = \int g_1 dx.$$

As ever define level sets $F_t^2 := \{x : f_2(x) > t\}$ and $G_t^2 := \{x : g_2(x) > t\}$. Note that $|F_t^1| = |F_t^2|$ and $|G_t^1| = |G_t^2|$, while $|\text{co}(F_t^1)| \geq |\text{co}(F_t^2)|$ and $|\text{co}(G_t^1)| \geq |\text{co}(G_t^2)|$, so that

$$\int_{t \in \mathbb{R}_{\geq 0}} (|\text{co}(F_t^2) \setminus F_t^2| + |\text{co}(G_t^2) \setminus G_t^2|) dt \leq \int_{t \in \mathbb{R}_{\geq 0}} (|\text{co}(F_t^1) \setminus F_t^1| + |\text{co}(G_t^1) \setminus G_t^1|) dt = O_{\lambda,p}(\delta).$$

Considering the corresponding transport map $T_2: \mathbb{R} \rightarrow \mathbb{R}; x \mapsto S_g^{-1}(T(S_f(x)))$, so that

$$\frac{dT_2}{dx}(x) = \frac{dT}{dS_f(x)}(S_f(x)) = \frac{f(S_f(x))}{g(T(S_f(x)))} = \frac{f_2(x)}{g(S_g(T_2(x)))} = \frac{f_2(x)}{g_2(T_2(x))}.$$

Since $S_f(x) \notin I$, we find that $\frac{dT_2}{dx}(x) \in [1/2, 3/2]$ for all $x \in \mathbb{R}$ and moreover

$$\begin{aligned} \int_{\mathbb{R}} f_2(x) \left| 1 - \frac{dT_2}{dx}(x) \right|^2 dx &= \int_{\mathbb{R}} f(S_f(x)) \left| 1 - \frac{dT}{dS_f(x)}(S_f(x)) \right|^2 dx \\ &= \int_{\mathbb{R} \setminus I} f(x) \left| 1 - \frac{dT}{dx}(x) \right|^2 dx \\ &\leq \int_{\mathbb{R}} f(x) \min \left\{ 1, \left| 1 - \sqrt{\frac{dT}{dx}(x)} \right|^2 \right\} dx = O_{\lambda,p}(\delta). \end{aligned}$$

Finally, we evaluate

$$\int_{\mathbb{R}} (|f_2 - f_1| + |g_2 - g_1|) dx = \int_{\mathbb{R}_{\geq 0}} (|F_t^2 \Delta F_t^1| + |G_t^2 \Delta G_t^1|) dt.$$

Since S_f has a fixed point where f_1 attains its supremum, we have $\text{co}(F_t^2) \subset \text{co}(F_t^1)$. As moreover $|F_t^2| = |F_t^1|$, we have $|F_t^2 \Delta F_t^1| \leq 2|\text{co}(F_t^1) \setminus F_t^1|$, so that

$$\int_{\mathbb{R}_{\geq 0}} |F_t^2 \Delta F_t^1| dt \leq \int_{\mathbb{R}_{\geq 0}} 2|\text{co}(F_t^1) \setminus F_t^1| dt = O_{\lambda,p}(\delta).$$

Analogously, we find

$$\int_{\mathbb{R}_{\geq 0}} |G_t^2 \Delta G_t^1| dt = O_{\lambda,p}(\delta).$$

Hence,

$$\int_{\mathbb{R}} (|f_2 - f_1| + |g_2 - g_1|) dx = \int_{\mathbb{R}_{\geq 0}} (|F_t^2 \Delta F_t^1| + |G_t^2 \Delta G_t^1|) dt = O_{\lambda,p}(\delta).$$

Next, we consider the functions given by filling in the level sets:

$$f_3 := \sup\{t : x \in \text{co}(F_t^2)\} \text{ and } g_3 := \sup\{t : x \in \text{co}(G_t^2)\},$$

so that letting (as always) $F_t^3 := \{x : f_3(x) > t\}$ and $G_t^3 := \{x : g_3(x) > t\}$, then $F_t^3 = \text{co}(F_t^2)$ and $G_t^3 = \text{co}(G_t^2)$. Hence, we immediately find

$$\int (|f_2 - f_3| + |g_2 - g_3|) dx = \int_{\mathbb{R}_{\geq 0}} (|F_t^2 \Delta F_t^3| + |G_t^2 \Delta G_t^3|) dt = \int_{t \in \mathbb{R}_{\geq 0}} (|\text{co}(F_t^2) \setminus F_t^2| + |\text{co}(G_t^2) \setminus G_t^2|) dt = O_{\lambda,p}(\delta).$$

Recall that $\int_{\mathbb{R}} f_2(x) \left| 1 - \frac{dT_2}{dx}(x) \right|^2 dx = O_{\lambda,p}(\delta)$, so that $\int_{\mathbb{R}} f_2(x) \left| 1 - \frac{dT_2}{dx}(x) \right| dx = O_{\lambda,p}(\sqrt{\delta})$. Since $\frac{dT_2}{dx}(x) \in [1/2, 3/2]$, we in fact also have

$$\int_{\mathbb{R}} |f_2(x) - g_2(T_2(x))| dx = \int_{\mathbb{R}} f_2(x) \left| 1 - \frac{1}{\frac{dT_2}{dx}(x)} \right| dx = O_{\lambda,p}(\sqrt{\delta}).$$

Translating this to f_3 and g_3 , we find

$$\begin{aligned} \int_{\mathbb{R}} |f_3(x) - g_3(T_2(x))| dx &\leq \int_{\mathbb{R}} (|f_2(x) - f_3(x)| + |f_2(x) - g_2(T_2(x))| + |g_2(T_2(x)) - g_3(T_2(x))|) dx \\ &\leq O_{\lambda,p}(\delta) + O_{\lambda,p}(\sqrt{\delta}) + 2 \int |g_2(x) - g_3(x)| dx \\ &= O_{\lambda,p}(\sqrt{\delta}) \end{aligned}$$

In other words, defining auxiliary function $k(x) := g_3(T_2(x))$, we have

$$\int_{\mathbb{R}} |f_3(x) - k(x)| dx = O_{\lambda,p}(\sqrt{\delta}).$$

Extending that $\int_{\mathbb{R}} f_2(x) \left|1 - \frac{dT_2}{dx}(x)\right| dx = O_{\lambda,p}(\sqrt{\delta})$ in a slightly different way, we also find

$$\begin{aligned} \int_{\mathbb{R}} k(x) \left|1 - \frac{dT_2}{dx}(x)\right| dx &\leq \int_{\mathbb{R}} |k(x) - f_3(x)| \cdot \left|1 - \frac{dT_2}{dx}(x)\right| dx + \int_{\mathbb{R}} |f_3(x) - f_2(x)| \cdot \left|1 - \frac{dT_2}{dx}(x)\right| dx \\ &\quad + \int_{\mathbb{R}} f_2(x) \left|1 - \frac{dT_2}{dx}(x)\right| dx \\ &\leq 2 \int_{\mathbb{R}} |k(x) - f_3(x)| dx + 2 \int_{\mathbb{R}} |f_3(x) - f_2(x)| dx + \int_{\mathbb{R}} f_2(x) \left|1 - \frac{dT_2}{dx}(x)\right| dx \\ &= O_{\lambda,p}(\sqrt{\delta}), \end{aligned}$$

and since $\frac{dT_2}{dx}(x) \in [1/2, 3/2]$ for almost all x , we also have

$$\int_{\mathbb{R}} k(x) \left|1 - \frac{d(T_2^{-1})}{dx}(x)\right| dx = O_{\lambda,p}(\sqrt{\delta}).$$

Our remaining aim is to show that

$$\int_{\mathbb{R}} |k(x) - g_3(x)| dx = O_{\lambda,p}(\sqrt{\delta}).$$

To this end define a final collection of level sets $K_t := \{x : k(x) > t\}$. Note that $K_t = T_2^{-1}(G_t^3)$. In terms of the level sets, we find

$$\int_{\mathbb{R}} |k(x) - g_3(x)| dx = \int_{\mathbb{R}_{\geq 0}} |K_t \Delta G_t^3| dt.$$

Write $k^{-1}(t)$ for $\max K_t$ (a slight abuse of notation) and $g_3^{-1}(t)$ for $\max G_t^3$, so that $T_2(k^{-1}(t)) = g_3^{-1}(t)$. Note that by translating f and g if necessary, we can guarantee that $k^{-1}(\sup g_1) = g_3^{-1}(\sup g_1) = 0 = T_2(0)$ and that $|K_t \Delta G_t^3|$ is upper bounded by $|k^{-1}(t) - g_3^{-1}(t)|$ plus the equivalent on the other end of the interval. Hence, it suffices to show $\int_{\mathbb{R}_{\geq 0}} |k^{-1}(t) - g_3^{-1}(t)| dt = O_{\lambda,p}(\sqrt{\delta})$ as the other end of the interval follows analogously. We compute as follows

$$\begin{aligned} \int_{\mathbb{R}_{\geq 0}} |k^{-1}(t) - g_3^{-1}(t)| dt &= \int_{\mathbb{R}_{\geq 0}} |k^{-1}(t) - T_2(k^{-1}(t))| dt \\ &= \int_{\mathbb{R}_{\geq 0}} \left(\int_0^{k^{-1}(t)} \left|1 - \frac{dT_2}{dx}(x)\right| dx \right) dt \\ &= \int_{\mathbb{R}} \left(\int_0^{k(x)} \left|1 - \frac{dT_2}{dx}(x)\right| dt \right) dx \\ &= \int_{\mathbb{R}} k(x) \left|1 - \frac{dT_2}{dx}(x)\right| dx \\ &= O_{\lambda,p}(\sqrt{\delta}). \end{aligned}$$

Hence, we find

$$\int_{\mathbb{R}} |k(x) - g_3(x)| dx = \int_{\mathbb{R}_{\geq 0}} |K_t \triangle G_t^3| dt = O_{\lambda,p}(\sqrt{\delta}).$$

Putting it all together, we find

$$\begin{aligned} \int |f - g| dx &\leq \int (|f - f_1| + |f_1 - f_2| + |f_2 - f_3| + |f_3 - k| + |k - g_3| + |g_3 - g_2| + |g_2 - g_1| + |g_1 - g|) dx \\ &= O_{\lambda,p}(\delta) + O_{\lambda,p}(\delta) + O_{\lambda,p}(\delta) + O_{\lambda,p}(\sqrt{\delta}) + O_{\lambda,p}(\sqrt{\delta}) + O_{\lambda,p}(\delta) + O_{\lambda,p}(\delta) + O_{\lambda,p}(\delta) \\ &= O_{\lambda,p}(\sqrt{\delta}), \end{aligned}$$

which concludes the proof of the proposition. \square

6 Proof of Theorem 6.1, i.e., Theorem 1.9 in \mathbb{R}^2

We will now prove the following theorem, which is the two-dimensional instance of Theorem 1.9.

Theorem 6.1. *Given $\lambda \in (0, 1/2]$ and $p \in (-1/2, \infty)$ there exists $d = d_{2,\lambda,p} > 0$ such that the following holds. Let $f, g, h: \mathbb{R}^2 \rightarrow [0, 1]$ be integrable functions such that $\int_{\mathbb{R}^2} f dx = \int_{\mathbb{R}^2} g dx = 1$ and for all $x, y \in \mathbb{R}^2$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$. If $\int_{\mathbb{R}^2} h dx = 1 + \delta$ for some $0 \leq \delta \leq d$, then, up to translation, $\int_{\mathbb{R}^2} |f - g| dx = O_{2,\lambda,p}(\sqrt{\delta})$.*

The first step is to leverage the 1-dimensional result to resolve the context of well behaved tubes as in the following proposition.

Proposition 6.2. *Given $\lambda \in (0, 1/2]$, $p \in (-1/2, \infty)$, $r, \ell, s_0, s_1 \in (0, \infty)$ there exists $d = d_{\lambda,p,\ell,s_0,s_1} > 0$ such that the following holds. Let $f: [0, 1] \times [0, \infty) \rightarrow [0, s_1]$, $g: [0, r] \times [0, \infty) \rightarrow [0, s_1]$, and $f, g, h: [0, \lambda + (1 - \lambda)r] \times [0, \infty) \rightarrow [0, s_1]$ be continuous functions with bounded support such that*

- $\int f dx = \int g dx$,
- for every $x \in [0, 1]$, $\int f(x, y) dy \leq \ell s_1$,
- for every $x \in [0, r]$, $\int g(x, y) dy \leq \ell s_1$,
- $|\{x \in [0, 1]^2 : f(x) < s_0\}| + |\{x \in [0, r] \times [0, 1] : g(x) < s_0\}| \leq \eta$
- for all $x \in [0, 1] \times [0, \infty)$, $y \in [0, r] \times [0, \infty)$ we have that $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$, and
- $\int h dx \leq (1 + \delta) \int f dx$ for some $0 \leq \delta \leq d$.

Then $\int |f - g| dx = O_{\lambda,p,\ell,s_0,s_1}(\sqrt{\delta} + \eta) \int f dx$

Using the result for tubes we can find the result for all points where the functions are not too small by showing that that region is not too big and hence can be covered effectively with tubes.

Proposition 6.3. *Given $\lambda \in (0, 1/2]$, $p \in (-1/2, \infty)$, there exists $d = d_{\lambda,p}$ such that the following holds. Let $f, g, h: \mathbb{R}^2 \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int f dx = \int g dx$,
- for all $x, y \in \mathbb{R}^2$ we have that $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$, and
- $\int h dx \leq (1 + \delta) \int f dx$ for some $0 \leq \delta \leq d$.

Let $F_{0.1} := \{x \in \mathbb{R}^2 : f(x) > 0.1\}$, then there is some $v \in \mathbb{R}^2$ so that $\int_{F_{0.1}} |f(x) - g(x+v)| dx = O_{\lambda,p}(\sqrt{\delta}) \int f dx$

6.1 Tubes: Proof of Proposition 6.2

We use the following quick consequence of the 1-dimensional result

Lemma 6.4. *Given $\lambda \in (0, 1/2]$, $p \in (-1/2, \infty)$, $\ell, s_0, s_1 \in (0, \infty)$ there exists $d = d_{\lambda, p, \ell, s_0, s_1}$ such that the following holds. Let $f: [0, \infty) \rightarrow [0, s_1]$, $g: [0, \infty) \rightarrow [0, s_1]$ and $h: [0, \infty) \rightarrow [0, s_1]$ be continuous functions with bounded support such that*

- $\int f dx = \int g dx$,
- $\int f(x) dx \leq \ell \int_{[0,1]} f(x) dx$,
- $\int g(x) dx \leq \ell \int_{[0,1]} g(x) dx$, and
- $|\{x \in [0, 1] : \min\{g(x), f(x)\} < s_0\}| \leq \xi$
- for all x, y we have that $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$, and
- $\int h dx \leq (1 + \delta) \int f dx$ for some $0 \leq \delta \leq d$.

Then $\int |f - g| dx = O_{\lambda, p, \ell, s_0, s_1}(\delta^{1/2} + \xi) \int f dx$.

Proof of Lemma 6.4. By Theorem 5.1, we find that there exists some $v \in \mathbb{R}$ so that

$$\int_{\mathbb{R}} |f(x) - g(x - v)| dx = O_{\lambda, p}(\sqrt{\delta}) \int f dx.$$

We aim to show that not much changes replacing v by 0. Assuming $v > 0$ (the case $v < 0$ follows analogously), we find that

$$\int_{\mathbb{R}} |f(x) - g(x - v)| dx \geq \int_{[0, v]} |f(x) - g(x - v)| dx = \int_{[0, v]} f(x) dx \geq \min\left\{\frac{\int f}{\ell}, (v - \xi) \cdot s_0\right\},$$

where the last minimum depends on whether $v \geq 1$ or $v < 1$. We can exclude the former case by choosing $d = d_{\lambda, p, \ell, s_0, s_1}$ sufficiently small and noting that it would imply

$$\frac{1}{\ell} \leq O_{\lambda, p}(\sqrt{\delta}) \leq O_{\lambda, p}(\sqrt{d}),$$

a contradiction. Hence, we find

$$v \leq \xi + O_{\lambda, p}\left(\frac{\sqrt{\delta}}{s_0}\right) \int f dx,$$

so the translate is small. To show that g doesn't change much when translated, consider level sets $G_t := \{x \in \mathbb{R} : g(x) > 0\}$. By Lemma 4.6 we may assume that they're mostly convex, viz

$$\int_0^{s_1} |\text{co}(G_t) \setminus G_t| dt = O_{\lambda, p}(\delta) \int f dx.$$

Hence, we find

$$\begin{aligned} \int |g(x) - g(x - v)| dx &= \int_0^{s_1} |G_t \Delta (G_t + v)| dt \\ &\leq \int_0^{s_1} (|G_t \Delta \text{co}(G_t)| + |\text{co}(G_t) \Delta (\text{co}(G_t) + v)| + |(\text{co}(G_t) + v) \Delta (G_t + v)|) dt \\ &\leq \int_0^{s_1} (|\text{co}(G_t) \setminus G_t| + v + |\text{co}(G_t) \setminus G_t|) dt \\ &\leq v \cdot s_1 + O_{\lambda, p}(\delta) \int f dx \leq \xi s_1 + O_{\lambda, p}\left(\frac{s_1 \sqrt{\delta}}{s_0}\right) \int f dx \leq O_{\lambda, p, s_0, s_1}(\sqrt{\delta} + \xi) \int f dx \end{aligned}$$

Combining these two results we find

$$\int |f(x) - g(x)| dx \leq \int (|f(x) - g(x - v)| + |g(x - v) - g(x)|) dx \leq O_{\lambda, p, s_0, s_1}(\sqrt{\delta} + \xi) \int f dx,$$

which proves the lemma. \square

With this lemma in place, we can prove Proposition 6.2.

Proof of Proposition 6.2 using Lemma 6.4. Consider the maps $T: [0, 1] \rightarrow [0, r]$ defined by $\frac{dT}{dx}(x) = \frac{\int f(x,y) dy}{\int g(T(x),y) dy}$, so that for all $t \in [0, 1]$, we have

$$\int_{[0,t] \times [0,\infty)} f(x,y) dx dy = \int_{[0,T(t)] \times [0,\infty)} g(x,y) dx dy$$

For fixed $x \in [0, 1]$ and $y_1, y_2 \in [0, \infty)$, we have

$$h(\lambda x + (1-\lambda)T(x), \lambda y_1 + (1-\lambda)y_2) \geq M_{\lambda,p}(f(x, y_1), g(T(x), y_2)).$$

By Borell-Brascamp-Lieb inequality in dimension 1, we deduce

$$\int_{\mathbb{R}_+} h(\lambda x + (1-\lambda)T(x), y) dy \geq M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right) \int_{\mathbb{R}_+} f(x, y) dy.$$

Hence, we can find $\delta_x \geq 0$ for every $x \in [0, 1]$ such that

$$\int_{\mathbb{R}_+} h(\lambda x + (1-\lambda)T(x), y) dy = (1 + \delta_x) M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right) \int_{\mathbb{R}_+} f(x, y) dy.$$

Integrating over x , we get

$$\begin{aligned} (1 + \delta) \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx &\geq \int_{[0, \lambda + (1-\lambda)r]} \int_{\mathbb{R}_+} h(x, y) dy dx \\ &= \int_{[0,1]} \left(\lambda + (1-\lambda) \frac{dT}{dx}(x) \right) \left[\int_{\mathbb{R}_+} h(\lambda x + (1-\lambda)T(x), y) dy \right] dx \\ &= \int_{[0,1]} \left(\lambda + (1-\lambda) \frac{dT}{dx}(x) \right) (1 + \delta_x) M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right) \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx \\ &= \int_{[0,1]} (1 + \delta_x) \frac{M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)}{M_{\lambda,-1} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)} \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx \end{aligned}$$

Hence,

$$\delta \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx \geq \int_{[0,1]} \left(-1 + (1 + \delta_x) \frac{M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)}{M_{\lambda,-1} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)} \right) \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx \quad (6.1)$$

Recall $p > -1/2$ and hence $p/(1+p) > -1$ and therefore $M_{\lambda,p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right) \geq M_{\lambda,-1} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)$. Hence, we may conclude

$$\delta \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx \geq \int_{[0,1]} \delta_x \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx. \quad (6.2)$$

With this we can “clean up” the interval $[0, 1]$ to find that $\frac{dT}{dx}$ is mostly well-behaved and in most fibres a significant part of the mass of f (and g) lies inside the box $[0, 1]^2$ (and $[0, r] \times [0, 1]$ respectively). Consider the following subset of $[0, 1]$ which we’ll show contain little mass of f ;

$$I := \{x \in [0, 1] : |\{y \in [0, 1] : f(x, y) < s_0\}| \leq 1/2 \text{ and } |\{y \in [0, 1] : g(T(x), y) < s_0\}| \leq 1/2\}$$

Indeed, by assumption, we find that

$$|[0, 1] \setminus I| \cdot \frac{1}{2} \leq |\{x \in [0, 1]^2 : f(x) < s_0\}| + |\{x \in [0, r] \times [0, 1] : g(x) < s_0\}| \leq \eta.$$

Note that since for all $x \in [0, 1]$, we have $\int f(x, y) dy \leq \ell s_1$, we find that

$$\int_{([0,1] \setminus I) \times \mathbb{R}^+} f(x, y) dx dy \leq 2\eta \ell s_1, \quad (6.3)$$

which is negligible for us, so we'll focus on what happens in I . One thing we know about $x \in I$, is that

$$\int f(x, y) dy, \int g(T(x), y) dy \geq \frac{1}{2}s_0,$$

and by assumption $\int f(x, y) dy, \int g(T(x), y) dy \leq \ell s_1$, so $\max \left\{ \frac{dT}{dx}(x), \frac{1}{\frac{dT}{dx}(x)} \right\} \leq \frac{\ell s_1}{\frac{1}{2}s_0} = O_{s_0, s_1, \ell}(1)$

Hence, for $x \in I$, we have $\frac{M_{\lambda, p/(1+p)} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)}{M_{\lambda, -1} \left(1, \frac{1}{\frac{dT}{dx}(x)} \right)} = 1 + \Theta_{\lambda, p, \ell, s_0, s_1} \left(\left| \frac{dT}{dx}(x) - 1 \right|^2 \right)$, so that Equation (6.1)

additionally implies

$$\delta \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx \geq \int_I \Theta_{\lambda, p, \ell, s_0, s_1} \left(\left| \frac{dT}{dx}(x) - 1 \right|^2 \right) \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx.$$

Given that for $x \in I$, we have $\int_{\mathbb{R}_+} f(x, y) dy = \Theta_{s_0, s_1, \ell}(1)$, we deduce

$$\Theta_{\lambda, p, s_0, s_1, \ell}(\delta) \geq \int_I \left| \frac{dT}{dx}(x) - 1 \right|^2 dx.$$

By the Cauchy-Schwarz inequality we get

$$\Theta_{\lambda, p, s_0, s_1, \ell}(\sqrt{\delta}) \geq \int_I \left| \frac{dT}{dx}(x) - 1 \right| dx. \quad (6.4)$$

Given $T(0) = 0$, by integrating, we get that for $x \in [0, 1]$, $|T(x) - x| \leq \Theta_{\lambda, p, s_0, s_1, \ell}(\delta^{1/2} + \eta)$.

For fixed $x \in I$, the functions $h(\lambda x + (1 - \lambda)T(x), y)$, $f(x, y)$ and $g(T(x), y)$ satisfy the hypothesis of Lemma 6.4 for some $\xi_x \geq 1/2$ with the property that $\int_{[0,1]} \xi_x dx \leq \eta$, so that

$$\int_{\mathbb{R}_+} \left| f(x, y) - \frac{dT}{dx}(x)g(T(x), y) \right| dy = O_{\lambda, p, \ell, s_0, s_1, \frac{dT}{dx}(x)}(\delta_x^{1/2} + \xi_x) \int_{\mathbb{R}_+} f(x, y) dy = O_{\lambda, p, \ell, s_0, s_1}(\delta_x^{1/2} + \xi_x) \int_{\mathbb{R}_+} f(x, y) dy, \quad (6.5)$$

where the last inequality follows from the fact that $\frac{dT}{dx}(x) = \Theta_{s_0, s_1, \ell}(1)$. Combining Equation (6.2) with Equation (6.5), and using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_I \int_{\mathbb{R}_+} \left| f(x, y) - \frac{dT}{dx}(x)g(T(x), y) \right| dy dx &\leq \int_I O_{\lambda, p, \ell, s_0, s_1}(\delta_x^{1/2} + \xi_x) \left[\int_{\mathbb{R}_+} f(x, y) dy \right] dx \\ &\leq O_{\lambda, p, \ell, s_0, s_1}(\delta^{1/2} + \eta) \int_I \int_{\mathbb{R}_+} f(x, y) dy dx \\ &\leq O_{\lambda, p, \ell, s_0, s_1}(\delta^{1/2} + \eta) \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx, \end{aligned}$$

where in the last inequality we used that almost no mass of f lies outside of I .

Combining the last inequality with Equation (6.4) and the fact that $\int_{\mathbb{R}_+} g(T(x), y) dy = \Theta_{\ell, s_0, s_1}(1)$, we conclude

$$\begin{aligned} \int_I \int_{\mathbb{R}_+} |f(x, y) - g(T(x), y)| dy dx &\leq \int_I \int_{\mathbb{R}_+} \left(\left| f(x, y) - \frac{dT}{dx}(x)g(T(x), y) \right| + g(T(x), y) \left| \frac{dT}{dx}(x) - 1 \right| \right) dy dx \\ &\leq O_{\lambda, p, \ell, s_0, s_1}(\delta^{1/2} + \eta) \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx. \end{aligned}$$

To translate this information to get control over $\int |f - g|$, we consider the auxiliary function $k: [0, 1] \times [0, \infty) \rightarrow \mathbb{R}_{\geq 0}$, $(x, y) \mapsto g(T(x), y)$, so that (recalling Equation (6.3))

$$\begin{aligned} \int_{[0,1]} \int_{\mathbb{R}_+} |f - k| dy dx &\leq \int_{[0,1] \setminus I} \int_{\mathbb{R}_+} (f + k) dy dx + \int_I \int_{\mathbb{R}_+} |f - k| dy dx \\ &\leq 4\eta \ell s_1 + \int_I \int_{\mathbb{R}_+} |f(x, y) - g(T(x), y)| dy dx \\ &= O_{\lambda, p, \ell, s_0, s_1}(\delta^{1/2} + \eta) \int_{[0,1]} \int_{\mathbb{R}_+} f(x, y) dy dx. \end{aligned}$$

We shall show that also $\int |g - k| dx$ is small. To that end, consider the level sets $G_t := \{x \in \mathbb{R}^2 : g(x) > t\}$ and $K_t := \{x \in \mathbb{R}^2 : k(x) > t\}$, so that

$$\int |g - k| dx = \int_0^{s_1} |G_t \Delta K_t| dt.$$

By Lemma 4.6, we may assume

$$\int_0^{s_1} |\text{co}(G_t) \setminus G_t| dt \leq O_{\lambda,p}(\delta) \int f dx.$$

Additionally, consider the map $\mathcal{T}: [0, 1] \times [0, \infty) \rightarrow [0, r] \times [0, \infty)$, $(x, y) \mapsto (T(x), y)$ so that $\mathcal{T}(K_t) = G_t$. Since $\frac{dT}{dx}(x) = \Theta_{s_0, s_1, \ell}(1)$ for $x \in I$ and $|[0, 1] \setminus I| \leq 2\eta$, we have that

$$\int_0^{s_1} |\mathcal{T}^{-1}(\text{co}(G_t)) \Delta K_t| dt = \int_0^{s_1} |\mathcal{T}^{-1}(\text{co}(G_t) \setminus G_t)| dt \leq O_{\lambda,p,s_0,s_1,\ell}(\delta + \eta) \int f dx.$$

Hence, it suffices to prove that

$$\int_0^{s_1} |\mathcal{T}^{-1}(\text{co}(G_t)) \Delta \text{co}(G_t)| dt \leq O_{\lambda,p,s_0,s_1,\ell}(\delta^{1/2} + \eta) \int f dx \quad (6.6)$$

is small. We shall approach this fibre by fibre as follows. For a given t , we can express $|\mathcal{T}(\text{co}(G_t)) \Delta \text{co}(G_t)|$ by considering the line segments $\text{co}(G_t)_y := \{x : (x, y) \in \text{co}(G_t)\}$, so that

$$\begin{aligned} |\mathcal{T}(\text{co}(G_t)) \Delta \text{co}(G_t)| &= \int_0^\infty |\text{co}(G_t)_y \Delta T(\text{co}(G_t)_y)| dy \\ &\leq \int_0^\infty (|\min \text{co}(G_t)_y - T(\min \text{co}(G_t)_y)| + |\max \text{co}(G_t)_y - T(\max \text{co}(G_t)_y)|) dy. \end{aligned}$$

Recall that (Equation (6.4)) for all $x \in [0, 1]$ we have $|T(x) - x| \leq O_{\lambda,p,s_0,s_1,\ell}(\delta^{1/2} + \eta)$, so that

$$|\mathcal{T}(\text{co}(G_t)) \Delta \text{co}(G_t)| \leq O_{\lambda,p,s_0,s_1,\ell}(\delta^{1/2} + \eta) |\{y : \text{co}(G_t)_y \neq \emptyset\}|,$$

and, integrating over t , that

$$\int_0^{s_1} |\mathcal{T}^{-1}(\text{co}(G_t)) \Delta \text{co}(G_t)| dt \leq O_{\lambda,p,s_0,s_1,\ell}(\delta^{1/2} + \eta) |\{(t, y) \in \mathbb{R}_{\geq 0}^2 : \text{co}(G_t)_y \neq \emptyset\}|.$$

Hence it suffices to show that $|\{(t, y) : \text{co}(G_t)_y \neq \emptyset\}|$ is not too big. Consider the function $m: [0, \infty) \rightarrow [0, s_1]$, $y \mapsto \max_{x \in [0, r]} g(x, y)$. Per the following claim it suffices to control $\int_0^\infty m dy$.

Claim 6.5. $\int_0^\infty m dy \geq \frac{1}{4} |\{(t, y) \in \mathbb{R}_{\geq 0}^2 : \text{co}(G_t)_y \neq \emptyset\}|$

Proof of Claim. By definition $\int_0^\infty m dy = |\{(t, y) \in \mathbb{R}_{\geq 0}^2 : (G_t)_y \neq \emptyset\}|$. Recall that $|G_t| \geq \frac{1}{2} |\text{co}(G_t)|$ for all $t \in \mathbb{R}_{\geq 0}$. For fixed t , $\{y \in \mathbb{R}_{\geq 0} : (G_t)_y \neq \emptyset\}$ is the projection of (fairly convex) $G_t \subset \mathbb{R}^2$ onto one of the coordinate axis. The projection preserves the approximate convexity as follows, so that for fixed t , we have

$$\frac{|\{y \in \mathbb{R}_{\geq 0} : (G_t)_y \neq \emptyset\}|}{|\{y \in \mathbb{R}_{\geq 0} : \text{co}(G_t)_y \neq \emptyset\}|} \geq 1 - \sqrt{\frac{|G_t|}{|\text{co}(G_t)|}} \geq 1 - \sqrt{1/2} \geq 1/4.$$

Integrating over t proves the claim. \square

If for a given y , we have $m(y) \geq \alpha$, then there exists an x_y so that $g(x_y, y) \geq \alpha$. By definition of h , we find that for at least $\frac{1}{2} \leq 1 - \eta$ of the $z \in [0, 1]^2$, we have $f(z) \geq s_0$. Hence, for all $(x', y') \in [0, \lambda + (1 - \lambda)r] \times [(1 - \lambda)y, \lambda + (1 - \lambda)y]$, we have some $z \in [0, 1]^2$ so that

$$h(x', y') \geq h(\lambda z + (1 - \lambda)(x_y, y)) \geq M_{\lambda,p}(f(z), g(x_y, y)) \geq M_{\lambda,p}(s_0, \alpha).$$

Hence, we find

$$\int_{[0, \lambda + (1 - \lambda)r]} \int_{[(1 - \lambda)y, \lambda + (1 - \lambda)y]} h(x', y') dy' dx' \geq \frac{\lambda^2}{2} M_{\lambda,p}(s_0, \alpha) \geq \frac{\lambda^2}{2} \min\{s_0, m(y)\} \geq \frac{\lambda^2 s_0}{2s_1} m(y).$$

Integrating this inequality over y , we find that

$$\frac{\lambda^2 s_0}{2s_1} \int_0^\infty m(y) dy \leq \int_0^\infty \left(\int_{[0, \lambda+(1-\lambda)r]} \int_{[(1-\lambda)y, \lambda+(1-\lambda)y]} h(x', y') dy' dx' \right) dy = \frac{\lambda}{1-\lambda} \int_{[0, \lambda+(1-\lambda)r]} \int_{[0, \infty)} h(x', y') dy' dx'$$

Since $\int_{\mathbb{R}^2} h dx \leq 2 \int_{\mathbb{R}^2} f dx$, this implies that

$$\int_0^\infty m dy = O_{\lambda, s_0, s_1} \left(\int_{\mathbb{R}^2} f dx \right).$$

Combining the last few steps, we find

$$\begin{aligned} \int_0^{s_1} |\mathcal{T}^{-1}(\text{co}(G_t)) \Delta \text{co}(G_t)| dt &\leq O_{\lambda, p, s_0, s_1, \ell} \left(\sqrt{\delta} \right) |\{(t, y) : \text{co}(G_t)_y \neq \emptyset\}| \\ &\leq O_{\lambda, p, s_0, s_1, \ell} \left(\delta^{1/2} + \eta \right) \int m dy \\ &\leq O_{\lambda, p, s_0, s_1, \ell} \left(\delta^{1/2} + \eta \right) \int f dx. \end{aligned}$$

Putting everything together, we now find

$$\begin{aligned} \int |f - g| dx &\leq \int (|f - k| + |k - g|) dx \\ &\leq O_{\lambda, p, \ell, s_0, s_1} \left(\delta^{1/2} + \eta \right) \int f dx + \int_0^{s_1} |G_t \Delta K_t| dt \\ &\leq O_{\lambda, p, \ell, s_0, s_1} \left(\delta^{1/2} + \eta \right) \int f dx + \int_0^{s_1} (|G_t \Delta \text{co}(G_t)| + |\text{co}(G_t) \Delta \mathcal{T}^{-1}(\text{co}(G_t))| + |\mathcal{T}^{-1}(\text{co}(G_t)) \Delta \mathcal{T}^{-1}(G_t)|) dt \\ &\leq O_{\lambda, p, \ell, s_0, s_1} \left(\delta^{1/2} + \eta \right) \int f dx, \end{aligned}$$

which concludes the proof of the proposition. \square

6.2 Large level sets: Proof of Proposition 6.3

In order to proof this proposition, we will show that there exists a large ball in $F_{0,1}$ in which $\text{co}(F_{0,1}) \setminus F_{0,1}$ is very small. Within this ball we can properly position the tubes needed to cover all of $F_{0,1}$ using the following lemma.

Lemma 6.6. *Given $\lambda \in (0, 1/2]$, $p \in (-1/2, \infty)$, $r, \ell \in (0, \infty)$, where r is sufficiently small, there exists $d = d_{\lambda, p, r, \ell} > 0$ such that for $0 \leq \delta < d$ the following holds. Let $w = (r/100, 0)$. Let $C := \{(x, y) : x > 0, |y| \leq x\}$. Let B' be the ball of radius r centred at the origin intersected with C . Let $f, g, h: C \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int_C f dx = \int_C g dx$,
- $\int_C f dx \leq \ell \int_{B'} f dx$,
- for all $x, y \in C$ we have $h(\lambda x + (1-\lambda)y) \geq M_{\lambda, p}(f(x), g(y))$,
- $\int_C h dx = (1 + \delta) \int_C f dx$,
- $|B' \setminus F_\beta| + |B' \setminus G_\beta| \leq \delta$

Then for every unit vectors $f_1, f_2 \in \mathbb{R}^2$ with $\text{span}\{f_1, f_2\} = \text{span}\{e_1, e_2\}$, $|\langle f_1, e_1 \rangle| \leq 1/100$ and $|\langle f_1, f_2 \rangle| \geq 1/200$, there exists $v \in B(o, O_{\lambda, p, r, \ell, \beta}(\sqrt{\delta}))$ and $r' \in [0, 2r]$ so that

- $\int_{w+R_{r'/100}^{f_1, f_2}} f dx = \int_{w+v+R_{r'/100}^{f_1, f_2}} g dx$, and
- $\int_{w+v/2+R_{(r+r')/200}^{f_1, f_2}} h dx \leq (1 + \delta) \int_{w+R_{r'/100}^{f_1, f_2}} f dx$.

To then cover the $F_{0,01}$ we use the following observation.

Observation 6.7. For every $m \in (0, \infty)$ there exists $j = O_r(m)$ such that the following holds. There exists a collection of pairs of unit vectors $\{f_1^i, f_2^i\}_{i=1}^j \subset \mathbb{R}^2$ such that $|\langle f_1^i, e_1 \rangle| \leq 1/100$ and $|\langle f_1^i, f_2^i \rangle| \geq 1/200$ and $\bigcup_i R_r^{f_1^i, f_2^i} \supset B(0, m)$.

We first prove Lemma 6.6

Proof of Lemma 6.6. We construct the appropriate r' and v by cutting with consecutive hyperplanes which we show to be close together. First consider the halfspace G_1^- containing the origin with defining hyperplane $G_1 := w + \mathbb{R} \times \{0\}$, where $\mathbb{R} \times \{0\}$ is considered in the basis f_1, f_2 , i.e., this is the hyperplane $w + \mathbb{R}f_1$ in either base. Let G_1' be the parallel hyperplane with the property that

$$\int_{C \cap G_1^-} f \, dx = \int_{C \cap G_1'^-} g \, dx.$$

We shall show that the distance between these two planes is at most $O_{\lambda, p, \ell}(\sqrt{\delta})$. First note that we have

$$\int_{C \cap (\lambda G_1^- + (1-\lambda)G_1'^-)} h \, dx - \int_{C \cap G_1^-} f \, dx \leq \delta \int_C f \, dx \leq O_\ell(\delta) \int_{C \cap G_1^-} f \, dx,$$

where in the last step we used that $C \cap G_1^- \subset C \cap B'$ and $|C \cap G_1^-| = \Omega(|C \cap B'|)$. Hence, by Lemma 4.8, we find that

$$|(C \cap G_1^-) \Delta (C \cap G_1'^-)| = |(supp(f|_{C \cap G_1^-}) \Delta (supp(g|_{C \cap G_1'^-}))| = O_{p, \lambda, \ell}(\sqrt{\delta}),$$

which of course implies the planes are distance at most $O_{p, \lambda, \ell}(\sqrt{\delta})$ apart.

We proceed similarly for the two other hyperplanes defining $w + R_{r/100}^{f_1, f_2}$. Consider the hyperplane $G_2 := w + \{0\} \times \mathbb{R}$ in the basis f_1, f_2 , i.e., the hyperplane $w + \mathbb{R}f_2$ in the basis e_1, e_2 . As before Let G_2' be the parallel hyperplane so that for the parallel halfspaces G_2^+ and $(G_2')^+$, we have

$$\int_{C \cap G_1^+ \cap G_2^+} f \, dx = \int_{C \cap (G_1')^+ \cap (G_2')^+} g \, dx.$$

In order to show that these hyperplanes are (again) close together, we would like to use (again) Lemma 4.8, so we restrict our attention further to the bit of the domain where $f(x) \geq 0.1$ as follows. Consider the hyperplane G_3 parallel to G_1 containing $2w$, i.e., $w + \mathbb{R}f_1$ and G_3' parallel to these hyperplanes so that

$$\int_{C \cap G_1^+ \cap G_2^+ \cap G_3^-} f \, dx = \int_{C \cap (G_1')^+ \cap (G_2')^+ \cap (G_3')^-} g \, dx.$$

Since $C \cap G_3^-$ and $C \cap (G_3')^-$ are subsets of $C \cap B$, we still have that the functions f and g take values in $[0.1, 1]$ on most of these domains. Since $|C \cap G_1^+ \cap G_2^+ \cap G_3^-| = \Omega_r(|C \cap B|)$, we have

$$\int_{C \cap G_1^+ \cap G_2^+ \cap G_3^-} f \, dx \geq \Omega_r(|C \cap B|) \geq \Omega_r \left(\int_{C \cap B} f \, dx \right) \geq \Omega_{r, \ell} \left(\int_C f \, dx \right),$$

so that

$$\int_{\lambda(C \cap G_1^+ \cap G_2^+ \cap G_3^-) + (1-\lambda)(C \cap (G_1')^+ \cap (G_2')^+ \cap (G_3')^-)} h \, dx - \int_{C \cap G_1^+ \cap G_2^+ \cap G_3^-} f \, dx \leq \delta \int_C f \, dx = \Omega_{r, \ell} \left(\int_{C \cap G_1^+ \cap G_2^+ \cap G_3^-} f \, dx \right).$$

Analogous to the previous application, by Lemma 4.8, we find that up to translation

$$|(C \cap G_1^+ \cap G_2^+ \cap G_3^-) \Delta (C \cap (G_1')^+ \cap (G_2')^+ \cap (G_3')^-)| = O_{p, \lambda, \ell}(\sqrt{\delta}).$$

In particular, that implies that indeed G_2 and G_2' are at most a distance $O_{p, \lambda, \ell}(\sqrt{\delta})$ apart. Finally, let G_4 be the remaining hyperplane defining $w + R_{r/100}^{f_1, f_2}$, i.e., the plane parallel to G_2 so that $w + R_{r/100}^{f_1, f_2} = G_1^+ \cap G_2^+ \cap G_4^-$. Analogously to G_2 and G_2' , we find that G_4 and G_4' are at distance at most $O_{p, \lambda, \ell}(\sqrt{\delta})$ from one another, which allows us to conclude.

Let $v \in B(o, O_{\lambda, p, r, \ell}(\sqrt{\delta}))$ so that $v + G_1 = G_1'$ and $v + G_2 = G_2'$. Let r' be so that $w + v + R_{r'/100}^{f_1, f_2} = (G_1')^+ \cap (G_2')^+ \cap (G_4')^-$, which thus implies $r' = r \pm O_{\lambda, p, r, \ell}(\sqrt{\delta})$ \square

Proof of Proposition 6.3. For notational convenience normalise so that $\int f dx = \int g dx = 1$ by rescaling the domain. We may assume $\sup f(x) = 1$.

Analogously to Section 4.2 where we show that Theorem 1.9 follows from Proposition 4.1, we may additionally assume that for $i = 1, 2, 3$ we have (for $C_i = C_n^i$ as defined in Definition 2.5)

$$\int_{C_i} f dx = \int_{C_i} g dx,$$

and for some $r, \beta = \Omega_{\lambda,p}(1)$, we have that if we let $B' = B(o, r)$ the ball of radius r centred at the origin, then

$$|B' \setminus F_\beta| + |B' \setminus G_\beta| = O_{\lambda,p}(\delta) \text{ and } |\text{co}(F_\beta)| \leq 2|F_\beta| \leq 2\beta^{-1} = O_{\lambda,p}(1).$$

Note that by Borell-Brascamb-Lieb, we have that $\int_{C_i} h dx \geq \int_{C_i} f dx$ and thus

$$\int_{C_i} h dx - \int_{C_i} f dx \leq \delta \int f dx \leq O_{\lambda,p}(\delta) \int_{C_i} f dx,$$

where in the last inequality we used that each of the cones contains a $\Omega_{\lambda,p}(1)$ proportion of $\int f dx$ (even when restricted to $C_i \cap B'$).

We consider each of the cones separately; in particular, we consider $C_1 = \{(x, y) : x > 0, |y| \leq x\}$ and the others will follow analogously. Let $w = (r/100, 0)$. Since $|B' \cap C_1 \cap F_\beta| \geq \Omega_{\lambda,p}(1)$ and $|\text{co}(F_\beta)| = O_{\lambda,p}(1)$, we find that there is an $m = O_{\lambda,p}(1)$ so that $F_\beta \subset B(w, m)$. Apply Observation 6.7 with parameters $r/100$ and m to find $j = O_{\lambda,p}(1)$ a collection $\{f_1^i, f_2^i\}_{i=1}^j$ so that $B' \cap C_1 \subset w + \bigcup_i R_{r/100}^{f_1^i, f_2^i}$. By Lemma 6.6, we can find $v_i \in B(o, O_{\lambda,p}(\sqrt{\delta}))$ so that

- $\int_{w+R_{r/100}^{f_1^i, f_2^i}} f dx = \int_{w+v_i+R_{r/100}^{f_1^i, f_2^i}} g dx$, and
- $\int_{w+v_i/2+R_{(r+r')/200}^{f_1^i, f_2^i}} h dx \leq (1+\delta) \int_{w+R_{r/100}^{f_1^i, f_2^i}} f dx$.

Since $w + R_{r/100}^{f_1^i, f_2^i}$ intersects B' in a big part, at least a $1/\ell = \Omega_{\lambda,p}(1)$ proportion of f and g is concentrated near the base of the tube $w + R_{r/100}^{f_1^i, f_2^i}$. Hence, we can apply Proposition 6.2 with $s_1 = 1$, $s_0 = \beta \geq \Omega_{\lambda,p}(1)$, and $\eta = O_{\lambda,p}(\delta)$, we find that

$$\int_{w+R_{r/100}^{f_1^i, f_2^i}} |f(x) - g(x - v_i)| dx = O_{\lambda,p}(\sqrt{\delta}) \int f dx$$

Using that $v_i \in B(o, O_{\lambda,p}(\sqrt{\delta}))$ we find by Lemma 4.9 that;

$$\int_{w+R_{r/100}^{f_1^i, f_2^i}} |f(x) - g(x)| dx = O_{\lambda,p}(\sqrt{\delta}) \int f dx$$

Summing this over all i gives that

$$\begin{aligned} \int_{F_\beta} |f - g| dx &\leq \int_{B(m, w)} |f - g| dx \leq \int_{w + \bigcup_i R_{r/100}^{f_1^i, f_2^i}} |f - g| dx \leq \sum_i \int_{w+R_{r/100}^{f_1^i, f_2^i}} |f - g| dx \\ &\leq j \cdot O_{\lambda,p}(\sqrt{\delta}) \int f dx = O_{\lambda,p}(\sqrt{\delta}) \int f dx. \end{aligned}$$

Since $F_{0.1} \subset F_\beta$, this concludes the proof of the proposition. \square

6.3 Gluing together level sets: Proof of Theorem 6.1

In this section, we'll see that applying Proposition 6.3 to the cut off functions $\min\{f, 2^i\}$ shows that all level sets of f have small symmetric difference to level sets of g up to translation. Careful analysis allows us to conclude that those translations may all be assumed to be the same. The main idea is that Proposition 6.3 implies that level sets are close together

Proof of Theorem 6.1. As ever, consider the level sets F_t, G_t, H_t and transport map $T: [0, 1] \rightarrow [0, 1]$ so that $\int_0^t |F_s| ds = \int_0^{T(t)} |G_s| ds$. Recall that by Lemma 4.6, we have that

$$\int (|\text{co}(F_t) \setminus F_t| + |\text{co}(G_t) \setminus G_t|) dt = O_{\lambda,p}(\delta) \int f dx$$

A particular consequence of Proposition 6.3 is that for some translate v_0 , we have

$$\int_{0.1}^1 |F_t \Delta(v_0 + G_t)| dt = O_{\lambda,p}(\sqrt{\delta}).$$

In order to extend this to all level sets, consider the functions

$$f_i := \min\{f, 2^{-i}\}, g_i := \min\{g, T(2^{-i})\}, h_i := \min\{h, M_{\lambda,p}(2^{-i}, T(2^{-i}))\},$$

so that $h_i \leq M_{\lambda,p}^*(f_i, g_i)$. Let $\delta_i := \delta \frac{f dx}{f_i dx}$ (so that δ_i is non-decreasing in i) and note that⁵

$$\int h_i dx \leq \int h dx - \int (h - h_i) dx \leq \int h dx - \int (f - f_i) dx \leq \int f_i dx + \delta \int f dx \leq (1 + \delta_i) \int f_i dx.$$

In the end we will only consider f_i so that $\int f_i dx \geq \sqrt{\delta} \int f dx$ (which we'll see has $i = O_{\lambda,p}(-\log(\delta))$); beyond that it's trivial to control the symmetric difference. Hence, we may continue to assume δ_i is uniformly sufficiently small in terms of λ and p whenever we need it.

Since the level sets of f_i are just the level sets of f up to 2^{-i} , we can again apply Proposition 6.3 to find a translate v_i

$$\int_{0.1 \cdot 2^{-i}}^{2^{-i}} |F_t \Delta(v_i + G_t)| dt = O_{\lambda,p}(\sqrt{\delta_i}) \int f_i dx.$$

To get control over v_i we note that consecutive integrals overlap significantly. Indeed, we have that

$$\int_{0.1 \cdot 2^{-i}}^{0.5 \cdot 2^{-i}} (|F_t \Delta(v_i + G_t)| + |F_t \Delta(v_{i+1} + G_t)|) dt \leq O_{\lambda,p}(\sqrt{\delta_{i+1}}) \int f_i dx,$$

so that by the triangle inequality, we have

$$\int_{0.1 \cdot 2^{-i}}^{0.5 \cdot 2^{-i}} |F_t \Delta(v_i - v_{i+1} + F_t)| dt \leq O_{\lambda,p}(\sqrt{\delta_{i+1}}) \int f_i dx,$$

and recalling that $\int |\text{co}(F_t) \setminus F_t| dt \leq O_{\lambda,p}(\delta) \int f dx$, this implies

$$\int_{0.1 \cdot 2^{-i}}^{0.5 \cdot 2^{-i}} |\text{co}(F_t) \Delta(v_i - v_{i+1} + \text{co}(F_t))| dt \leq O_{\lambda,p}(\sqrt{\delta_{i+1}}) \int f_i dx + O_{\lambda,p}(\delta) \int f dx \leq O_{\lambda,p}(\sqrt{\delta_{i+1}}) \int f_i dx.$$

This implies that $v_i - v_{i+1}$ is small from the perspective of $F_{0.1 \cdot 2^{-i}}$. Indeed, if some vector v is not in ηK for some (centered) convex set $K \subset \mathbb{R}^2$, then for any subset $X \subset K$, we have $|X \Delta(v + X)| \geq \Omega(\eta)|X|$.

Hence, combining the above inequality with $\int_{0.1 \cdot 2^{-i}}^{0.5 \cdot 2^{-i}} |\text{co}(F_t)| dt = \Omega_{\lambda,p}(\int f_i dx)$ and $\text{co}(F_t) \subset \text{co}(F_{0.1 \cdot 2^{-i}})$ for all $t \in [0.1 \cdot 2^{-i}, 0.5 \cdot 2^{-i}]$, we derive that

$$v_i - v_{i+1} \in O_{\lambda,p}(\sqrt{\delta_i}) \text{co}(F_{0.1 \cdot 2^{-i}}).$$

By induction, we can derive that (aiming to have the same translate throughout)

$$v_j - v_0 = \sum_{i=0}^{j-1} v_{i+1} - v_i \in \sum_{i=0}^{j-1} O_{\lambda,p}(\sqrt{\delta_i}) \text{co}(F_{0.1 \cdot 2^{-i}}) \subset j \cdot O_{\lambda,p}(\sqrt{\delta_j}) \text{co}(F_{0.1 \cdot 2^{-j}}),$$

where in the last containment we use that δ_i is increasing in i and $\text{co}(F_{0.1 \cdot 2^{-i}})$ form a nested sequence of sets. Returning to the symmetric difference, this implies

$$\int_{2^{-i}}^{0.5 \cdot 2^{-i}} |F_t \Delta(v_0 + G_t)| dt \leq \int_{2^{-i}}^{0.5 \cdot 2^{-i}} (|F_t \Delta(v_i - v_0 + F_t)| + |F_t \Delta(v_i + G_t)|) dt,$$

⁵For more elaboration on the second inequality see Equation (4.1).

where the latter is easily bounded and the former is controlled since (again) $|\text{co}(F_t) \setminus F_t|$ is small. Indeed, we have

$$|F_t \Delta(v_i - v_0) + F_t| \leq |\text{co}(F_t) \Delta(v_i - v_0) + \text{co}(F_t)| + 2|\text{co}(F_t) \setminus F_t|,$$

and for all $t \geq 0.1 \cdot 2^{-i}$, since $v_i - v_0 \in i \cdot O_{\lambda,p}(\sqrt{\delta_i}) \text{co}(F_{0.1 \cdot 2^{-i}})$ we have

$$|\text{co}(F_t) \Delta(v_i - v_0) + \text{co}(F_t)| \leq i \cdot O_{\lambda,p}(\sqrt{\delta_i}) |\text{co}(F_{0.1 \cdot 2^{-i}})| \leq i \cdot O_{\lambda,p}(\sqrt{\delta_i}) |F_{0.1 \cdot 2^{-i}}|.$$

Hence, we find

$$\int_{2^{-i}}^{0.5 \cdot 2^{-i}} |F_t \Delta(v_i - v_0) + F_t| dt \leq 2^{-i} \cdot i \cdot O_{\lambda,p}(\sqrt{\delta_i}) |F_{0.1 \cdot 2^{-i}}| \leq i \cdot O_{\lambda,p}(\sqrt{\delta_i}) \int f_i dx,$$

and thus

$$\int_{2^{-i}}^{0.5 \cdot 2^{-i}} |F_t \Delta(v_0 + G_t)| dt \leq (i+1) \cdot O_{\lambda,p}(\sqrt{\delta_i}) \int f_i dx + O_{\lambda,p}(\delta) \int f dx \leq i \cdot O_{\lambda,p}(\sqrt{\delta_i}) \int f_i dx.$$

Recalling the definition of δ_i and summing over all $i \leq i_0$, where we pick i_0 minimal so that $\int f_i \leq \sqrt{\delta} \int f dx$, we find

$$\begin{aligned} \int_0^1 |F_t \Delta(G_t + v_0)| dt &= \int f_{i_0} dx + \sum_{i=0}^{i_0} \int_{2^{-(i+1)}}^{2^{-i}} |F_t \Delta(v_0 + G_t)| dt \\ &\leq \int f_{i_0} dx + \sum_{i=0}^{i_0} i \cdot O_{\lambda,p}(\sqrt{\delta_i}) \int f_i dx \\ &\leq O_{\lambda,p}(\sqrt{\delta}) \int f dx \cdot \left[\sum_{i=0}^{i_0} i \cdot \sqrt{\frac{\int f_i dx}{\int f dx}} \right]. \end{aligned}$$

As we've used several times before, as long as $\int h_i dx \leq (1+\eta) \int f_i dx$ for some $\eta \ll_{\lambda,p} 1$ (which is the case for $i \leq i_0$), we find that

$$\int f_i dx - \int f_{i+1} dx = \int_{2^{-(i+1)}}^{2^{-i}} |F_t| dt \geq \Omega_{\lambda,p} \left(\int f_i dx \right).$$

Hence, the $\int f_i dx$ form a geometric sequence (as long as $i \leq i_0$) and thus $\sum_{i=0}^{i_0} i \cdot \sqrt{\frac{\int f_i dx}{\int f dx}} = O_{\lambda,p}(1)$. This allows us to conclude

$$\int_{x \in \mathbb{R}^2} |f(x) - g(x - v_0)| dx = \int_0^1 |F_t \Delta(G_t + v_0)| dt = O_{\lambda,p}(\sqrt{\delta}) \int f dx.$$

□

7 Proof of Theorem 1.9 in \mathbb{R}^n

7.1 Reduction from Theorem 1.9 to Proposition 7.1

Thanks to the arguments in Section 4, to show the validity of Theorem 1.9 in \mathbb{R}^n it suffices to prove Proposition 4.1. We first show that the latter is implied by the following result.

Proposition 7.1. *Given $n \in \mathbb{N}$, $i \in [0, n]$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $r, \beta \in (0, \infty)$ there exists $d = d_{n,\lambda,p,r,\beta} > 0$ such that the following holds. Let $f, g, h: C_n^i \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int_{C_n^i} f dx = \int_{C_n^i} g dx = 1$,
- $\int_{C_n^i} h dx = 1 + \delta$ for some $0 \leq \delta \leq d$,
- $|\{x \in rS_n \cap C_n^i : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \delta$, and
- for all $x, y \in \mathbb{R}^n$ we have $h(\lambda x + (1-\lambda)y) \geq M_{\lambda,p}(f(x), g(y))$.

Then $\int_{C_n^i} |f - g| dx = O_{n,\lambda,p,r,\beta}(\sqrt{\delta})$.

Proof that Proposition 7.1 implies Proposition 4.1. Consider f and g as in Proposition 4.1, and let f_i (resp. g_i) be the function f (resp g) restricted to C_n^i renormalized, i.e.,

$$f_i(x) := \frac{f(x)\mathbf{1}_{C_n^i}(x)}{\int_{C_n^i} f dy},$$

so that $\int_{C_n^i} f_i(x) dx = \int_{C_n^i} g_i(x) dx = 1$.

Let $h_i := M_{\lambda,p}^*(f_i, g_i)$. We'll show that h has integral not much larger than 1. Note that the support of h_i falls in C_n^i , so that by Borell-Brascamb-Lieb $\int_{C_n^i} h_i dx \geq 1$. Moreover, since the supports of the h_i are essentially distinct, we have that $\sum_{i=0}^n \left(\int_{C_n^i} f dy\right) h_i(x) \leq h(x)$ for almost all x . Hence, integrating over $x \in \mathbb{R}^n$, we find

$$\sum_{i=0}^n \left(\int_{C_n^i} f dy\right) \int_{C_n^i} h_i(x) dx \leq \int_{\mathbb{R}^n} h(x) dx \leq 1 + \delta.$$

Isolating h_{i_0} for some fixed i_0 and using $\int_{C_n^{i_0}} h_{i_0} dx \geq 1$, we find

$$\int_{C_n^{i_0}} h_{i_0}(x) dx \leq \frac{(1 + \delta) - \sum_{i \neq i_0} \left(\int_{C_n^i} f dy\right) \int_{C_n^i} h_i(x) dx}{\int_{C_n^{i_0}} f dy} \leq \frac{(\int_{\mathbb{R}^n} f dy + \delta) - \sum_{i \neq i_0} \left(\int_{C_n^i} f dy\right)}{\int_{C_n^{i_0}} f dy} = 1 + \frac{\delta}{\int_{C_n^{i_0}} f dy}.$$

Finally, we use that f is big on rS_n to find that

$$\int_{C_n^{i_0}} f dy \geq \int_{C_n^{i_0} \cap rS_n} f dy \geq \frac{\beta}{2} |C_n^{i_0} \cap rS_n| = \Omega_{n,r,\beta}(1).$$

Hence, we can choose $d_{n,\lambda,p,r,\beta}$ in Proposition 4.1 sufficiently small so that $\frac{2\delta}{\beta |C_n^{i_0} \cap rS_n|}$ is smaller than the d in Proposition 7.1.

Now that we've found that the f_i, g_i, h_i satisfy the conditions of Proposition 7.1, we can apply that proposition in each of the cones and conclude that $\int_{C_n^i} |f_i - g_i| dx = O_{n,\lambda,p,r} \left(\frac{2\delta}{\beta |C_n^{i_0} \cap rS_n|}\right)^{1/2} = O_{n,\lambda,p,r,\beta}(\sqrt{\delta})$. Summing over the cones, we find

$$\int_{\mathbb{R}^n} |f - g| dx = \sum_{i=0}^n \int_{C_n^i} f_i dy \int_{C_n^i} |f_i - g_i| dx \leq \sum_{i=0}^n \int_{C_n^i} |f_i - g_i| dx \leq O_{n,\lambda,p,r,\beta}(\sqrt{\delta}),$$

so that Proposition 4.1 follows. \square

7.2 Reduction from Proposition 7.1 to Proposition 7.2

For the next sections we shall assume that the simplex S_n and the basis e_1, \dots, e_n of \mathbb{R}^n are such that $e_1 \perp H_n^1$ where we recall H_n^1 is the supporting hyperplane of the face F_n^1 of the simplex S_n .

Proposition 7.2. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $r, s, \ell, \beta \in (0, \infty)$, there exists $d = d_{n,\lambda,p,r,\ell,\beta}^{7.2} > 0$ so that for all $\delta \in (0, d)$, there exists $e = e_{n,\lambda,p,r,s,\ell,\beta,\delta} > 0$ so that for all $\epsilon \in (0, e)$ the following holds. Let $C \subset C_n^1$ be a cone. Let $f, g, h: C \rightarrow [0, 1]$ be continuous functions such that*

- $|\int_C (f - g) dx| \leq \epsilon \int_C f dx,$
- $\int_C h dx = (1 + \delta) \int_C f dx,$
- $\int_C f dx \leq \ell \int_{C \cap rS_n} f dx,$
- $|\{x \in rS_n \cap C : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma |C \cap rS_n|,$
- for every $x \notin sS_n$ we have $f(x) = g(x) = 0,$
- for all $x, y \in C$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y)),$ and

- for all $z, w \in \mathbb{R}$ and $x, y \in C^{z,w}$ we have $|f(x) - f(y)| \leq \varepsilon$ and $|g(x) - g(y)| \leq \varepsilon$.

Then $\int_C |f - g| dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta + \gamma}) \int_C f dx$.

Lemma 7.3. Let $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $r, \beta \in (0, \infty)$, then there exist $\ell = \ell_{n,\lambda,p,r,\beta}^{7.3}$ and $d = d_{n,\lambda,p,r,\beta}^{7.3} > 0$ so that the following hold. Let $f, g, h: \mathbb{R}^n \rightarrow [0, 1]$ be continuous functions with bounded support and a cone C such that

- $\int_{\mathbb{R}^n} f dx = \int_{\mathbb{R}^n} g dx = 1$,
- $\int_{\mathbb{R}^n} h dx \leq 1 + d$,
- $\int_C f dx = \int_C g dx$,
- $\int_C h dx \leq (1 + d) \int_C f dx$,
- $|\{x \in rS_n : f(x) < \beta \text{ or } g(x) < \beta\}| \leq 0.1|rS_n|$,
- $|\text{co}(\{x \in \mathbb{R}^n : f(x) > t\})| \leq 2|\{x \in \mathbb{R}^n : f(x) > t\}|$ for all $t \in [0, 1]$, and
- for all $x, y \in \mathbb{R}^n$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$.

Then we have $\int_C f dx \leq \ell|C \cap rS_n|$.

Definition 7.4. Given a subcone $C \subset C_n^1$ and a basis f_1, \dots, f_n with $e_1 = f_1$, define the slice $C^{z,w} := C \cap (z, w) \times \mathbb{R}^{n-2}$.

Lemma 7.5. Let $n \in \mathbb{N}$ and let $f, g: C_n^1 \rightarrow [0, 1]$ be continuous functions with bounded support such that $\int_{C_n^1} f dx = \int_{C_n^1} g dx$. Then for every $\varepsilon > 0$ there exists a family $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ of cones partitioning the cone C_n^1 such that

- $\sum_{C \in \mathcal{F}_1} \int_C f dx < \varepsilon$.
- For every $C \in \mathcal{F}_2$, $|\int_C (f - g) dx| \leq \varepsilon \int_C f dx$.
- For every $C \in \mathcal{F}_2$, there exists a basis f_1, \dots, f_n depending on C with $f_1 = e_1$, such that for all $z, w \in \mathbb{R}$ we have $\text{diam}(C^{z,w}) \leq \varepsilon|z|$.

7.2.1 Proof of Reduction

Proof that Proposition 7.2 implies Proposition 7.1. We consider the cone C_n^1 , the other C_n^i follow analogously by rotation.

Given that f and g have bounded support, there exists an s so that for $x \notin sS_n$ we have $f(x) = g(x) = 0$.

Apply Lemma 7.5 with parameter $\eta > 0$ sufficiently small in terms of all of $f, g, n, \lambda, p, r, s, \ell, \beta, \varepsilon$, to find essential partition $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ of C_n^1 . Further subdivide \mathcal{F}_2 into two sets as follows. For every cone $C \in \mathcal{F}_2$ define

$$\delta_C := \frac{\int_C h dx}{\int_C f dx} - 1,$$

so that

$$\sum_{C \in \mathcal{F}_2} \delta_C \int_C f dx \leq \sum_{C \in \mathcal{F}_2} \int_C (h - f) dx \leq \int_{\mathbb{R}^n} (h - f) dx + 2\eta \leq 2\delta,$$

where in the penultimate inequality we used that $\int_C (h - f) dx \geq -\eta \int_C f dx$ for $C \in \mathcal{F}$ and $\sum_{C \in \mathcal{F}_1} \int (h - f) dx \geq \sum_{C \in \mathcal{F}_1} \int (-f) dx \geq -\eta$.

Analogously define

$$\gamma_C := \frac{|\{x \in rS_n \cap C : f(x) < \beta \text{ or } g(x) < \beta\}|}{|rS_n \cap C|},$$

so that

$$\sum_{C \in \mathcal{F}} \gamma_C |rS_n \cap C| = \sum_{C \in \mathcal{F}} |\{x \in rS_n \cap C : f(x) < \beta \text{ or } g(x) < \beta\}| = |\{x \in rS_n : f(x) < \beta \text{ or } g(x) < \beta\}| \leq O_{n,\lambda,p,r}(\delta).$$

Partition $\mathcal{F}_2 = \mathcal{F}_\gamma^+ \sqcup \mathcal{F}_\delta^+ \sqcup \mathcal{F}^-$ as follows. If $\delta_C \geq \frac{1}{2} \min \left\{ d_{n,\lambda,p,r,\beta}^{7.3}, d_{n,\lambda,p,r,\beta,\ell_{n,\lambda,p,r}^{7.3}}^{7.2} \right\}$ let $C \in \mathcal{F}^+$. If $\gamma_C \geq \sqrt{\delta}$ (and $C \notin \mathcal{F}_\delta^+$), let $C \in \mathcal{F}_\gamma^+$. Finally, let $\mathcal{F}^- := \mathcal{F}_2 \setminus (\mathcal{F}_\gamma^+ \sqcup \mathcal{F}_\delta^+)$

It's easy to address the cones in $\mathcal{F}_1 \sqcup \mathcal{F}_\gamma^+ \sqcup \mathcal{F}_\delta^+$ as they contain little mass. First for $C \in \mathcal{F}_1$ by definition

$$\sum_{C \in \mathcal{F}_1} \int_C |f - g| dx \leq \sum_{C \in \mathcal{F}_1} \int_C (f + g) dx \leq \eta.$$

For the $C \in \mathcal{F}_\delta^+$, first note that for all $C' \in \mathcal{F}_2$, we have by the Borell-Brascamb-Lieb inequality that

$$\int_{C'} h dx \geq \min \left\{ \int_{C'} f dx, \int_{C'} g dx \right\} \geq \int_{C'} f dx - \left| \int_{C'} (f - g) dx \right| \geq (1 - \eta) \int_{C'} f dx,$$

Hence, we can find a bound on

$$\begin{aligned} \int_{\mathbb{R}^n} h dx &\geq \sum_{C \in \mathcal{F}_\delta^+} \int_C h dx + \sum_{C' \in \mathcal{F}_2 \setminus \mathcal{F}_\delta^+} \int_{C'} h dx \\ &\geq \left(1 + \frac{1}{2} \min \left\{ d_{n,\lambda,p,r,\beta}^{7.3}, d_{n,\lambda,p,r,\beta,\ell_{n,\lambda,p,r}^{7.3}}^{7.2} \right\} \right) \sum_{C \in \mathcal{F}_\delta^+} \int_C f dx + \sum_{C' \in \mathcal{F}_2 \setminus \mathcal{F}_\delta^+} (1 - \eta) \int_{C'} f dx \\ &\geq \left[\int_{\mathbb{R}^n} f dx - \sum_{C \in \mathcal{F}_1} \int_C f dx \right] + \frac{1}{2} \min \left\{ d_{n,\lambda,p,r,\beta}^{7.3}, d_{n,\lambda,p,r,\beta,\ell_{n,\lambda,p,r}^{7.3}}^{7.2} \right\} \sum_{C \in \mathcal{F}_\delta^+} \int_C f dx - \eta \sum_{C' \in \mathcal{F}_2 \setminus \mathcal{F}_\delta^+} \int_{C'} f dx \\ &\geq [1 - \eta] + \frac{1}{2} \min \left\{ d_{n,\lambda,p,r,\beta}^{7.3}, d_{n,\lambda,p,r,\beta,\ell_{n,\lambda,p,r}^{7.3}}^{7.2} \right\} \sum_{C \in \mathcal{F}_\delta^+} \int_C f dx - \eta \end{aligned}$$

Recalling that $\int_{\mathbb{R}^n} h dx \leq 1 + \delta$ and noting that $\frac{1}{2} \min \left\{ d_{n,\lambda,p,r,\beta}^{7.3}, d_{n,\lambda,p,r,\beta,\ell_{n,\lambda,p,r}^{7.3}}^{7.2} \right\} = \Omega_{n,\lambda,p,r,\beta}(1)$, we can rearrange this inequality to find

$$\sum_{C \in \mathcal{F}_\delta^+} \int_C |f - g| dx \leq 3 \sum_{C \in \mathcal{F}_\delta^+} \int_C f dx \leq O_{n,\lambda,p,r,\beta}(\delta + 2\eta) = O_{n,\lambda,p,r,\beta}(\delta).$$

For $C \in \mathcal{F}_\gamma^+$, we find that

$$\sqrt{\delta} \sum_{C \in \mathcal{F}_\gamma^+} |rS_n \cap C| \leq \sum_{C \in \mathcal{F}_\gamma^+} \gamma_C |rS_n \cap C| \leq O_{n,\lambda,p,r,\beta}(\delta),$$

so that $\sum_{C \in \mathcal{F}_\gamma^+} |rS_n \cap C| \leq O_{n,\lambda,p,r,\beta}(\sqrt{\delta})$. By Lemma 7.3, we find that for these C , that $\int_C f dx \leq O_{n,\lambda,p,r,\beta}(|C \cap rS_n|)$. Hence,

$$\sum_{C \in \mathcal{F}_\gamma^+} \int_C |f - g| dx \leq 3 \sum_{C \in \mathcal{F}_\gamma^+} \int_C f dx \leq 3\ell \sum_{C \in \mathcal{F}_\gamma^+} |C \cap rS_n| \leq O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta}).$$

Finally, we turn our attention to cones $C \in \mathcal{F}^-$ to which we first apply Lemma 7.3⁶ to find that $\int_C f dx \leq \ell_{n,\lambda,p,r,\beta}^{7.3} \int_{C \cap rS_n} f dx$. Since f and g are bounded continuous functions with bounded supports, they are uniformly continuous, i.e., for any $\epsilon > 0$ we can choose η sufficiently small in f, g , and ϵ , so that we have $|x - y| < \eta$ implies $|f(x) - f(y)|, |g(x) - g(y)| \leq \epsilon$. Moreover, the parameter s in Proposition 7.2 depends only on f and g , so we can pick $\epsilon < e_{n,\lambda,p,r,s,\ell,\beta,\delta}$ and still find an appropriate η . Hence, we can apply Proposition 7.2 with parameters $n, \lambda, p, r, s = s_{f,g}, \ell = \ell_{n,\lambda,p,r,\beta}^{7.3}$, and $\delta = \delta_C < d_{n,\lambda,p,r,\beta,\ell}^{7.2}$ and $\gamma = \gamma_C$ to the cones $C \in \mathcal{F}^-$, to find that

$$\int_C |f - g| dx = O_{n,\lambda,p,r,\ell_{n,\lambda,p,r,\beta}^{7.3}}(\sqrt{\delta_C + \gamma_C}) \int_C f dx = O_{n,\lambda,p,r,\beta}(\sqrt{\delta_C + \gamma_C}) \int_C f dx.$$

Recall that $\sum_{C \in \mathcal{F}_2} \delta_C \int_C f dx \leq 2\delta$, and by Lemma 7.3,

$$\sum_{C \in \mathcal{F}^-} \gamma_C \int_C f dx \leq O_{n,\lambda,p,r,\beta} \left(\sum_{C \in \mathcal{F}^-} \gamma_C |C \cap rS_n| \right) \leq O_{n,\lambda,p,r,\beta}(\delta).$$

⁶Note that though f and g don't have the same integral in C , we can switch to $f' \leq f$ and $g' \leq g$ removing at most $\epsilon \int_C f dx$ to equalize the integral and apply Lemma 7.3.

Hence, we conclude by convexity of the square root that

$$\begin{aligned} \int_{\mathbb{R}^n} |f - g| dx &= \sum_{C \in \mathcal{F}_1 \cup \mathcal{F}^+} \int_C |f - g| dx + \sum_{C \in \mathcal{F}^-} \int_C |f - g| dx \\ &\leq O_{n,\lambda,p,r,\beta}(\delta) + \sum_{C \in \mathcal{F}^-} O_{n,\lambda,p,r,\beta}(\sqrt{\delta_C + \gamma_C}) \int_C f dx \\ &\leq O_{n,\lambda,p,r,\beta}(\sqrt{\delta}). \end{aligned}$$

This finishes the proof of the reduction. \square

7.2.2 Proof of Lemmas

Proof of Lemma 7.3. For a contradiction let $c_{n,\lambda,p,r,\beta}$ sufficiently large in its parameters to be determined later and C a cone so that

$$\int_C f dx > c_{n,\lambda,p,r,\beta} |C \cap rS_n|$$

Consider the level sets F_t^C, G_t^C, H_t^C for the restrictions of the f, g, h to C , i.e., $F_t^C := \{x \in C : f(x) > t\}$. By Lemma 4.6, we find some $\beta_C = \Omega_{n,\lambda,p}(1)$ so that

$$\int_{F_{\beta_C}^C} f dx \geq 0.9 \int_C f dx \geq 0.9 c_{n,\lambda,p,r,\beta} |C \cap rS_n|.$$

On the other hand, we have $\int_{F_{\beta_C}^C} f dx \leq |F_{\beta_C}^C|$. Since $F_{\beta_C}^C$ is so large, in particular it is not contained in $C \cap (0.9 c_{n,\lambda,p,r,\beta})^{1/n} rS_n$, so we can find a point $v \notin (0.9 c_{n,\lambda,p,r,\beta})^{1/n} rS_n$ with $f(v) \geq \beta_C$. Consider the level set $F_{\beta_C} \supset F_{\beta_C}^C$, which by assumption has $|F_{\beta_C}| \geq \frac{1}{2} |\text{co}(F_{\beta_C})|$. We may assume that $\beta_C \leq \beta$, so that we also have $|F_{\beta_C} \cap rS_n| \geq 0.9 |rS_n|$. This implies that that $|\text{co}(F_{\beta_C})| \geq |\text{co}((F_{\beta_C} \cap rS_n) \cup \{v\})| \rightarrow \infty$ as $c_{n,\lambda,p,r,\beta} \rightarrow \infty$. On the other hand, we have the trivial bound that $|\text{co}(F_{\beta_C})| \leq 2|F_{\beta_C}| \leq \beta_C^{-1} \int f dx = O_{n,\lambda,p,r}(1)$, which for sufficiently large $c_{n,\lambda,p,r,\beta}$ yields a contradiction. \square

For the proof of Lemma 7.5, we use one of the technical results from [FvHT23]. First we need the following definitions from that paper.

Definition 7.6. Let \mathcal{C}^n be the family of cones in \mathbb{R}^n and let \mathcal{T}_k^n be the set of codimension k subspaces of \mathbb{R}^n .

Definition 7.7. Say a function $k: \mathcal{C}^n \times \mathcal{T}_2^n \rightarrow \mathcal{T}_1^n$ is a respectful function if $L \subset k(C, L)$. A respectful function f induces functions $k^-, k^+: \mathcal{C}^n \times \mathcal{T}_2^n \rightarrow \mathcal{C}^n$, where $k^-(C, L), k^+(C, L)$ are the cones the hyperplane $k(C, L)$ partitions C into.

Definition 7.8. Given a respectful function $k: \mathcal{C}^n \times \mathcal{T}_2^n \rightarrow \mathcal{T}_1^n$ and a cone C , we say \mathcal{F} is a valid partition of C into cones if there exists a sequence of families $\{C\} = \mathcal{G}_0, \dots, \mathcal{G}_j = \mathcal{F}$ such that if $\mathcal{G}_i = \{C_1, C_2, \dots, C_k\}$, then there exists codimension-two subspaces L_1, \dots, L_m such that $\mathcal{G}_{i+1} = \bigcup_{j=1}^m R_j$, where $R_j = \{k^+(C_j, L_j), k^-(C_j, L_j)\}$ or $R_j = \{C_j\}$.

Now recall the following slightly weakened version of theorem 4.11 from [FvHT23].

Theorem 7.9 (Theorem 4.11 from [FvHT23] reformulated). For every $\eta > 0$ the following holds. Given a respectful $f: \mathcal{C}^n \times \mathcal{T}_2^n \rightarrow \mathcal{T}_1^n$, there exists a valid partition \mathcal{F} of \mathcal{C}_n^1 that can be written as $\mathcal{F}' = \mathcal{F}'_0 \sqcup \mathcal{F}'_1 \sqcup \mathcal{F}'_2$ such that

1. $\sum_{C \in \mathcal{F}'_0} |C \cap S_n| \leq \eta$.
2. For every cone $C \in \mathcal{F}'_1$, we find that $C \cap \partial S_n$ has diameter at most ϵ .
3. For every cone $C \in \mathcal{F}'_2$ there exists a sub-cone C' of C with $|C' \cap S_n| \geq (1 - \eta) |C \cap S_n|$ and there exists a basis f_1, \dots, f_n depending on C with $f_1 = e_1$, such that for all $z, w \in \mathbb{R}$ we have $\text{diam}((C')^{z,w}) \leq \eta \|z\|$.

With these in place, let us prove the lemma.

Proof of Lemma 7.5. To apply Theorem 7.9, we need to define a respectful function $k: \mathcal{C}^n \times \mathcal{T}_2^n \rightarrow \mathcal{T}_1^n$. To this end consider a cone $C \in \mathcal{C}^n$ and a subspace $L \in \mathcal{T}_2^n$. First, if $\int_C f dx \neq \int_C g dx$, let $k(C, L)$ be any hyperplane containing L ; this case will not occur for us.

Hence, assume $\int_C f dx = \int_C g dx$. Fix a hyperplane $H_0 \in \mathcal{T}_1^n$ so that $L \subset H_0$. Note that all such hyperplanes can be obtained from H_0 by a rotation in the two dimensional subspace orthogonal to L . Hence, let H_θ be the hyperplane obtained by rotating by θ angle in that orthogonal plane, so that e.g. $H_\pi = H_0$. Consider the induced H_0^+ and H_0^- and their rotated counterparts, e.g. $H_\pi^+ = H_0^-$.

Claim 7.10. *There exists a $\theta \in [0, \pi]$ so that $\int_{C \cap H_\theta^+} f dx = \int_{C \cap H_\theta^+} g dx$ and $\int_{C \cap H_\theta^-} f dx = \int_{C \cap H_\theta^-} g dx$*

Proof of claim. This follows easily by continuity of each of these integrals in θ . Indeed, consider the discrepancy function $D: \mathbb{R} \rightarrow \mathbb{R}$ defined by $D(\theta) := \int_{C \cap H_\theta^+} f dx - \int_{C \cap H_\theta^+} g dx$, which is periodic and continuous. Note that

$$\begin{aligned} D(\pi) &= \int_{C \cap H_\pi^+} f dx - \int_{C \cap H_\pi^+} g dx \\ &= \int_{C \cap H_0^-} f dx - \int_{C \cap H_0^-} g dx \\ &= \left(\int_C f dx - \int_{C \cap H_0^+} f dx \right) - \left(\int_C g dx - \int_{C \cap H_0^+} g dx \right) \\ &= \int_{C \cap H_0^+} g dx - \int_{C \cap H_0^+} f dx = -D(0). \end{aligned}$$

Hence, by continuity there exists some $\theta \in [0, \pi]$ with $D(\theta) = 0$, i.e., $\int_{C \cap H_\theta^+} f dx = \int_{C \cap H_\theta^+} g dx$. This immediately implies that also $\int_{C \cap H_\theta^-} f dx = \int_{C \cap H_\theta^-} g dx$. \square

For every pair (C, L) , let $k(C, L)$ be the hyperplane H_θ given by this claim.

With this respectful function defined, we can apply Theorem 7.9 with $\eta = \eta_{f,g,\epsilon}$ chosen sufficiently small depending on f, g and ϵ , to find valid partition $\mathcal{F}' = \mathcal{F}'_0 \sqcup \mathcal{F}'_1 \sqcup \mathcal{F}'_2$ and for every $C \in \mathcal{F}'_2$ let $C' \subset C$ be the subcone as specified by the theorem. From this partition, we create partition $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ as follows. Let $\mathcal{F}_2 := \mathcal{F}'_1 \sqcup \{C' : C \in \mathcal{F}'_2\}$ and let $\mathcal{F}_1 := \mathcal{F}'_0 \sqcup \{C \setminus C' : C \in \mathcal{F}'_2\}$.

To check the first condition, consider the contribution from \mathcal{F}'_0 . Since f and g have bounded support, there exists an $R = R_{f,g}$, so that $\text{supp}(f), \text{supp}(g) \subset RS_n$, i.e., $\int_{RS_n} f + g dx = \int_{\mathbb{R}^n} f + g dx$. Hence, we find

$$\sum_{C \in \mathcal{F}'_0} \int_C f dx \leq \sum_{C \in \mathcal{F}'_0} |C \cap RS_n| \leq \eta R^n \leq \epsilon/2,$$

for η sufficiently small in R, n , and ϵ . For the contribution from $\{C \setminus C' : C \in \mathcal{F}'_2\}$, we use

$$\sum_{(C \setminus C') \in \{C \setminus C' : C \in \mathcal{F}'_2\}} |(C \setminus C') \cap S_n| \leq \sum_{C \in \mathcal{F}'_2} \eta |C \cap S_n| \leq \eta |S_n| < \epsilon/2,$$

as long as η is small in terms of n and ϵ . Hence, the first condition is satisfied.

The second condition is the most tricky one. For $C \in \mathcal{F}'_1$ we have $\int_C f dx = \int_C g dx$ so the condition is immediately satisfied. For the other cones we need the following crude observations. Since $f, g \leq 1$, for any cone $C \in \mathcal{F}'_2$, we have

$$\int_{C \setminus C'} f dx \leq |(C \setminus C') \cap RS_n| = \left(\frac{R}{r}\right)^n |(C \setminus C') \cap rS_n| \leq \frac{\eta}{1-\eta} \left(\frac{R}{r}\right)^n |C' \cap rS_n| \leq \frac{\epsilon}{2} \cdot 0.1 |C' \cap rS_n| \leq \frac{\epsilon}{2} \int_{C'} f dx,$$

where in the penultimate inequality we used that η is sufficiently small in terms of R (i.e., f, g) and ϵ . To conclude, we find that as $\int_C f dx = \int_C g dx$, we have

$$\left| \int_{C'} (f - g) dx \right| = \left| \int_{C \setminus C'} (f - g) dx \right| \leq \int_{C \setminus C'} f dx + \int_{C \setminus C'} g dx \leq \epsilon \int_{C'} f dx.$$

The third condition is immediate for the $C' \in \{C' : C \in \mathcal{F}'_2\}$ and also easy to verify for $C \in \mathcal{F}'_1$, since $C^{z,w} \subset C \cap (|z|\partial S_n) = |z|(C \cap \partial S_n)$, so $\text{diam}(C^{z,w}) \leq |z| \text{diam}(C \cap \partial S_n) \leq \epsilon|z|$. Hence, the lemma follows. \square

7.3 Reduction from Proposition 7.2 to Proposition 7.11

Proposition 7.11. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $r, \beta, \ell \in (0, \infty)$, there exists $d = d_{n,\lambda,p,r,\beta,\ell} > 0$ such that the following holds. Let $C \subset C_n^1$ be a cone. Let $f, g, h: C \rightarrow [0, 1]$ be continuous functions with bounded support such that*

- $\int_C f \, dx = \int_C g \, dx$,
- $\int_C f \, dx \leq \ell \int_{C \cap rS_n} f \, dx$,
- $|\{x \in rS_n \cap C : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma |C \cap rS_n|$,
- for all $x, y \in C$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f(x), g(y))$,
- for all $z, w \in \mathbb{R}$ and $x, y \in C^{z,w}$ we have $f(x) = f(y)$ and $g(x) = g(y)$, and
- $\int_C h \, dx = (1 + \delta) \int_C f \, dx$ for some $0 \leq \delta \leq d$

Then $\int_C |f - g| \, dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta + \gamma}) \int_C f \, dx$.

Proof that Proposition 7.11 implies Proposition 7.2. We construct f', g' to which we can apply Proposition 7.11 by shrinking these functions a tiny bit (proportional to ϵ). First, construct $f_1 \leq f$ and $g_1 \leq g$ by taking the infimum of f in every slice $C^{z,w}$

$$f_1(x) := \inf \{f(y) : \exists z, w \in \mathbb{R} : x, y \in C^{z,w}\}$$

and g_1 analogously. Since $f(x) - f_1(x) \leq \epsilon$, we find that,

$$\int_C f_1 \, dx \geq \int_C f \, dx - \epsilon |supp(f)| \geq \int_C f \, dx - \epsilon |sS_n| = \int_C f \, dx - O_{n,s}(\epsilon),$$

and analogously for g_1 . Combining these, we find

$$\left| \int_C (f_1 - g_1) \, dx \right| \leq \left| \int_C (f - g) \, dx \right| + \left| \int_C (g - g_1) \, dx \right| + \left| \int_C (f - f_1) \, dx \right| = O_{n,\ell,r}(\epsilon) + 2 \cdot O_{n,s}(\epsilon) = O_{n,\ell,r,s}(\epsilon).$$

Thus we can decrease one of the two functions while keeping the other the same to create $f' \leq f_1, g' \leq g_1$ with $\int_C f' \, dx = \int_C g' \, dx$ while maintaining all the properties required to apply Proposition 7.11 (e.g. only reduce all points in a $C^{z,w}$ simultaneously and only reduce outside of $rS_n/2$), we can do this with

$$\int_C (f_1 - f') \, dx + \int_C (g_1 - g') \, dx = \left| \int_C (f_1 - g_1) \, dx \right| = O_{n,\ell,r,s}(\epsilon).$$

Since, $f' \leq f$ and $g' \leq g$, we still have for all $x, y \in C$ that $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f'(x), g'(y))$. We can evaluate

$$\int_C h \, dx \leq (1 + \delta) \int_C f \, dx \leq \frac{1 + \delta}{1 - O_{n,\ell,r,s}(\epsilon)} \int_C f' \, dx \leq (1 + 2\delta) \int_C f' \, dx,$$

where the last inequality follows from choosing $e_{n,\lambda,p,r,s,\ell,\delta}$ sufficiently small. Since $e_{n,\lambda,p,r,s,\ell,\delta}$ is definitely smaller than β , we find that $|\{x \in rS_n : f'(x) < \beta/2 \text{ or } g'(x) < \beta/2\}| \leq |\{x \in rS_n : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma$.

Now we can apply Proposition 7.11 to functions f', g', h , and parameters n, λ, p, ℓ as in Proposition 7.2 and $\beta/2, r/2$ to give

$$\int_C |f' - g'| \, dx = O_{n,\lambda,p,\beta/2,r/2,\ell}((2\delta)^{1/2} + \gamma) \int_C f' \, dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta} + \gamma) \int_C f \, dx.$$

Bringing this back to f and g , we find

$$\int_C |f - g| \, dx \leq \int_C |f' - g'| \, dx + \int_C |f - f'| \, dx + \int_C |g - g'| \, dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta} + \gamma) \int_C f \, dx,$$

which concludes the proof of the reduction. □

7.4 Reduction from Proposition 7.11 to Theorem 6.1

Definition 7.12. Consider the three cones K_1, K_2, K_3 partitioning \mathbb{R}^2 ;

$$K_1 := \{(x, y) : x < 0, |y| \leq |x|\}, K_2 = \{(x, y) : y \geq \max\{0, -x\}\}, K_3 = \{(x, y) : y \leq \min\{0, x\}\},$$

and corresponding cones K_1^n, K_2^n, K_3^n partitioning \mathbb{R}^n :

$$K_i^n = K_i \times \mathbb{R}^{n-2}.$$

The following lemma shows that if partition f inside cone C somewhat evenly into parallel to K_i^n , then the corresponding partition of g almost coincides. The proof is very similar to the proof of Lemma 6.6.

Lemma 7.13. Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $r, \beta, \ell \in (0, \infty)$ where r is sufficiently small in terms of n , there exists $d = d_{n, \lambda, p, r, \beta, \ell} > 0$ such that for $0 \leq \delta < d$ the following holds. Let $w = (r/2, 0, \dots, 0)$. Let C be a cone such that $B(e_1, 1) \cap H_n^1 \subset C \cap H_n^1 \subset B(e_1, n) \cap H_n^1$. Let $f, g, h: C \rightarrow [0, 1]$ be continuous functions with bounded support such that

- $\int_C f dx = \int_C g dx$,
- $\int_C f dx \leq \ell \int_{C \cap r(H_n^1)^-} f dx$,
- $|\{x \in (H_n^1)^- \cap C : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma |C \cap (H_n^1)^-|$,
- for all $x, y \in C$ we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$,
- $\int_C h dx = (1 + \delta) \int_C f dx$

there exists $v \in B(o, O_{n, \lambda, p, r, \beta, \ell}(\sqrt{\delta + \gamma}))$ so that for all $i = 1, 2, 3$

- $\int_{w+K_i^n} f dx = \int_{w+v+K_i^n} g dx$, and
- $\int_{w+(1-\lambda)v+K_i^n} h dx \leq (1 + O_{n, \lambda, p, r, \beta, \ell}(\delta)) \int_{w+K_i^n} f dx$.

Recall $e_1 \perp H_n^1$ and actually $e_1 \in H_n^1$ where H_n^1 is the supporting hyperplane of the face F_n^1 of S_n .

Lemma 7.14. Given $n \in \mathbb{N}$ and a cone $C \subset C_n^1$, there exists a bijective linear map $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

- $\alpha(e_1) = e_1$,
- $\alpha(e_2) = \xi e_2$ for some $\xi \in \mathbb{R}^*$, and
- $B(e_1, 1) \cap H_n^1 \subset \alpha(C) \cap H_n^1 \subset B(e_1, n) \cap H_n^1$.

Note that for all $z, w \in \mathbb{R}$, $(\alpha C)^{z, \xi w} = \alpha(C^{z, w})$.

Lemma 7.15. Given $n \in \mathbb{N}$ and $r, \beta \in (0, 1)$, there exist $s_1, s_0 \in (0, \infty)$ such that the following holds. Let C be a cone such that $B(e_1, 1) \cap H_n^1 \subset C \cap H_n^1 \subset B(e_1, n) \cap H_n^1$. Let $f, g: C \rightarrow [0, 1]$ be continuous functions with bounded support such that

- $|\{x \in C \cap r(H_n^1)^- : f(x) < \beta \text{ or } g(x) < \beta\}| \leq \gamma |C \cap r(H_n^1)^-|$, and
- for all $(z, w) \in \mathbb{R}^2$, and all $x, y \in C^{z, w}$ we have $f(x) = f(y)$ and $g(x) = g(y)$.

Let $C' = \pi_{1,2}(C) = \{(z, w) \in \mathbb{R}^2 : C^{z, w} \neq \emptyset\}$ and $f', g', h': C' \rightarrow \mathbb{R}_{\geq 0}$ be defined by

$$f'(z, w) = f(x)|C^{z, w}|, \text{ and } g'(z, w) = g(x)|C^{z, w}|,$$

for some $x = x_{z, w} \in C^{z, w}$. Then for every $(z, w) \in B((r/2, 0), r/10n) \subset \mathbb{R}^2$ we have

$$|\{(z, w) \in B((r/2, 0), r/10n) : f'(z, w), g'(z, w) \in [s_0, s_1]\}| \geq (1 - O_{n, r, \beta}(\gamma)) |B((r/2, 0), r/10n)|.$$

Lemma 7.16. Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $q = \frac{p}{1+np}$ the following holds. Let $C \subset \mathbb{R}^n$ be a cone. Let $f, g, h: C \rightarrow [0, 1]$ be continuous functions with bounded support such that

- for all $x, y \in C$, we have $h(\lambda x + (1 - \lambda)y) \geq M_{\lambda, p}(f(x), g(y))$, and

- for all $z, w \in \mathbb{R}$ and $x, y \in C^{z,w}$ we have $f(x) = f(y)$ and $g(x) = g(y)$.

Let $C' = \pi_{1,2}(C) = \{(z, w) \in \mathbb{R}^2 : C^{z,w} \neq \emptyset\}$ be a cone in \mathbb{R}^2 . Let $f', g', h' : C' \rightarrow \mathbb{R}_{\geq 0}$ be defined by

$$f'(z, w) = f(x)|C^{z,w}|, \quad g'(z, w) = g(x)|C^{z,w}|, \quad \text{and} \quad h'(z, w) = h(x)|C^{z,w}|$$

for some $x = x_{z,w} \in C^{z,w}$. Then, for $x, y \in C'$

- $h'(\lambda x + (1 - \lambda)y) \geq M_{\lambda,q}(f'(x), g'(y))$,
- $\int_{C'} f' dx = \int_C f dx$,
- $\int_{C'} g' dx = \int_C g dx$,
- $\int_{C'} h' dx = \int_C h dx$, and
- $\int_{C'} |f' - g'| dx = \int_C |f - g| dx$.

Furthermore, for any vector $w' \in \mathbb{R}^2$ and corresponding vector $w = (w'_1, w'_2, 0, \dots, 0) \in \mathbb{R}^n$ and any $i = 1, 2, 3$, we have

- $\int_{w'+K_i} f' dx = \int_{w+K_i^n} f dx$,
- $\int_{w'+K_i} g' dx = \int_{w+K_i^n} g dx$, and
- $\int_{w'+K_i} h' dx = \int_{w+K_i^n} h dx$.

Lemma 7.17. Given $n \in \mathbb{N}^*$, $\lambda \in (0, 1/2]$, $p \in (-1/(n+2), 0)$ and $q = \frac{p}{1+pn} \in (-1/2, 0)$ the following holds. If $a, b, c, u, v \in (0, \infty)$ satisfy $b^{1/n} \geq \lambda a^{1/n} + (1 - \lambda)c^{1/n}$, then

$$b \left(\lambda u^p + (1 - \lambda)v^p \right)^{1/p} \geq \left(\lambda (au)^q + (1 - \lambda)(cv)^q \right)^{1/q},$$

i.e., $bM_{\lambda,p}(u, v) \geq M_{\lambda,q}(au, cv)$.

7.4.1 Proof of reduction

Proof that Theorem 6.1 implies Proposition 7.11. By Lemma 7.14, there exists a bijective linear map $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that if we denote $C_1 = \alpha(C)$ the cone in \mathbb{R}^n , then

1. $\alpha(e_1) = e_1$,
2. $\alpha(e_2) = \xi e_2$ for some $\xi \in \mathbb{R}^*$, and
3. $B(e_1, 1) \cap H_n^1 \subset C_1 \cap H_n^1 \subset B(e_1, n) \cap H_n^1$.
4. for all $z, w \in \mathbb{R}$, $(C_1)^{z,w} = \alpha(C^{z,\xi^{-1}w})$.

Let and let $f_1, g_1, h_1 : \mathbb{R}^n \rightarrow [0, 1]$ be continuous functions with bounded support defined by $f_1 = f \circ \alpha$, $g_1 = g \circ \alpha$ and $h_1 = h \circ \alpha$. As we shall explain below, it is easy to check that the following properties hold:

1. $\int_{C_1} f_1 dx = \int_{C_1} g_1 dx$,
2. $\int_{C_1} f_1 dx \leq \ell \int_{C_1 \cap r(H_n^1)^-} f_1 dx$,
3. for every $|\{x \in C_1 \cap r(H_n^1)^- : f_1(x) < \beta \text{ or } g_1(x) < \beta\}| \leq \gamma |C_1 \cap r(H_n^1)^-|$,
4. for all $x, y \in C_1$ we have $h_1(\lambda x + (1 - \lambda)y) \geq M_{\lambda,p}(f_1(x), g_1(y))$,
5. for all $z, w \in \mathbb{R}$ and $x, y \in C_1^{z,w}$ we have $f_1(x) = f_1(y)$ and $g_1(x) = g_1(y)$, and
6. $\int_{C_1} h_1 dx = (1 + \delta) \int_{C_1} f_1 dx$ for some $0 \leq \delta \leq d$,
7. $\int_{C_1} |f - g| dx / \int_{C_1} f dx = \int_{C_1} |f_1 - g_1| dx / \int_{C_1} f_1 dx$.

Indeed, the first point follows from the fact $\int_{C_1} f_1 dx = |\det(\alpha)| \int_C f dx$ and $\int_{C_1} g_1 dx = |\det(\alpha)| \int_C g dx$.

Recall H_n^1 is the supporting hyperplane of the face F_n^1 of the simplex S_n and e_1 is orthogonal to H_n^1 . Because $\alpha(e_1) = e_1$, it follows that $r(H_n^1) = \alpha(r(H_n^1))$ and $\alpha(C \cap rS_n) = \alpha(C \cap r(H_n^1)^-) = C_1 \cap r(H_n^1)^-$. Thus, the second point follows from the fact that $|\det(\alpha)| \int_{C \cap rS_n} f dx = \int_{C_1 \cap r(H_n^1)^-} f_1 dx$.

The third point follows from the fact that all $\alpha(C \cap rS_n) = C_1 \cap r(H_n^1)^-$.

The fourth point follows from the fact that all $x, y \in C_1$ satisfy $x = \alpha(z)$ and $y = \alpha(w)$ for some $z, w \in C$. Note $h_1(\lambda x + (1 - \lambda)y) = h_1(\lambda z + (1 - \lambda)w) \geq M_{\lambda,p}(f(z), g(w)) = M_{\lambda,p}(f_1(x), g_1(y))$.

The fifth point follows from the fact that for all $z, w \in \mathbb{R}$ and all $x, y \in C_1^{z,w}$ there exist $a, b \in C^{z, \xi^{-1}w}$ such that $x = \alpha(a)$ and $y = \alpha(b)$. Note $f_1(x) = f(\alpha(a)) = f(\alpha(b)) = f_1(y)$.

Finally, the last two points follow in a similar way to the first point.

Now, let $C_2 = \pi_{1,2}(C_1) = \{(z, w) \in \mathbb{R}^2 : C_1^{z,w} \neq \emptyset\}$ be a cone in \mathbb{R}^2 . Let $f_2, g_2, h_2 : C_2 \rightarrow \mathbb{R}_{\geq 0}$ be defined by

$$f_2(z, w) = f_1(x)|C_1^{z,w}|, \quad g_2(z, w) = g_1(x)|C_1^{z,w}| \quad \text{and} \quad h_2(z, w) = h_1(x)|C_1^{z,w}|$$

for some $x = x_{z,w} \in C^{z,w}$. Note that

$$\frac{\int |f_2 - g_2| dx}{\int f_2 dx} = \frac{\int |f_1 - g_1| dx}{\int f_1 dx} = \frac{\int |f - g| dx}{\int f dx}.$$

By Lemma 7.15 combined with the third and fifth point, there exist $s_0 = s_0(n, r, \beta) > 0$ and $s_1 = s_1(n, r, \beta) > 0$ such that

$$|\{(z, w) \in B((r/2, 0), r/10n) : f_2(z, w), g_2(z, w) \in [s_0, s_1]\}| \geq (1 - O_{n,r,\beta}(\gamma))|B((r/2, 0), r/10n)|.$$

Let $w_0 = (r/2, 0, \dots, 0)$ as in Lemma 7.13. By Lemma 7.13 combined with Lemma 7.16 there exists $v \in B(o, O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta} + \gamma)) \cap (\mathbb{R}^2 \times \{0\}^{n-2})$ so that for all $i = 1, 2, 3$, we have

- $\int_{w_0 + K_i^n} f_1 dx = \int_{w_0 + v + K_i^n} g_1 dx$, and
- $\int_{w_0 + (1-\lambda)v + K_i^n} h_1 dx \leq (1 + O_{n,\lambda,p,r,\beta,\ell}(\delta)) \int_{w_0 + K_i^n} f_1 dx$.

In a slight abuse of notation, since w_0 and v are both in $\mathbb{R}^2 \times \{0\}^{n-2}$, we also use w_0 for $(r/2, 0)$ and v for the projection of v onto its first two coordinates. Hence, let f_2^i, g_2^i, h_2^i be defined by

$$f_2^i = f_2 \cdot \mathbf{1}_{w_0 + K_i}, \quad g_2^i = g_2 \cdot \mathbf{1}_{w_0 + v + K_i}, \quad h_2^i = h_2 \cdot \mathbf{1}_{w_0 + (1-\lambda)v + K_i}.$$

By Theorem 6.1, we find that there exist translations v_i (for $i = 1, 2, 3$) so that

$$\int_{\mathbb{R}^2} |f_2^i(x) - g_2^i(x + v_i)| dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta} + \gamma).$$

We'll show that the same is true without the translation v_i , by showing that v_i is not much different from $-v$. For simplicity, consider K_1 and v_1 the others will follow analogously. Consider the translation $v_1 + v$. Since the defining lines $e_2 - e_1$ and $-e_2 - e_1$ of the cone K_1 are at a right angle, one of the two makes a big angle with $v_1 + v$, so that

$$|(w_0 + K_1 \Delta (w_0 + v_1 + v + K_1)) \cap B(w_0, r/10n)| \geq \Omega_{n,r}(\|v_1 + v\|).$$

Since in this region the functions are large (except for in the small set $|\{(z, w) \in B((r/2, 0), r/10n) : f_2(z, w), g_2(z, w) < s_0\}| \leq O_{n,\lambda,p,r,\beta}(\gamma)$), this gives a lower bound on the previously established symmetric difference;

$$\begin{aligned} & s_0 \cdot \left(|(w_0 + K_1 \Delta (w_0 + v_1 + v + K_1)) \cap B(w_0, r/10n)| - O_{n,\lambda,p,r,\beta}(\gamma) \right) \\ & \leq \int_{(w_0 + K_1 \Delta (w_0 + v_1 + v + K_1)) \cap B(w_0, r/10n)} |f_2^1(x) - g_2^1(x + v_1)| dx \\ & \leq \int_{\mathbb{R}^2} |f_2^1(x) - g_2^1(x + v_1)| dx, \end{aligned}$$

so that recalling that $s_0 = \Omega_{n,r}(1)$, we have

$$\|v_1\| \leq \|v\| + \|v_1 + v\| = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta} + \gamma).$$

Since this translate is small, we find by Lemma 4.9 that shifting the function by this amount doesn't change the symmetric difference much, i.e.

$$\int_{\mathbb{R}^2} |f_2^i(x) - g_2^i(x)| dx = O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta + \gamma}).$$

Combining these, we conclude

$$\frac{\int_{\mathbb{R}^n} |f - g| dx}{\int_{\mathbb{R}^n} f dx} = \frac{\int_{\mathbb{R}^2} |f_2 - g_2| dx}{\int_{\mathbb{R}^2} f_2 dx} \leq \frac{\sum_{i=1}^3 \int_{\mathbb{R}^2} |f_2^i - g_2^i| dx}{\int_{\mathbb{R}^2} f_2 dx} = O_{n,\lambda,p,r,\beta,\ell,s_0,s_1}(\sqrt{\delta + \gamma}).$$

The conclusion follows. \square

7.4.2 Proof of Lemmas

Proof of Lemma 7.13. By Lemma 4.4, there exists a v so that for $i = 1, 2, 3$, we have

$$\int_{w+K_i^n} f dx = \int_{w+v+K_i^n} g dx.$$

Our task will be to show that $v \in B(o, O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta + \gamma}))$.

First note that we have

$$\int_{C \cap (w+(1-\lambda)vK_1)} h dx - \int_{C \cap (w+K_1)} f dx \leq \delta \int_C f dx \leq O_{n,\ell}(\delta) \int_{C \cap K_1} f dx,$$

where in the last step we used that $C \cap (w+K_1) \subset C \cap (H_n^1)^-$ and $|C \cap (w+K_1)| = \Omega_n(|C \cap (H_n^1)^-|)$. Hence, by Lemma 4.8, we find that

$$|(C \cap w+K_1) \Delta (C \cap w+v+K_1)| = |(supp(f|_{C \cap (w+K_1)}) \Delta (supp(g|_{C \cap (w+v+K_1)}))| = O_{n,p,\lambda,\beta,\ell}(\sqrt{\delta + \gamma}),$$

which immediately implies that $v \in B(o, O_{n,\lambda,p,r,\beta,\ell}(\sqrt{\delta + \gamma}))$. \square

Proof of Lemma 7.14. Consider the convex set $K := H_n^1 \cap C$ and let $K' := K - e_1 \subset \{0\} \times \mathbb{R}^{n-1}$. By John's theorem there exists an ellipsoid $E \subset \{0\} \times \mathbb{R}^{n-1}$ so that $E \subset K' \subset (n-1)E$. There exists a bijective linear map $\zeta: \{0\} \times \mathbb{R}^{n-1} \rightarrow \{0\} \times \mathbb{R}^{n-1}$ so that $\zeta(E) = \{0\} \times B^{n-1}(o, 1)$, and thus $\zeta(K') \subset \zeta((n-1)E) = B(o, n-1)$. Now consider $\zeta(e_2)$ and let $\gamma: \{0\} \times \mathbb{R}^{n-1} \rightarrow \{0\} \times \mathbb{R}^{n-1}$ be the rotation (a bijective linear map) bringing $\zeta(e_2)$ parallel to e_2 . Let $\alpha': \{0\} \times \mathbb{R}^{n-1} \rightarrow \{0\} \times \mathbb{R}^{n-1}$ be the bijective linear map $\alpha' = \gamma \circ \zeta$. Finally let

$$\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^n; (x_1, \dots, x_n) \mapsto (x_1, \alpha'(x_2, \dots, x_n)),$$

be the bijective linear map applying α' to the last $n-1$ coordinates. This map clearly satisfies the conditions in the lemma. \square

Proof of Lemma 7.15. First note that $B((r/2, 0), r/10n) \subset C \cap r(H_n^1)^-$ and conversely $|B((r/2, 0), r/10n)| \geq \Omega_{n,r}(|C \cap r(H_n^1)^-|)$.

Consider $(z, w) \in B((r/2, 0), r/10n)$ so that $f(x), g(x) \geq \beta$ for all $x \in C^{z,w}$, so all we need to do is control $|C^{z,w}|$. For the upper bound, simply use that since $(z, w) \in B((r/2, 0), r/10n)$ is on the same side of the hyperplane H_n^1 as the origin, we have $C^{z,w} \subset \text{co}((B(e_1, n) \cap H_n^1) \cup \{o\})$. Since this last set only depends on n , we immediately get $|C^{z,w}| = O_n(1)$.

For the lower bound, we similarly use that $\text{co}((B(e_1, 1) \cap H_n^1) \cup \{o\}) \subset C$. By convexity, this implies for $z \in r/2 + [-r/10n, r/10n]$, we have

$$B(ze_1, z) \cap zH_n^1 \subset C \cap d\{z\} \times \mathbb{R}^{n-1},$$

so that we have

$$|C^{z,w}| \geq |B(ze_1, z) \cap (z, w) \times \mathbb{R}^{n-2}| \geq \left(\sqrt{z^2 - w^2}\right)^{n-2} b_{n-2} \geq \left(\sqrt{(r/2 - r/10n)^2 - (r/10n)^2}\right)^{n-2} b_{n-2} = \Omega_{n,r}(1),$$

where b_{n-2} is the volume of the $n-2$ dimensional ball of radius 1. Hence, choices $s_0 = \Omega_{n,r,\beta}(1)$ and $s_1 = \Omega_{n,r,\beta}(1)$ as in the lemma exist. \square

Proof of Lemma 7.16. The equalities for the integrals follow from the simple observation that if we fix for every $z, w \in \mathbb{R}$, some point $x_{z,w}$ in $C^{z,w}$, then

$$\int_{C'} f'(z, w) d(z, w) = \int_{C'} f(x_{z,w}) |C^{z,w}| d(z, w) = \int_{C'} \int_{C^{z,w}} f(x) dx d(z, w) = \int_C f(x) dx.$$

Indeed the equalities for g and h follow analogously. By the definitions, we have that $|f'(z, w) - g'(z, w)| = |f(x_{z,w}) - g(x_{z,w})| \cdot |C^{z,w}|$, so that integral follows analogously as well. As do the integrals involving $R_r^{f_1, f_2}$ with the note that $R_r^{f_1, f_2} = \pi_{1,2}(R_r^{f_1, f_2})$, where the former is a subset of \mathbb{R}^2 and the latter of \mathbb{R}^n .

Finally, we turn our attention to the lower bound on h' . Consider two points $(z, w), (z', w') \in C'$. The cone C is a convex set, so that $C^{\lambda(z,w)+(1-\lambda)(z',w')} \supset \lambda C^{(z,w)} + (1-\lambda)C^{(z',w')}$ and thus by the Brunn-Minkowski inequality, we have

$$\left| C^{\lambda(z,w)+(1-\lambda)(z',w')} \right|^{1/n} \geq \lambda |C^{z,w}|^{1/n} + (1-\lambda) |C^{z',w'}|^{1/n}.$$

Hence, we can apply Lemma 7.17 with $b = \left| C^{\lambda(z,w)+(1-\lambda)(z',w')} \right|$, $a = |C^{z,w}|$, $c = |C^{z',w'}|$, $u = f(x_{z,w})$, and $v = g(x_{z',w'})$, to find that

$$\begin{aligned} h'(\lambda(z, w) + (1-\lambda)(z', w')) &= \left| C^{\lambda(z,w)+(1-\lambda)(z',w')} \right| \cdot h(\lambda(z, w) + (1-\lambda)(z', w')) \\ &\geq \left| C^{\lambda(z,w)+(1-\lambda)(z',w')} \right| \cdot M_{\lambda,p}(f(x_{z,w}), g(x_{z',w'})) \\ &\geq M_{\lambda,q} \left(|C^{z,w}| \cdot f(x_{z,w}), |C^{z',w'}| \cdot g(x_{z',w'}) \right) \\ &= M_{\lambda,q}(f'(z, w), g'(z', w')). \end{aligned}$$

This concludes the proof of the lemma. □

Proof of Lemma 7.17. We need to show that

$$\left(\lambda a^{1/n} + (1-\lambda)c^{1/n} \right)^n \left(\lambda u^p + (1-\lambda)v^p \right)^{1/p} \geq \left(\lambda (au)^q + (1-\lambda)(cv)^q \right)^{1/q}.$$

After rearranging, this is equivalent to

$$\left(\lambda a^{1/n} + (1-\lambda)c^{1/n} \right)^n \left(\lambda (au)^q + (1-\lambda)(cv)^q \right)^{-1/q} \geq \left(\lambda u^p + (1-\lambda)v^p \right)^{-1/p}.$$

After more rearranging, this is equivalent to

$$\left(\lambda a^{1/n} + (1-\lambda)c^{1/n} \right)^{-pn} \left(\lambda (au)^q + (1-\lambda)(cv)^q \right)^{p/q} \geq \lambda u^p + (1-\lambda)v^p$$

also to

$$\left(\lambda (a^{-p})^{-1/pn} + (1-\lambda)(c^{-p})^{-1/pn} \right)^{-pn} \left(\lambda ((au)^p)^{q/p} + (1-\lambda)((cv)^p)^{q/p} \right)^{p/q} \geq \lambda u^p + (1-\lambda)v^p.$$

Note that p/q and $-pn$ are positive numbers with $-pn + p/q = -pn + 1 + pn = 1$. By Hölder's inequality we get

$$\left(\lambda (a^{-p})^{-1/pn} + (1-\lambda)(c^{-p})^{-1/pn} \right)^{-pn} \left(\lambda ((au)^p)^{q/p} + (1-\lambda)((cv)^p)^{q/p} \right)^{p/q} \geq \lambda a^{-p} (au)^p + (1-\lambda)c^{-p} (cv)^p.$$

The conclusion now follows. □

8 Theorem 1.10: structure of the proof

To prove Theorem 1.10, the main difficulty is finding the correct p -concave function. To that end consider a maximizer $f' : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ of the following quantity:

$$\int (M_{\lambda,p}^*(f, f) - M_{\lambda,p}^*(f', f')) dx - (1+c) \int (f - f') dx, \quad (8.1)$$

where $c = c_{n,\lambda,p} > 0$ is some small constant and with the constraint that $f' \leq f$ pointwise. Intuitively, this can be seen as removing any parts of f whose removal decreases $M_{\lambda,p}^*(f, f)$ more than f .

The bulk of the work is to show that if we let ℓ be the minimal p -concave function with $\ell \geq f'$ pointwise (i.e., the p -concave hull of f'), then $\int (\ell - f') dx = O_{n,\lambda,p}(\delta) \int f dx$. Indeed, note that we can rewrite the above expression as

$$\begin{aligned} \int [(M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f')] dx - c \int (f - f') dx &\leq \int (M_{\lambda,p}^*(f, f) - f) dx - c \int (f - f') dx \\ &\leq \delta \int f dx - c \int (f - f') dx. \end{aligned}$$

Since we maximize the left-hand side and we can always choose $f' = f$ as a competitor, it follows that the right-hand side is at least zero. In particular

$$\int (f - f') dx \leq c^{-1} \delta \int f dx.$$

Hence, it suffice to show that $\int (\ell - f') dx = O_{n,\lambda,p}(\delta) \int f dx$.

To use that f' is a maximizer of 8.1, we use that f' is of the form $f' = \min\{f, \ell\}$. For any p -concave function ℓ' (in particular small alterations of ℓ), the induced function $f'' = \min\{f, \ell'\}$ is in some sense worse than f' . To create an interesting ℓ' to compare to, we take the following approach to “shaving off” near the faces (slightly moving one of the faces of ℓ).

1. Consider a non-degenerate (i.e., full-dimensional) face F of ℓ , i.e., a subset of the support of ℓ on which ℓ^p is linear. Then, $\ell|_F = L|_F$ for some linear function L^p on \mathbb{R}^n .
2. The fact that f' is a maximizer of Equation (8.1) implies the minimality of ℓ in the following sense: If we consider $f'' := \min\{f', L - \epsilon\}$ (or equivalently $\ell' = \min\{\ell, L - \epsilon\}$ and $f'' = \min\{f, \ell'\}$) for some very small $\epsilon > 0$ (i.e., f'' is obtained from f' by “shaving off” the points near the face), then

$$\int (M_{\lambda,p}^*(f', f') - M_{\lambda,p}^*(f'', f'')) dx \leq (1+c) \int (f' - f'') dx.$$

3. Let $X := \{x \in F : f'(x) = \ell(x)\} \subset \mathbb{R}^n$ and note that (assuming that f is well-behaved)

$$\int (f' - f'') dx = \epsilon(1 + o_\epsilon(1))|X|,$$

as $\epsilon \rightarrow 0$ (where $o_\epsilon(1) \rightarrow 0$ as $\epsilon \rightarrow 0$).

4. On the other hand, a closer look shows that

$$\int (M_{\lambda,p}^*(f', f') - M_{\lambda,p}^*(f'', f'')) dx = \epsilon(1 + o(1))|\lambda X + (1 - \lambda)X|.$$

5. Combining these, we find that

$$|\lambda X + (1 - \lambda)X| \leq (1 + 2c)|X|,$$

with c sufficiently small in terms of n, λ, p , so we can apply the stability of the Brunn-Minkowski inequality to deduce⁷

$$|\text{co}(X) \setminus X| \ll_{n,\lambda,p} |X|.$$

In addition, one can note that $\text{co}(X) = F$.

⁷Here and in the sequel, given positive constants a and b , by $a \ll_{n,\lambda,p} b$ we mean that $a \leq cb$ for some constant $c > 0$ that depends on n, λ, p and that can be made arbitrary small.

Given the steps above, we first establish the result in the case when L varies by at most a constant factor (say 100) in the face F . To this end, we use the stability of the Brunn-Minkowski inequality in each of the level sets combined with the information that $|\text{co}(X) \setminus X| \ll_{n,\lambda,p} |X|$. We will use a refined version of the above proof to reduce to this case. Indeed we find a triangulation of the support of f' into simplices in which ℓ^p is essentially linear, ℓ varies by at most a factor 100, and with vertices v on which $f'(v) = \ell(v)$. We achieve this as follows.

1. Consider a one-dimensional face $F = [0, 1]$ (i.e., an edge) of ℓ , so that by an argument akin to the one above we have $|\text{co}(X) \setminus X| \ll_{n,\lambda,p} |X|$, where $\text{co}(X)$ is the entire interval $[0, 1]$.
2. Consider a one-dimensional face $F = [0, 1] \times \{0\}^{n-1}$ (an edge) of ℓ , i.e. so that there exists a function $L \geq \ell$ so that L^p is linear on \mathbb{R}^n , $L = \ell$ on F , $L > \ell$ outside F , and $L = \ell = f'$ on $\{0, 1\} \times \{0\}^{n-1}$ the endpoints of F .
3. By an argument akin to the one above, we have $|\text{co}(X) \setminus X| \ll_{n,\lambda,p} |X|$ (in the 1-dimensional Hausdorff measure), where $\text{co}(X) = F$.
4. Our analysis will look at arbitrarily small shavings so that we may restrict our attention to what happens in this (one-dimensional!) face.
5. For a contradiction, assume there is some big gap in X , i.e., there are $a, b \in [0, 1]$ so that $f'(a \times \{0\}^{n-1}) > 100 f'(b \times \{0\}^{n-1})$ and $X \cap ([a, b] \times \{0\}^{n-1}) = \{a, b\} \times \{0\}^{n-1}$.
6. Rather than pushing down L everywhere, we shave by rotating L around a as per the following definition: Consider the function L_ϵ so that

- $L_\epsilon = L = \ell$ on $(-\infty, a] \times \mathbb{R}^{n-1}$,
- L_ϵ^p is linear on $[a, \infty) \times \mathbb{R}^{n-1}$,
- $L_\epsilon(b) = \ell - \epsilon$.

Note that $L_\epsilon \rightarrow L$ pointwise as $\epsilon \rightarrow 0$.

7. Similarly to before, consider the function $f'' = \min\{f', L_\epsilon\}$ (or equivalently $\ell' = \min\{\ell, L_\epsilon\}$ and $f'' = \min\{f, \ell'\}$), and use the minimality of f' to find that we cannot remove much of $M_{\lambda,p}^*(f', f')$ compared to f' .
8. Note that the removed mass $\int (f' - f'') dx$ all lies in $[b, 1] \times \mathbb{R}^{n-1}$, where $\ell - L$ is small, while the mass of

$$\int (M_{\lambda,p}^*(f', f') - M_{\lambda,p}^*(f'', f'')) dx$$

also lies in $[a, b] \times \mathbb{R}^{n-1}$, where $\ell - L$ is (relatively) large. A careful inspection of this discrepancy leads to a contradiction.

Hence, for every edge F of ℓ we find that the set $\ell(X \cap F)$ does not leave big gaps in $\ell(F)$. We can thus find a triangulation of the domain using elements of X on the edges so that in each of the simplices, the ℓ varies by at most a factor 100. Using the previously described approach, this allows us to conclude.

9 Proof of Theorem 1.10

9.1 Reduction from Theorem 1.10 to Proposition 9.1

Proposition 9.1. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ there exists $d = d_{n,\lambda,p} > 0$ such that the following holds. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a continuous function with bounded support such that*

- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f, f) dx \leq (1 + \delta) \int f dx$ for some $0 \leq \delta \leq d$.

Then there exists a p -concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

- $\int_{\mathbb{R}^n} |f - \ell| dx = O_{n,\lambda,p}(\delta) \int_{\mathbb{R}^n} f dx$.

Proof Proposition 9.1 implies Theorem 1.10. We can find a continuous functions with bounded support $f': \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that $\int_{\mathbb{R}^n} |f - f'| dx \leq \delta \int_{\mathbb{R}^n} f dx$ and $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f', f') - M_{\lambda,p}^*(f, f)) dx \leq \delta \int_{\mathbb{R}^n} f dx$. It follows that $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f', f') dx \leq (1 + 5\delta) \int_{\mathbb{R}^n} f' dx$. By Proposition 9.1, we get that there exists a p -concave function $\ell: \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $\int_{\mathbb{R}^n} |f' - \ell| dx = O_{n,\lambda,p}(5\delta) \int_{\mathbb{R}^n} f' dx$. We conclude $\int_{\mathbb{R}^n} |f - \ell| dx = O_{n,\lambda,p}(\delta) \int_{\mathbb{R}^n} f dx$. \square

9.2 Reduction from Proposition 9.1 to Proposition 9.9, 9.10, 9.11

Before we can state our results we collect some definitions in the subsection below.

9.2.1 Definitions

For the rest of this section we use the following definitions.

Definition 9.2. Given a compact convex set (possibly degenerate) $K \subset \mathbb{R}^n$ we denote by $\text{int}(K)$ the relative interior of K i.e. letting H be the affine subspace spanned by K , we define $\text{int}(K) = \{x \in K : \exists r > 0, B(x, r) \cap H \subset K\}$.

Definition 9.3. Given a compact convex set (possibly degenerate) $K \subset \mathbb{R}^n$, we say that $v \in K$ is a vertex of K if whenever $v = tx + (1-t)y$ for some $t \in (0, 1)$ and $x, y \in K$, we have $v = x = y$.

Definition 9.4. Given two points $a, b \in \mathbb{R}^n$, we denote by $[a, b] := \{ta + (1-t)b : t \in [0, 1]\}$ the line segment from a to b .

Definition 9.5. Given a compact convex set (possibly degenerate) $K \subset \mathbb{R}^n$, we say that $[a, b] \subset K$ is an edge of K if whenever $c \in [a, b]$, $t \in (0, 1)$ and $d, e \in K$ are such that $c = td + (1-t)e$, we have $d, e \in [a, b]$.

Definition 9.6. Given $n \in \mathbb{N}$, $p \in (-1/n, 0)$, $y \in \mathbb{R}^n$, and $d \in \mathbb{R}$, a p -plane with parameters y and d is a function $h_{y,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined by

$$h_{y,d}(x) = \begin{cases} 0 & \text{if } p > 0 \text{ and } \langle x, y \rangle + d \leq 0 \\ \infty & \text{if } p < 0 \text{ and } \langle x, y \rangle + d \leq 0 \\ (\langle x, y \rangle + d)^{1/p} & \text{if } p \neq 0 \text{ and } \langle x, y \rangle + d > 0 \\ \exp(\langle x, y \rangle + d) & \text{if } p = 0 \end{cases}.$$

Remark 9.7. A p -plane $h_{y,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is a continuous p -concave function. Moreover, if we denote by W the $(n-1)$ -dimensional subspace that is the orthogonal complement of y , then for any $x \in \mathbb{R}^n$, we have $h_{y,d}(x) = h_{y,d}(x + W)$ i.e. $h_{y,d}$ is constant on $x + W$.

Definition 9.8. Let $n \in \mathbb{N}$ and $p \in (-1/n, \infty)$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a continuous p -concave function with bounded support. We say that a compact convex set $F \subset \mathbb{R}^n$ with $\text{int}(F) \subset \text{supp}(g)$ is a p -face of g if there exists a p -plane $h: \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $h(x) = g(x)$ for all $x \in F$ and $h(x) > g(x)$ for all $x \in \text{supp}(g) \setminus F$. In this case, we say h is a tangent p -plane.

9.2.2 Main results

We are now ready to state the main results in this subsection.

Proposition 9.9. Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $m \in (1, \infty)$ there exists $d = d_{n,\lambda,p,m} > 0$ such that the following holds. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a continuous function with bounded support such that

- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f, f) dx \leq (1 + \delta) \int f dx$ for some $0 \leq \delta \leq d$
- for any $x \in \text{co}(\text{supp}(f))$, there exists a p -face $F \subset \overline{\text{co}(\text{supp}(f))}$ of $\text{co}_p(f)$ and a simplex $T \subset F$ (not necessarily full dimensional) such that $x \in \text{int}(T)$ and for any $y \in V(T)$ we have $f(y) = \text{co}_p(f)(y)$ and $\text{co}_p(f)(x)/\text{co}_p(f)(y) \in [1/m, m]$.

Then

- $\int_{\mathbb{R}^n} (\text{co}_p(f) - f) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$.

Proposition 9.10. Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $c \in (0, 1)$ there exists $d = d_{n,\lambda,p,c} > 0$ such that the following holds. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be continuous functions with bounded support such that

- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f, f) dx = (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $0 \leq \delta \leq d$.

There exists a p -concave function $\ell': \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $f': \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f' = \min(f, \ell')$ satisfies

- $\int_{\mathbb{R}^n} \left[-c(f - f') + (M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f') \right] dx \geq 0$.

Moreover,

- $\int |f - f'| dx \leq O_c(\delta) \int f dx$
- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f', f') dx = (1 + 3\delta) \int_{\mathbb{R}^n} f' dx$.

Furthermore, for any p -concave function $\ell'' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, the function $f'' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f'' = \min(f', \ell'')$ satisfies

- either $f'' = f'$
- or $\int_{\mathbb{R}^n} \left[-c(f' - f'') + (M_{\lambda,p}^*(f', f') - f') - (M_{\lambda,p}^*(f'', f'') - f'') \right] dx < 0$.

Proposition 9.11. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, there exists $c = c_{n,\lambda,p} > 0$ and $m = m_{n,\lambda,p} > 0$ such that the following holds. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a continuous functions with bounded support such that for any p -concave function $\ell' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, the function $f' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f' = \min(f, \ell')$ satisfies*

- either $f' = f$,
- or $\int_{\mathbb{R}^n} \left[-c(f - f') + (M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f') \right] dx < 0$.

Then for any p -face F of $\text{co}_p(f)$ and any edge $[p, q] \subset F$, with $\text{co}_p(f)(p) > \text{co}_p(f)(q)$ there exists a partition of the segment $[p, q] = [x_1, x_2] \sqcup [x_2, x_3] \sqcup \dots$ such that $f(x_i) = \text{co}_p(f)(x_i)$ and $f(x_i)/f(x_{i+1}) \in [1/m, m]$ for all i .

Proof Proposition 9.9, 9.10, 9.11 imply Proposition 9.1. Let f be a function satisfying the hypothesis of Proposition 9.1. Fix c, m as in Proposition 9.11. By Proposition 9.10, there exists a p -concave function $\ell' : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $f' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f' = \min(f, \ell')$ satisfies

- $\int |f - f'| dx \leq O_c(\delta) \int f dx$
- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f', f') dx = (1 + 3\delta) \int_{\mathbb{R}^n} f' dx$.

Furthermore, for any p -concave function $\ell'' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, the function $f'' : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f'' = \min(f', \ell'')$ satisfies

- either $f'' = f'$
- or $\int_{\mathbb{R}^n} \left[-c(f' - f'') + (M_{\lambda,p}^*(f', f') - f') - (M_{\lambda,p}^*(f'', f'') - f'') \right] dx < 0$.

By Proposition 9.11, for any p -face F of $\text{co}_p(f')$ and any edge $[p, q] \subset F$, with $\text{co}_p(f')(p) > \text{co}_p(f')(q)$ there exists a partition of the segment $[p, q] = [x_1, x_2] \sqcup [x_2, x_3] \sqcup \dots$ such that $f'(x_i) = \text{co}_p(f')(x_i)$ and $f'(x_i)/f'(x_{i+1}) \in [1/m, m]$ for all i .

This immediately implies for any $x \in \text{co}(\text{supp}(f'))$, there exists a face $F \subset \overline{\text{co}(\text{supp}(f'))}$ of $\text{co}_p(f')$ and a simplex $T \subset F$ (not necessarily full dimensional) such that $x \in \text{int}(T)$ and for any $y \in V(T)$ we have $f'(y) = \text{co}_p(f')(y)$ and $\text{co}_p(f')(x)/\text{co}_p(f')(y) \in [1/m, m]$.

By Proposition 9.9 combined with the second bullet point, we deduce that

$$\int_{\mathbb{R}^n} (\text{co}_p(f') - f') dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f' dx$$

By the first bullet point, we conclude that

$$\int_{\mathbb{R}^n} (|\text{co}_p(f') - f|) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$$

□

9.3 Reduction from Proposition 9.9 to Proposition 9.12

Proposition 9.12. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$, $m \in (1, \infty)$ and $k \in (1, \infty)$ there exist $d = d_{n,\lambda,p,m,k} > 0$ and $s = s_{n,\lambda,p,m,k} \geq 2k$ such that the following holds. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function with bounded support such that*

- f is continuous on $\overline{\text{co}(\text{supp}(f))}$
- there exists a compact set $Y \subset \mathbb{R}^n$ such that $\text{co}_p(f) = \text{co}_p(1_Y \times f)$ and for $y \in Y$ we have $f(y) \in [1/m, m]$

- $\int_{\mathbb{R}^n} f dx > k^{-1} \int_{\mathbb{R}^n} \text{co}_p(f) dx$
- letting $S = \{x: f(x) \geq 1/ms\}$, we have $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f \times 1_S, f \times 1_S) dx \leq (1 + \delta) \int_{\mathbb{R}^n} (f \times 1_{\lambda S + (1-\lambda)S}) dx$ for some $0 \leq \delta \leq d$.

Then

- $\int_{\mathbb{R}^n} (\text{co}_p(f) - f) dx = O_{n,\lambda,p,k,m}(\delta) \int_{\mathbb{R}^n} f dx \leq \int_{\mathbb{R}^n} f dx$.

We need the following lemma.

Lemma 9.13. *Let $n \in \mathbb{N}$, $p \in (-1/n, \infty)$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a p -concave function. Assume without loss of generality that $g(o) = 1$ and $g(x) \leq 1$ for every $x \in \mathbb{R}^n$. Let $S = \{x: g(x) \geq 1/2\}$. Then $\int_S g dx = \Theta_{n,p}(1) \int_{\mathbb{R}^n} g dx$.*

Proof of Lemma 9.13. Note that S is convex and contains o . Because g is p -concave, it follows that for $\lambda \geq 1$ and $x \in \partial(\lambda S)$ (so that $\lambda^{-1}x \in \partial(S)$), we have $g(\lambda^{-1}x) \geq M_{\lambda^{-1},p}(g(x), g(o))$. Assume $g(x) \neq 0$.

First assume $p \in (-1/n, 0)$. We get $g(x) \leq \lambda^{1/p}(1/2^p - 1 + \lambda^{-1})^{1/p} \leq \lambda^{1/p}$. Therefore, $\int_{S^c} g dx \leq \int_{\lambda \geq 1} \lambda^{1/p} n \lambda^{n-1} |S| d\lambda = \Theta_{n,\lambda,p}(|S|)$.

Second assume $p = 0$. We get $g(x) \leq 1/2^\lambda$. Hence, $\int_{S^c} g dx \leq \int_{\lambda \geq 1} 2^{-\lambda} n \lambda^{n-1} |S| d\lambda = \Theta_{n,\lambda,p}(|S|)$.

Third assume $p > 0$. We get $g(x) \leq \lambda^{1/p}(1/2^p - 1 + \lambda^{-1})^{1/p}$, so $\lambda \leq (1 - 1/2^p)^{-1}$. Hence, $\int_{S^c} g dx |1 - 1/2^p|^{-1} |S| = (1 - 1/2^p)^{-n} |S|$.

We have $\int_S g dx \geq 2^{-1} |S|$. We conclude $\int_S g dx = \Theta_{n,p}(1) \int_{\mathbb{R}^n} g dx$. \square

Proof Proposition 9.12 implies Proposition 9.9. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function that satisfies the hypothesis in Proposition 9.9. Assume without loss of generality that $f(o) = 1$ and $f(x) \leq 1$ for every $x \in \mathbb{R}^n$. As f is continuous with bounded support, it follows that $\text{co}_p(f)$ is continuous with bounded support. Construct the level sets $K_i = \{x: \text{co}_p(f) \geq m^{-i}\}$. Because $\text{co}_p(f)$ is continuous with bounded support, we deduce that K_i are compact. Note that the level sets K_i of $\text{co}_p(f)$ are convex and nested $K_i \subset K_{i+1}$. Note that for $i \geq 1$, $\text{int}(K_i) = \{x: \text{co}_p(f) > m^{-i}\}$ is an open set and $K_{i-1} \subset \text{int}(K_i)$.

Define the compact subset $Y_i \subset K_{i+1} \setminus \text{int}(K_{i-2})$, $Y_i := \{y \in K_{i+1} \setminus \text{int}(K_{i-2}): f(y) = \text{co}_p(f)(y)\}$. By the second hypothesis, it follows that for $x \in K_i \setminus \text{int}(K_{i-1})$, $\text{co}_p(f)(x) = \text{co}_p(f \times 1_{Y_i})(x)$.

Define $f_i = \min(f, \text{co}_p(1_{Y_i} \times f))$. Note that for $y \in Y_i$ we have $f_i(y) = f(y) = \text{co}_p(f)(y) \in [m^{-1-i}, m^{2-i}]$. Moreover $\text{co}_p(f_i) = \text{co}_p(1_{Y_i} \times f_i)$. Indeed, on the one hand we trivially have $\text{co}_p(f_i) \geq \text{co}_p(1_{Y_i} \times f_i)$. On the other hand we have $\text{co}_p(f_i) \leq \text{co}_p(1_{Y_i} \times f) = \text{co}_p(1_{Y_i} \times f_i)$. Furthermore, as f is continuous and as $\text{co}_p(1_{Y_i} \times f)$ is continuous on its support which is convex and compact, we deduce that f_i is continuous on $\text{co}(\text{supp}(f_i))$. Additionally, for $x \in K_i \setminus \text{int}(K_{i-1})$, $\text{co}_p(f_i)(x) = \text{co}_p(f)(x)$. Indeed, $\text{co}_p(f_i)(x) = \text{co}_p(1_{Y_i} \times f_i)(x) = \text{co}_p(1_{Y_i} \times f)(x) = \text{co}_p(f)(x)$.

Claim 9.14. *If $\int_{\mathbb{R}^n} f_i dx \geq 2^{-1} \int_{\mathbb{R}^n} \text{co}_p(f_i) dx$, then $\int_{\mathbb{R}^n} f_{i+1} dx \geq k^{-1} \int_{\mathbb{R}^n} \text{co}_p(f_{i+1}) dx$ for some $2 \leq k = O_{n,p,m}(1)$.*

Proof of Claim. First, by construction, for $x \in K_{i-1}$, $\text{co}_p(f_i)(x) \in [m^{1-i}, m^{2-i}]$, for $x \in K_i \setminus K_{i-1}$, $\text{co}_p(f_i) = \text{co}_p(f)(x) \in [m^{-i}, m^{1-i}]$, for $x \in K_{i+1} \setminus K_i$, $\text{co}_p(f_i)(x) \leq m^{-i}$ and for $x \in K_{i+1}^c$, $\text{co}_p(f_i)(x) = 0$. Hence, $\text{co}_p(f_{i+1}) = \Omega_m(1) \text{co}_p(f_i)$ and for $x \in K_i$, $\text{co}_p(f_{i+1})(x) = O_m(1) \text{co}_p(f_i)$. In particular,

$$\int_{K_i} \text{co}_p(f_{i+1}) dx \leq O_m(1) \int_{K_i} \text{co}_p(f_i) dx.$$

By construction, for $x \in \mathbb{R}^n$, $f_i(x) = \min(f(x), \text{co}_p(f_i)(x))$. By the first paragraph, we get

$$\int_{\mathbb{R}^n} f_{i+1} dx = \Omega_m(1) \int_{\mathbb{R}^n} f_i dx. \quad (9.1)$$

Second, $\int_{K_i^c} \text{co}_p(f_{i+1}) dx = O_{n,p,m} \int_{K_i} \text{co}_p(f_i) dx$. To see this, consider the function $g_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as follows: for $x \in K_i$, $g_i(x) = \text{co}_p(f_i)(x)$ and for $x \in K_i^c$, $g_i = \text{co}_p(f_{i+1})(x)$. As observed above, for $x \in K_i \setminus K_{i-1}$, we have $g_i(x) = \text{co}_p(f_i)(x) = \text{co}_p(f)(x)$ and for $x \in K_{i+1} \setminus K_i$, we have $g_i(x) = \text{co}_p(f_{i+1})(x) = \text{co}_p(f)(x)$. As $\text{co}_p(f_i)$, $\text{co}_p(f_{i+1})$ and $\text{co}_p(f)$ are p -concave, and as being p -concave is a local property and g_i is obtained by gluing together these functions, we deduce that g_i is also p -concave. By the first paragraph, for $x \in K_{i-1}$, we

have $g_i(x) \in [m^{1-i}, m^{2-i}]$ and for $x \in K_i^c$, we have $g_i(x) \leq m^{-i}$. By Lemma 9.13 (assuming wlog $m \geq 2$), we deduce $\int_{K_i^c} g_i(x) dx = \Omega_{n,p,m}(1) \int_{K_i^c} g_i(x) dx$. In other words,

$$\int_{K_i^c} \text{co}_p(f_{i+1}) dx = O_{n,p,m} \int_{K_i} \text{co}_p(f_i) dx.$$

Putting everything together, we conclude

$$\int_{\mathbb{R}^n} f_{i+1} dx \geq \Omega_m(1) \int_{\mathbb{R}^n} f_i dx \geq \Omega_m(1) \int_{\mathbb{R}^n} \text{co}_p(f_i) dx \geq \Omega_{n,p,m}(1) \int_{\mathbb{R}^n} \text{co}_p(f_{i+1}) dx.$$

□

Let $d = d_{n,\lambda,p,m^2,k} > 0$ and $s = s_{n,\lambda,p,m^2,k} > 0$ be the output of Proposition 9.12 with input n, λ, p and m^2 (instead of m) and k .

For $i \in \mathbb{N}$, define the set $S_i = \{x : f_i(x) \geq m^{-i-2}s^{-1}\}$ and define $\delta_i \geq 0$ such that

$$\int_{\mathbb{R}^n} M_{\lambda,p}^*(f_i \times 1_{S_i}, f_i \times 1_{S_i}) dx = (1 + \delta_i) \int_{\mathbb{R}^n} (f_i \times 1_{\lambda S_i + (1-\lambda)S_i}) dx.$$

Finally, let $i_0 = \inf\{i : \delta_i > d\}$.

Claim 9.15. For $1 \leq i < i_0$, we have $\int_{\mathbb{R}^n} \text{co}_p(f_i) dx \leq (1 + O_{n,\lambda,p,k,m}(\delta_i)) \int_{\mathbb{R}^n} f_i dx \leq 2 \int_{\mathbb{R}^n} f_i dx$.

Proof of Claim. We first consider the case $i = 1$. By a weak stability result (see e.g. Lemma 4.6, Corollary 4.7), we have $\int_{\mathbb{R}^n} f_1 dx \geq 2^{-1} \int_{\mathbb{R}^n} \text{co}_p(f_1) dx$. Recalling the properties of f_1 , we see that mf_1 satisfies the hypothesis of Proposition 9.12 with parameters n, λ, p and m^2 (instead of m) and k and thus we can apply Proposition 9.12 to conclude the case $i = 1$.

Now assume that the claim holds for $i - 1$. Then $m^i f_i$ satisfies the hypothesis of Proposition 9.12 with parameters n, λ, p and m^2 (instead of m) and k . Indeed, the first two hypotheses hold by the discussion preceding the first claim. The third hypothesis holds by combining the first claim with the inductive hypothesis. The fourth hypothesis holds by the definition of δ_i and the assumption that $\delta_i < d$. Thus we can apply Proposition 9.12 to conclude. □

Claim 9.16. $\sum_{i < i_0} \int_{K_i \setminus K_{i-1}} (\text{co}_p(f)(x) - f(x)) dx \leq O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f(x) dx$.

Proof of Claim. Recall that for $x \in K_i \setminus K_{i-1}$, we have $\text{co}_p(f)(x) = \text{co}_p(f_i)(x)$. Also, $f_i = \min(f, \text{co}_p(f_i))$. Hence,

$$\int_{K_i \setminus K_{i-1}} (\text{co}_p(f)(x) - f(x)) dx = \int_{K_i \setminus K_{i-1}} (\text{co}_p(f_i)(x) - f_i(x)) dx \leq \int_{\mathbb{R}^n} (\text{co}_p(f_i)(x) - f_i(x)) dx.$$

By the second claim, we have

$$\sum_{i < i_0} \int_{K_i \setminus K_{i-1}} (\text{co}_p(f)(x) - f(x)) dx \leq \sum_{i < i_0} O_{n,\lambda,p,m}(\delta_i) \int_{\mathbb{R}^n} f_i(x) dx$$

By the second claim, we also have

$$\int_{\mathbb{R}^n} \text{co}_p(f_i) dx \leq 2 \int_{\mathbb{R}^n} f_i dx.$$

Recall that for $x \in \text{supp}(\text{co}_p(f_i))$, we have $\text{co}_p(f_i) \in [m^{-i-1}, m^{2-i}]$ and $s \geq 2k \geq 4$ and $S_i = \{x : f_i(x) \geq 1/m\}$. Hence, $S_i \supset \{x : f_i(x) \geq 4^{-1} \text{co}_p(f_i)\}$. It follows that

$$\int_{\mathbb{R}^n} f_i \times 1_{\lambda S_i + (1-\lambda)S_i} dx \geq 2^{-1} \int_{\mathbb{R}^n} f_i dx.$$

Combining the last equation and the third last equation, we deduce

$$\sum_{i < i_0} \int_{K_i \setminus K_{i-1}} (\text{co}_p(f)(x) - f(x)) dx \leq \sum_{i < i_0} O_{n,\lambda,p,m}(\delta_i) \int_{\mathbb{R}^n} f_i \times 1_{\lambda S_i + (1-\lambda)S_i} dx.$$

By the definition of δ_i , we further get

$$\sum_{i < i_0} \int_{K_i \setminus K_{i-1}} (\text{co}_p(f)(x) - f(x)) dx = O_{n,\lambda,p,m}(1) \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_i \times 1_{S_i}, f_i \times 1_{S_i}) - f_i \times 1_{\lambda S_i + (1-\lambda)S_i}) dx.$$

Given that

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx \leq \delta \int_{\mathbb{R}^n} f dx,$$

it is enough to argue that

$$\sum_i \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_i \times 1_{S_i}, f_i \times 1_{S_i}) - f_i \times 1_{\lambda S_i + (1-\lambda)S_i}) dx \leq O_{n,\lambda,p,m}(1) \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx.$$

As $f_i = \min(f, \text{co}_p(f_i))$ is the restriction of f to a p -concave function, we have

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_i \times 1_{S_i}, f_i \times 1_{S_i}) - f_i \times 1_{\lambda S_i + (1-\lambda)S_i}) dx \leq \int_{\mathbb{R}^n} \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i}) - f \times 1_{\lambda S_i + (1-\lambda)S_i}\right) dx.$$

Therefore, it is enough to argue that

$$\sum_i \int_{\mathbb{R}^n} \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i}) - f \times 1_{\lambda S_i + (1-\lambda)S_i}\right) dx \leq O_{n,\lambda,p,m}(1) \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx.$$

Fix any $x \in \mathbb{R}^n$. It is enough to show that

$$\sum_i \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i})(x) - f \times 1_{\lambda S_i + (1-\lambda)S_i}(x)\right) \leq O_{n,\lambda,p,m}(1) \left(M_{\lambda,p}^*(f, f)(x) - f(x)\right).$$

Let $I = \{i: x \in \lambda S_i + (1-\lambda)S_i\}$. It is enough to show that

$$\sum_{i \in I} \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i})(x) - f(x)\right) \leq O_{n,\lambda,p,m}(1) \left(M_{\lambda,p}^*(f, f)(x) - f(x)\right).$$

Note that for $x \in S_i$, we have $f(x) \in [m^{-2-i} s^{-1}, m^{2-i}]$. Indeed, on the one hand, we have $S_i \subset \text{supp}(f_i) \subset K_{i+1} \setminus K_{i-2}$ and for $x \in S_i \subset K_{i-2}^c$, we have $f(x) \leq \text{co}_p(f)(x) \leq m^{2-i}$. Additionally, for $x \in S_i$, we have $f_i(x) \geq m^{-2-i} s^{-1}$. As $f_i = \min(f, \text{co}_p(f_i)) \leq f$, for $x \in S_i$, we have $f(x) \geq m^{-2-i} s^{-1}$.

Therefore, if $i_1, i_2 \in I$ satisfy

$$\frac{M_{\lambda,p}^*(f \times 1_{S_{i_1}}, f \times 1_{S_{i_1}})(x)}{M_{\lambda,p}^*(f \times 1_{S_{i_2}}, f \times 1_{S_{i_2}})(x)} \in [1/2, 2],$$

then

$$|i_1 - i_2| = O_{n,\lambda,p,m}(1).$$

Moreover,

$$\sup_i \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i})(x) - f(x)\right) \leq M_{\lambda,p}^*(f, f)(x) - f(x).$$

Combining the last two equations, we conclude that

$$\sum_{i \in I} \max\left(0, M_{\lambda,p}^*(f \times 1_{S_i}, f \times 1_{S_i})(x) - f(x)\right) \leq O_{n,\lambda,p,m}(1) \left(M_{\lambda,p}^*(f, f)(x) - f(x)\right).$$

This finishes the proof of the claim □

Claim 9.17. *If $i_0 < \infty$, then $\int_{\mathbb{R}^n} \text{co}_p(f_{i_0}) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$.*

Proof of Claim. By construction,

$$\int_{\mathbb{R}^n} M_{\lambda,p}^*(f_{i_0} \times 1_{S_{i_0}}, f_{i_0} \times 1_{S_{i_0}}) dx \geq (1+d) \int_{\mathbb{R}^n} f_{i_0} \times 1_{\lambda S_{i_0} + (1-\lambda)S_{i_0}} dx \geq (1+d) \int_{\mathbb{R}^n} f_{i_0} \times 1_{S_{i_0}} dx.$$

Hence,

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_{i_0} \times 1_{S_{i_0}}, f_{i_0} \times 1_{S_{i_0}}) - f_{i_0} \times 1_{S_{i_0}}) dx \geq d \int_{\mathbb{R}^n} f_{i_0} \times 1_{S_{i_0}} dx.$$

We have

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_{i_0}, f_{i_0}) - f_{i_0}) dx \geq \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_{i_0} \times 1_{S_{i_0}}, f_{i_0} \times 1_{S_{i_0}}) - f_{i_0} \times 1_{S_{i_0}}) dx$$

As $f_{i_0} = \min(f, \text{co}_p(f_{i_0}))$ is f capped by a p -concave function, we get

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx \geq \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f_{i_0}, f_{i_0}) - f_{i_0}) dx.$$

By hypothesis, we have

$$\delta \int_{\mathbb{R}^n} f dx \geq \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx.$$

Putting everything together, we get that

$$\int_{\mathbb{R}^n} f_{i_0} \times 1_{S_{i_0}} dx \leq d^{-1} \delta \int_{\mathbb{R}^n} f dx$$

By combining the first and second claims, we also have

$$\int_{\mathbb{R}^n} f_{i_0} dx \geq k^{-1} \int_{\mathbb{R}^n} \text{co}_p(f_{i_0}) dx.$$

Recall that for $x \in \text{supp}(\text{co}(f_{i_0}))$, we have $\text{co}(f_{i_0})(x) \in [m^{-1-i_0}, m^{2-i_0}]$. Recall that $S_{i_0} = \{x: f_{i_0}(x) \geq m^{-i_0-2}s^{-1}\}$ and $s \geq 2k$. We deduce that $S_{i_0} \supset \{x: f_{i_0}(x) \geq (2k)^{-1} \text{co}_p(f_{i_0})\}$. It follows that

$$\int_{\mathbb{R}^n} f_{i_0} \times 1_{S_{i_0}} dx = (2k)^{-1} \int_{\mathbb{R}^n} \text{co}_p(f_{i_0}) dx.$$

Therefore, recalling $k = O_{n,p,m}(1)$ and $d = O_{n,p,m,\lambda,k}(1)$, we get

$$\int_{\mathbb{R}^n} \text{co}_p(f_{i_0}) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$$

□

Claim 9.18. *If $i_0 < \infty$, then $\int_{K_{i_0-1}^c} \text{co}_p(f) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$.*

Proof of Claim. By the previous claim, we have

$$\int_{\mathbb{R}^n} \text{co}_p(f_{i_0}) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx.$$

Recall that for $x \in K_{i_0} \setminus K_{i_0-1}$, we have $\text{co}_p(f_{i_0})(x) = \text{co}_p(f)(x)$; hence, we also have

$$\int_{K_{i_0} \setminus K_{i_0-1}} \text{co}_p(f) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$$

Now construct the function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ as follows. For $x \in K_i$, let $g(x) = \text{co}_p(f_i)(x)$ and for $x \in K_{i-1}^c$, let $g(x) = \text{co}_p(f)(x)$. Both $\text{co}_p(f_i)$ and $\text{co}_p(f)$ are p -concave. Since being p -concave is a local property and g is obtained by gluing together these functions, we also get that g is p -concave.

Recall that for $x \in K_{i_0}^c$, we have $g(x) = \text{co}_p(f)(x) \leq m^{-i_0}$ and for $x \in K_{i_0-1}$, we have $g(x) \geq m^{-i_0+1}$. By Lemma 9.13 (assuming wlog $m \geq 2$) together with the previous claim, we deduce

$$\int_{K_{i_0}^c} \text{co}_p(f) dx = \int_{K_{i_0}^c} g(x) dx = O_{n,\lambda,p,m}(1) \int_{K_{i_0}} g(x) dx = O_{n,\lambda,p,m}(1) \int_{K_{i_0}} \text{co}_p(f_{i_0})(x) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx.$$

Putting everything together, the conclusion of the claim follows. □

Combining the third and the last claims we conclude that

$$\int_{\mathbb{R}^n} (\text{co}_p(f)(x) - f(x)) dx \leq O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f(x) dx.$$

This finishes the proof of the proposition. □

9.4 Reduction from Proposition 9.12 to Proposition 9.19

Proposition 9.19. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $m \in (1, \infty)$ there exists $d = d_{n,\lambda,p,m} > 0$ such that the following holds. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function with compact support such that*

- *$\text{supp}(f)$ is compact and f is continuous on $\text{supp}(f)$,*
- *there exists a compact set Y such that $\text{co}_p(f) = \text{co}_p(1_Y \times f)$ and for $y \in Y$ we have $f(y) \in [1/m, m]$,*
- *$\int_{\mathbb{R}^n} f \, dx > 0$, and*
- *$\int_{\mathbb{R}^n} M_{\lambda,p}^*(f, f) \, dx \leq (1 + \delta) \int_{\mathbb{R}^n} f \, dx$ for some $0 \leq \delta \leq d$.*

Then

- $\int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f \, dx$.

Proof Proposition 9.19 implies Proposition 9.12. Pick $s \geq 2k$ large enough. As f is continuous on $\overline{\text{co}(\text{supp}(f))}$, it follows that $\text{supp}(f \times 1_S)$ is compact and $f \times 1_S$ is continuous on $\text{supp}(f \times 1_S)$. Moreover, $\int_{\mathbb{R}^n} f \times 1_S \, dx > \frac{1}{2} \int_{\mathbb{R}^n} f \, dx > (2k)^{-1} \int_{\mathbb{R}^n} \text{co}_p(f) \, dx$. Furthermore, $f \times 1_S(x) \in [1/ms, m]$ for $x \in \text{supp}(f \times 1_S)$.

As $\text{co}_p(f)(x) \geq m^{-1}$ for $x \in \text{supp}(f)$ and $f(x) \leq 1/ms$ for $x \notin S$, we get

$$\int_{\mathbb{R}^n} f \times 1_{\lambda S + (1-\lambda)S} \, dx \leq \int_{\mathbb{R}^n} f \times 1_S \, dx + s^{-1} \int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx.$$

As $\text{co}_p(f) = \text{co}_p(1_Y \times f)$ and for $y \in Y$ we have $f(y) = f \times 1_S(y) \in [1/m, m]$, we have that $\text{co}_p(f) = \text{co}_p(f \times 1_S) = \text{co}_p(f \times 1_S \times 1_Y)$.

We deduce that

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f \times 1_S, f \times 1_S) - f \times 1_S) \, dx \leq \delta \int_{\mathbb{R}^n} f \times 1_S \, dx + 2s^{-1} \int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx.$$

In particular,

$$\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f \times 1_S, f \times 1_S) - f \times 1_S) \, dx \leq (\delta + 4ks^{-1}) \int_{\mathbb{R}^n} f \times 1_S \, dx.$$

By taking $\delta > 0$ and $s > 0$ sufficiently small, we can make $\delta + 4ks^{-1}$ arbitrary small. By Proposition 9.19, we deduce that

$$\int_{\mathbb{R}^n} (\text{co}_p(f \times 1_S) - f \times 1_S) \, dx = O_{n,\lambda,p,k,m}(1) \left(\delta \int_{\mathbb{R}^n} f \times 1_S \, dx + 2s^{-1} \int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx \right).$$

As $\text{co}_p(f \times 1_S) = \text{co}_p(f)$ and $f \times 1_S \leq f$, this implies

$$\int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx = O_{n,\lambda,p,k,m}(1) \left(\delta \int_{\mathbb{R}^n} f \, dx + 2s^{-1} \int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx \right).$$

By taking s larger than $O_{n,\lambda,p,k,m}(1)$, we conclude

$$\int_{\mathbb{R}^n} (\text{co}_p(f) - f) \, dx = O_{n,\lambda,p,k,m}(\delta) \int_{\mathbb{R}^n} f \, dx.$$

□

9.5 Reduction from Proposition 9.19 to Proposition 9.20

Proposition 9.20. *Given $n \in \mathbb{N}$, $\lambda \in (0, 1/2]$, $p \in (-1/n, \infty)$ and $m \in (1, \infty)$ there exists $d = d_{n,\lambda,p,m} > 0$ such that the following holds. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function with compact support such that*

- *$\text{supp}(f)$ is compact and f is continuous on $\text{supp}(f)$,*
- *$f(x) \in [m^{-1}, m]$ for $x \in \text{supp}(f)$,*
- *$\int_{\mathbb{R}^n} f \, dx > 0$, and*

- $\int_{\mathbb{R}^n} M_{\lambda,p}^*(f, f) dx \leq (1 + \delta) \int_{\mathbb{R}^n} f dx$ for some $0 \leq \delta \leq d$.

Then

$$\bullet \int_{\mathbb{R}^n} (\text{co}_p(f) - f) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx.$$

Proof Proposition 9.20 implies Proposition 9.19. Let $A_t := \{x: f(x) \geq t\}$ and $B_t := \{x: M_{\lambda,p}^*(f, f)(x) \geq t\}$ and $C_t := \{x: \text{co}_p(f)(t) \geq t\}$. The sets A_t, B_t, C_t are nested. For $t \geq 0$, we have

$$C_t \supset B_t \supset \lambda A_t + (1 - \lambda)A_t \supset A_t.$$

Moreover, for $t \geq m$, we have $C_t = \emptyset$ and for $t \leq m^{-1}$, we have $C_t = \text{co}(Y)$ by the second hypothesis.

The third and fourth hypotheses imply $\int_{\mathbb{R}^n} f dx \geq 2^{-1} \int_{\mathbb{R}^n} \text{co}_p(f) dx$. We have $\int_{\mathbb{R}^n} \text{co}_p(f) dx \geq m^{-1} |\text{co}(Y)|$. We also have $\int_{\mathbb{R}^n} f dx \leq m |A_{4^{-1}m^{-1}}| + 4^{-1}m^{-1} |\text{co}(Y)|$. We deduce $|A_{4^{-1}m^{-1}}| \geq 4^{-1}m^{-2} |\text{co}(Y)|$. So for $t \leq 4^{-1}m^{-1}$, we have $|A_t| \geq 4^{-1}m^{-2} |\text{co}(Y)|$. Moreover, by the second hypothesis, $\text{co}(A_t) = \text{co}(Y)$.

By the fourth hypothesis, we get $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx \leq \delta \int_{\mathbb{R}^n} f dx \leq \delta m |\text{co}(Y)|$. Therefore,

$$\int (|\lambda A_t + (1 - \lambda)A_t| - |A_t|) dt \leq \delta m |\text{co}(Y)|.$$

In particular, $\int_{t \leq 4^{-1}m^{-1}} (|\lambda A_t + (1 - \lambda)A_t| - |A_t|) dt \leq \delta m |\text{co}(Y)|$.

As for $t \leq 4^{-1}m^{-1}$, we have $|A_t| \geq 4^{-1}m^{-2} |\text{co}(Y)|$ and $\text{co}(A_t) = \text{co}(Y)$, by Theorem 1.1, we deduce that $|\lambda A_t + (1 - \lambda)A_t| - |A_t| = \Theta_{n,\lambda,m}(1) |\text{co}(A_t) \setminus A_t|$.

Combining the last two inequalities, we get

$$\int_{t \leq 4^{-1}m^{-1}} |\text{co}(A_t) \setminus A_t| dt = O_{n,\lambda,m}(\delta) |\text{co}(Y)| = O_{n,\lambda,m}(\delta) \int_{\mathbb{R}^n} f dx.$$

Let $S = \{x: f(x) \geq 4^{-1}m^{-1}\}$ and let $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be defined by $g(x) = f(x) \times 1_S$. By the last inequality, we get

$$\int_{\mathbb{R}^n} (f - g) dx = \int_{t \leq 4^{-1}m^{-1}} |\text{co}(A_t) \setminus A_t| dt = O_{n,\lambda,m}(\delta) \int_{\mathbb{R}^n} f dx.$$

As $g \leq f$, by the fourth hypothesis, we further deduce that

$$\int_{\mathbb{R}^n} M_{\lambda,p}^*(g, g) dx \leq (1 + \delta) \int_{\mathbb{R}^n} f dx \leq (1 + O_{n,\lambda,m}(\delta)) \int_{\mathbb{R}^n} g dx.$$

By the first hypothesis, we deduce that $S = \text{supp}(g)$ is compact and g is continuous on S .

By the second hypothesis and the definition of S , we also get that $g(x) \in [4^{-1}m^{-1}, m]$ for $x \in S = \text{supp}(g)$ and additionally, $\text{co}_p(g) = \text{co}_p(f)$.

Finally, as $|A_{4^{-1}m^{-1}}| > 0$, we infer that $\int_{\mathbb{R}^n} g dx > 0$.

By Proposition 9.20, we get that $\int_{\mathbb{R}^n} (\text{co}_p(g) - g) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} g dx$. As $\text{co}_p(f) = \text{co}_p(g)$ and additionally $\int_{\mathbb{R}^n} f dx \leq (1 + O_{n,\lambda,m}(\delta)) \int_{\mathbb{R}^n} g dx$, we conclude $\int_{\mathbb{R}^n} (\text{co}_p(f) - f) dx = O_{n,\lambda,p,m}(\delta) \int_{\mathbb{R}^n} f dx$. \square

9.6 Proof of Proposition 9.20

Proof of Proposition 9.20. By the first hypothesis, f , $M_{\lambda,p}^*(f, f)$, and $\text{co}_p(f)$ are integrable. After scaling and replacing m by $m^{1/2}$ we can assume $f(x) \in [2, m]$ for $x \in \text{supp}(f)$. Moreover, $M_{\lambda,p}^*(f, f)(x) \in [2, m]$ for $x \in \text{supp}(M_{\lambda,p}^*(f, f))$ and $\text{co}_p(f) \in [2, m]$ for $x \in \text{supp}(\text{co}_p(f))$.

Define the function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ as follows.

$$g(x) = \begin{cases} 0 & \text{if } x \notin \text{supp}(f) \\ \text{sign}(p) f^p(x) + 1 & \text{if } p \neq 0 \text{ and } x \in \text{supp}(f) \\ \log(f(x)) + 1 & \text{if } p = 0 \text{ and } x \in \text{supp}(f) \end{cases}$$

Note that the function $M_{\lambda,1}^*(g, g): \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfies

$$M_{\lambda,1}^*(g, g)(x) = \begin{cases} 0 & \text{if } x \notin \text{supp}(M_{\lambda,p}^*(f, f)) \\ \text{sign}(p)(M_{\lambda,p}^*(f, f)(x))^p + 1 & \text{if } p \neq 0 \text{ and } x \in \text{supp}(M_{\lambda,p}^*(f, f)) \\ \log(M_{\lambda,p}^*(f, f)(x)) + 1 & \text{if } p = 0 \text{ and } x \in \text{supp}(M_{\lambda,p}^*(f, f)) \end{cases}$$

Also note that the function $\text{co}_1(g): \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfies

$$\text{co}_1(g)(x) = \begin{cases} 0 & \text{if } x \notin \text{supp}(\text{co}_p(f)) \\ \text{sign}(p) \text{co}_p^p(f) + 1 & \text{if } p \neq 0 \text{ and } x \in \text{supp}(\text{co}_p(f)) \\ \log(\text{co}_p(f)) + 1 & \text{if } p = 0 \text{ and } x \in \text{supp}(\text{co}_p(f)) \end{cases}$$

By the above, $\text{supp}(g) = \text{supp}(f)$, $\text{supp}(M_{\lambda,1}^*(g, g)) = \text{supp}(M_{\lambda,p}^*(f, f))$, and $\text{supp}(\text{co}_1(g)) = \text{supp}(\text{co}_p(f))$. Moreover, for every $x \in \mathbb{R}^n$, we have

$$g(x) = \Theta_{n,p,m}(1)f(x), \quad M_{\lambda,1}^*(g, g)(x) = \Theta_{n,p,m}(1)M_{\lambda,p}^*(f, f)(x), \quad \text{co}_1(g)(x) = \Theta_{n,p,m}(1)\text{co}_p(f)(x).$$

Furthermore,

$$M_{\lambda,1}^*(g, g)(x) - g(x) = \Theta_{n,p,m}(1)(M_{\lambda,p}^*(f, f)(x) - f(x)) \text{ and } \text{co}_1(g)(x) - g(x) = \Theta_{n,p,m}(1)(\text{co}_p(f)(x) - f(x)).$$

As f , $M_{\lambda,p}^*(f, f)$, and $\text{co}_p(f)$ are integrable, so are g , $M_{\lambda,1}^*(g, g)$, and $\text{co}_1(g)$. Additionally, as $\int f \, dx > 0$, we also get $\int g \, dx > 0$.

Putting everything together, we see that

$$\begin{aligned} \int (M_{\lambda,1}^*(g, g)(x) - g(x)) \, dx &= \Theta_{n,m,p}(1) \int (M_{\lambda,p}^*(f, f)(x) - f(x)) \, dx \\ &\leq \Theta_{n,m,p}(\delta) \int f(x) \, dx \\ &\leq \Theta_{n,m,p}(\delta) \int g(x) \, dx. \end{aligned}$$

By Theorem 1.1, we deduce that

$$\int (\text{co}_1(g)(x) - g(x)) \, dx \leq O_{n,\lambda,p,m}(\delta) \int g(x) \, dx,$$

from which we conclude that

$$\int (\text{co}_p(f)(x) - f(x)) \, dx \leq O_{n,\lambda,p,m}(\delta) \int f(x) \, dx.$$

□

9.7 Proof of Proposition 9.10

The purpose of this section is to prove Proposition 9.10. We will make use of the following remarks.

Remark 9.21. *Let $n \in \mathbb{N}$, $p \in (-1/n, \infty)$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a p -concave function. Then f has convex support and convex level sets and is continuous on the interior of its support.*

Remark 9.22. *Let $n \in \mathbb{N}$, $p \in (-1/n, \infty)$ and let $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a p -concave function. Then f is a measurable function.*

Remark 9.23. *Let $n \in \mathbb{N}$ and let $p \in (-1/n, \infty)$. Let I be a set of indices. For $i \in I$, let $\ell_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a p -concave function. Then, the function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$, $g(x) = \inf_i \ell_i(x)$ is p -concave.*

Proof of Proposition 9.10. Let \mathcal{L} be the collection of p -concave functions $\ell': \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the function $f': \mathbb{R}^n \rightarrow \mathbb{R}_+$, $f' = \min(f, \ell')$ satisfies $\int_{\mathbb{R}^n} [-c(f - f') + (M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f')] \, dx \geq 0$.

Put on \mathcal{L} the partial order $\ell_1 \leq \ell_2$ if $\ell_1(x) \leq \ell_2(x)$ for all $x \in \mathbb{R}^n$. To show that \mathcal{L} has a minimal element, by Zorn's lemma it is enough to show that any chain $C \subset \mathcal{L}$ has a lower bound in \mathcal{L} .

Let C be a chain \mathcal{L} . Let $\ell_0: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be defined by $\ell_0(x) = \inf_{\ell \in C} \ell(x)$. By Remark 9.23, it follows that ℓ_0 is also a p -concave function. Clearly, ℓ_0 is a lower bound for C . It remains to show that $\ell_0 \in \mathcal{L}$.

By Remark 9.22, ℓ_0 is measurable and any $\ell \in C$ is measurable. We will first show that there exist a sequence $\ell_1 \geq \ell_2 \geq \dots$ in C such that $\int_{\mathbb{R}^n} \ell_i(x) \, dx \rightarrow \int_{\mathbb{R}^n} \ell_0(x) \, dx$ as $i \rightarrow \infty$.

For a function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $c > 0$, let $g^c = \{x \in \mathbb{R}^n: g(x) \geq c\}$. By Remark 9.21, for $c > 0$, the family $\{\ell^c: \ell \in C\}$ is a chain of convex sets under the subset order. Moreover, ℓ_0^c is a convex set. Furthermore, for every $\ell \in C$, $\ell_0^c \subset \ell^c$. Finally, for any $p \notin \ell_0^c$, there exists $\ell \in C$ such that $p \notin \ell^c$.

It follows that for a fixed $c > 0$, there exist $\ell_1 \geq \ell_2 \geq \dots$ in C such that $\ell_1^c \supset \ell_2^c \supset \dots$ and $|\ell_i^c| \rightarrow |\ell_0^c|$ as $i \rightarrow \infty$. It further follows that there exist $\ell_1 \geq \ell_2 \geq \dots$ in C such that for every $c \in \mathbb{Q}$, $\ell_1^c \supset \ell_2^c \supset \dots$ and $|\ell_i^c| \rightarrow |\ell_0^c|$ as $i \rightarrow \infty$.

Note $\int_{\mathbb{R}^n} \ell_i(x) dx = \int_{c \geq 0} |\ell_i^c| dc$. Therefore to show $\int_{\mathbb{R}^n} \ell_i(x) dx \rightarrow \int_{\mathbb{R}^n} \ell_0(x) dx$ as $i \rightarrow \infty$, it is enough to show $\int_{c \geq 0} |\ell_i^c| dc \rightarrow \int_{c \geq 0} |\ell_0^c| dc$ as $i \rightarrow \infty$.

For each i , the function $|\ell_i^c|$ is left continuous and decreasing in c . Moreover, the sequence of functions $|\ell_i^c|$ is pointwise decreasing and it is lower bounded by $|\ell_0^c|$. Furthermore, for any $c \in \mathbb{Q}$, we have $|\ell_i^c| \rightarrow |\ell_0^c|$ as $i \rightarrow \infty$. By a triangle inequality, it follows that for any $c \in \mathbb{R}$, we have $|\ell_i^c| \rightarrow |\ell_0^c|$ as $i \rightarrow \infty$. Finally, by pointwise convergence, we get $\int_{c \geq 0} |\ell_i^c| dc \rightarrow \int_{c \geq 0} |\ell_0^c| dc$ as $i \rightarrow \infty$.

We infer that there exist a sequence $\ell_1 \geq \ell_2 \geq \dots$ in C such that $\int_{\mathbb{R}^n} \ell_i(x) dx \rightarrow \int_{\mathbb{R}^n} \ell_0(x) dx$ as $i \rightarrow \infty$. Let $f_i: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be defined by $f_i = \min(\ell_i, f)$. Then it follows that $\int_{\mathbb{R}^n} f_i(x) dx \rightarrow \int_{\mathbb{R}^n} f_0(x) dx$ as $i \rightarrow \infty$. We also have $M_{\lambda,p}^*(f_i, f_i) \geq M_{\lambda,p}^*(f_0, f_0)$. Recalling the definition of \mathcal{L} , from the fact that $\ell_i \in \mathcal{L}$ for all i , by taking the limit, we conclude the first bullet point $\ell_0 \in \mathcal{L}$.

By the previous bullet point, $M_{\lambda,p}^*(f', f') \geq f'$, and the hypothesis, we conclude that

$$\int_{\mathbb{R}^n} (f - f') dx \leq c^{-1} \int_{\mathbb{R}^n} [(M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f')] dx \leq c^{-1} \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx \leq c^{-1} \delta \int_{\mathbb{R}^n} f dx.$$

By the previous bullet point, we also get that $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f', f') - f') dx \leq \int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - f) dx$. By hypothesis, we deduce that $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f', f') - f') dx \leq \delta \int_{\mathbb{R}^n} f dx$. By the previous assertion, provided $d \leq c/10$, we conclude that $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f', f') - f') dx \leq 3\delta \int_{\mathbb{R}^n} f dx$.

For the last bullet point, assume $f'' \neq f'$. Let $\ell = \min(\ell', \ell'')$. Then $\ell \leq \ell'$ and $\ell \neq \ell'$. Moreover, $f'' = \min(f, \ell)$. By the minimality of ℓ' , we deduce that

$$\int_{\mathbb{R}^n} [-c(f - f'') + (M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f'', f'') - f'')] dx < 0.$$

As

$$\int_{\mathbb{R}^n} [-c(f - f') + (M_{\lambda,p}^*(f, f) - f) - (M_{\lambda,p}^*(f', f') - f')] dx \geq 0,$$

we reach our desired conclusion that

$$\int_{\mathbb{R}^n} [-c(f' - f'') + (M_{\lambda,p}^*(f', f') - f') - (M_{\lambda,p}^*(f'', f'') - f'')] dx < 0.$$

□

9.8 Proof of Proposition 9.11

The purpose of this subsection is to prove the Proposition 9.11. Before we begin the proof, we need the following lemma.

Lemma 9.24. *Let $n \in \mathbb{N}$, $p \in (-1/n, \infty)$, $y \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a continuous p -concave function with bounded support. Let $F \subset \mathbb{R}^n$ be a compact convex set such that F is a p -face of g with tangent p -plane $h_{y,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$. Let $[o, v]$ be an edge of F and assume $h_{y,d}(o) > h_{y,d}(v)$. Then $\text{sign}(p) \langle y, v \rangle < 0$ and there exists a sequence of vectors $z_i \rightarrow o$ as $i \rightarrow \infty$ with $\langle z_i, v \rangle = 0$ such that the p -planes $h_{y+z_i,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ have the following property. For $x \in [o, v]$, we have $h_{y+z_i,d}(x) = h_{y,d}(x) = g(x)$. Moreover, the set $S_i = \{x \in \overline{\text{supp}(g)}: g(x) \geq h_{y+z_i,d}(x)\}$ satisfies $S_i \subset [o, v] + B(o, 1/i)$.*

Proof of Lemma 9.24. Given $z, x \in \mathbb{R}^n$, if $\langle x, z \rangle > 0$, then $\text{sign}(p)h_{y+z,d}(x) \geq \text{sign}(p)h_{y,d}(x)$ and if $\langle x, z \rangle < 0$, then $\text{sign}(p)h_{y+z,d}(x) \leq \text{sign}(p)h_{y,d}(x)$ and if $\langle x, z \rangle = 0$, then $h_{y+z,d}(x) = h_{y,d}(x)$. Moreover, as $z \rightarrow 0$, $h_{y+z,d} \rightarrow h_{y,d}$ locally uniformly.

Recall that $\text{supp}(g)$ is a bounded open convex set and $F \subset \overline{\text{supp}(g)}$. We claim that for $x \in \overline{\text{supp}(g)} \setminus F$, we have $h_{y,d}(x) > g(x)$. Indeed, assume for a contradiction that $h_{y,d}(x) \leq g(x)$. We have $o, x \in \text{supp}(g)$ so $[o, x] \in \overline{\text{supp}(g)}$. Moreover, $o \in \text{supp}(g)$ so $[o, x] \in \text{supp}(g)$. We also have $o \in F$ and $x \notin F$ so there exists $\lambda \in [0, 1)$ such that for $t \in [0, \lambda]$, we have $tx \in F$ and for $t \in (\lambda, 1)$, we have $tx \notin F$. By hypothesis, for $t \in (\lambda, 1)$, we have $tx \in \text{supp}(g) \setminus F$, so $h_{y,d}(tx) > g(tx)$. Additionally, $h_{y,d}(\lambda x) = g(\lambda x)$. Given that for $t \in [\lambda, 1]$ we have that $g(tx)$ is p -concave and continuous and $h_{y,d}(tx)$ is p -convex and continuous, we conclude that $h_{y,d}(x) > g(x)$.

Fix $i \in \mathbb{N}$ sufficiently small and consider the compact set $(F + B(o, 1/i))^c$. There exists $\varepsilon_i > 0$ such that for $x \in \text{supp}(g) \setminus (F + B(o, 1/i))$ we have $g(x) \leq h_{y,d}(x) - \varepsilon_i$. Therefore, there exists $\delta_i > 0$ sufficiently small such that for $|z| < \delta_i$ and $x \in \text{supp}(g) \setminus (F + B(o, 1/i))$, we have $h_{y+z,d}(x) > g(x)$.

To conclude, it is enough to show that for any j we can find i and z_i with $|z_i| < \delta_i$ such that the half spaces $H_i^+ = \{x \in \mathbb{R}^n : \text{sign}(p) \langle x, z_i \rangle > 0\}$ satisfy $(H_i^+)^c \cap (F + B(o, 1/i)) \subset [0, v] + B(o, 1/j)$.

Indeed, fix $x \in \text{supp}(g)$ such that $g(x) \geq h_{y+z_i,d}(x)$. Then $x \in (F + B(o, 1/i)) \cap \text{supp}(g)$. Clearly $x \in \text{supp}(h_{y,d})$. If additionally $x \in H_i^+$, then $g(x) \leq h_{y,d}(x) < h_{y+z_i,d}(x)$, a contradiction. Hence, $x \in (F + B(o, 1/i)) \cap (H_i^+)^c$ and we are done provided $(H_i^+)^c \cap (F + B(o, 1/i)) \subset [0, v] + B(o, 1/j)$.

For a fixed j and a fixed closed half-space H^- , if i is sufficiently large, then $H^- \cap (F + B(o, 1/i)) \subset (H^- \cap F) + B(o, 1/2j)$.

Therefore, it is enough to show that for any fixed j , there exists $z \neq o$, such that the closed half-space $H^- = \{x \in \mathbb{R}^n : \text{sign}(p) \langle x, z \rangle \leq 0\}$ satisfies $H^- \cap F \subset [0, v] + B(o, 1/2j)$.

In other words, it is enough to show that for any fixed j , there exists a half-space H^- with the hyperplane H containing $[0, v]$ such that $H^- \cap F \subset [0, v] + B(o, 1/2j)$.

Assuming for a contradiction this is not the case, we deduce that there exists $\text{co}(F \setminus ([0, v] + B(o, 1/2j))) \cap \text{span}(v) \neq \emptyset$. As $[0, v]$ is an edge of F we obtain the desired contradiction.

This concludes the proof of the lemma. \square

Proof of Proposition 9.11. Fix $m = m_{n,\lambda,p} > 0$ large and $c = c_{n,\lambda,p} > 0$ small.

As f is continuous with bounded support, so is $\text{co}_p(f)$. Let F be a p -face of $\text{co}_p(f)$ and let $[o, v]$ be an edge of F (after translation) such that $\text{co}_p(o) > \text{co}_p(v)$. Let $h_{y,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be the tangent p -plane corresponding to F .

Assume for a contradiction that there exist $0 \leq a < b \leq 1$ such that $f(av) > mf(bv)$, $f(av) = \text{co}_p(f)(av)$ and $f(bv) = \text{co}_p(f)(bv)$ and for every $t \in (a, b)$ we have $f(tv) < \text{co}_p(f)(tv)$.

By Lemma 9.24, $\text{sign}(p) \langle y, v \rangle < 0$ and there exists a sequence of vectors $z_i \rightarrow o$ as $i \rightarrow \infty$ with $\langle z_i, v \rangle = 0$ such that the p -planes $h_{y+z_i,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ have the following property. For $x \in [o, v]$, we have $h_{y+z_i,d}(x) = h_{y,d}(x) = \text{co}_p(x)$. Moreover, the set $S_i = \{x \in \text{supp}(\text{co}_p(f)) : \text{co}_p(f)(x) \geq h_{y+z_i,d}(x)\}$ satisfies $S_i \subset [o, v] + B(o, 1/i)$.

Recall that f and $\text{co}_p(f)$ are continuous and $f(tv) < \text{co}_p(f)(tv)$ for $t \in (a, b)$. Hence, there exists a sequence of vectors $z_i \rightarrow o$ as $i \rightarrow \infty$ with $\langle z_i, v \rangle = 0$ such that the p -planes $h_{y+z_i,d}: \mathbb{R}^n \rightarrow \mathbb{R}_+$ have the following property. For $x \in [o, v]$, we have $h_{y+z_i,d}(x) = h_{y,d}(x) = \text{co}_p(x)$. Moreover, the set $S_i = \{x \in \text{supp}(f) : f(x) \geq h_{y+z_i,d}(x)\}$ satisfies $S_i \subset ([o, av] + B(o, 1/i)) \cup ([bv, v] + B(o, 1/i))$. Furthermore, $f(av) = \text{co}_p(f)(av) = h_{y,d}(av) = h_{y+z_i,d}(av)$.

There exists $c \in (a, b)$ sufficiently close to b such that $\text{co}_p(f)(av) > 2^{-1}m \text{co}_p(f)(cv)$.

Partition the space in three regions $\mathbb{R}^n = H \sqcup H^+ \sqcup H^-$. Here $H = \{x : \langle x, y \rangle = c < \langle v, y \rangle\}$, $H^+ = \{x : \text{sign}(p) \langle x, y \rangle > \text{sign}(p)c < \langle v, y \rangle\}$ and $H^- = \{x : \text{sign}(p) \langle x, y \rangle < \text{sign}(p)c < \langle v, y \rangle\}$. Recalling $\text{sign}(p) \langle v, y \rangle < 0$, we note $cv \in H$, $[o, av] \in H^+$ and $[bv, v] \in H^-$. Moreover, for i sufficiently large, $[o, av] + B(o, 1/i) \subset H^+$ and $[bv, v] + B(o, 1/i) \subset H^-$.

For $s \in \mathbb{R}_+$, construct the p -plane $h_s = h_{y+z_i+sy, d-s\langle v, y \rangle}$. Note that for $x \in H$, we have $h_s(x) = h_{y+z_i,d}(x)$; for $x \in H^+$, we have $h_s(x) > h_{y+z_i,d}(x)$ and for $x \in H^-$, we have $h_s(x) \leq h_{y+z_i,d}(x)$ (actually $h_s(x) < h_{y+z_i,d}(x)$ provided $h_{y+z_i,d} > 0$). Moreover, for $x \in H^+$, we have $h_s(x) \rightarrow \infty$ increasing as $s \rightarrow \infty$.

Let $s_i \in \mathbb{R}_+$ and $p_i \in [o, av] + B(o, 1/i)$ be such that $h_{s_i}(p_i) = f(p_i)$ and for every $p \in [o, av] + B(o, 1/i)$, $h_{s_i}(p) \geq f(p)$.

Remark 9.25. $f'_i = \min(f, h_{s_i})$ is continuous and $f' \neq f$

Proof of Remark. This follows from the fact that

$$\text{co}_p\left(\frac{b+c}{2}v\right) = h_{y+z_i,d}\left(\frac{b+c}{2}v\right) > h_{s_i}\left(\frac{b+c}{2}v\right)$$

as $\frac{b+c}{2}v \in H^-$ and $h_{y+z_i,d}\left(\frac{b+c}{2}v\right) = \text{co}_p(f)\left(\frac{b+c}{2}v\right) > 0$. \square

Claim 9.26. For i large enough, the p -concave continuous functions $h_{s_i}: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ have the following properties

- $h_{s_i}(p_i) = f(p_i) > 2^{-1}f(av) > 0$, and
- if $f(q) > h_{s_i}(q)$, then $f(p_i) \geq 8^{-1}m \text{co}_p(f)(q) \geq 8^{-1}mf(q)$.

Proof of Claim. The first part follows from the fact that f is continuous, $f(av) > 0$ and $p_i \in \overline{[o, av] + B(o, 1/i)}$.

The second part follows in several steps. First, note that $q \in H^-$. Indeed, by construction, if $q \in \overline{[o, av] + B(o, 1/i)}$, then $f(q) \leq h_{s_i}(q)$, which is a contradiction. If $q \in \overline{[bv, v] + B(o, 1/i)}$, then $q \in H^-$ as noted above. If $q \in (H \cup H^+) \cap ([0, av] + B(o, 1/i))^c \cap ([bv, v] + B(o, 1/i))^c$, we have $H_{s_i}(q) \geq H_{y+z_i, d}(q) \geq f(q)$, which is a contradiction.

Second note that $h_{s_i}(cv) \geq h_{s_i}(q)$ by the definition of h_{s_i} and the fact that $\text{sign}(p) < v, y > 0$.

Third, note that $h_{s_i}(p_i) \geq 2^{-1}f(av) = 2^{-1}\text{co}_p(f)(av) \geq 4^{-1}m\text{co}_p(f)(cv) = 4^{-1}h_{y+z_i, d}(cv) = 4^{-1}h_{s_i}(cv)$. Here, the first relation holds by the first part, the second and third relations hold by hypothesis. The last two relations hold by construction.

Combining the second and third observations, we get $h_{s_i}(p_i) \geq 4^{-1}mh_{s_i}(q)$.

To conclude the second part, it would be enough to argue that $h_{s_i}(p_i) \geq 4^{-1}m(\text{co}_p(f)(q) - h_{s_i}(q))$. Actually, by the first part, it is enough to show $f(av) \geq 2^{-1}m(\text{co}_p(f)(q) - h_{s_i}(q))$.

The last part follows from the fact that as $i \rightarrow \infty$, we have $z_i \rightarrow 0$ and $[o, av] + B(o, 1/i) \rightarrow [o, av]$. These force $s_i \rightarrow 0$. Thus $h_{s_i} \rightarrow h_{y, d}$ and $\text{co}_p(f) \leq h_{y, d}$. This concludes the claim. \square

Claim 9.27. *If $p \in (-1/n, 1]$ and i is sufficiently large, then*

$$\int_{\mathbb{R}^n} (M_{\lambda, p}^*(f, f) - M_{\lambda, p}^*(f'_i, f'_i)) dx \geq (1+c) \int_{\mathbb{R}^n} (f - f'_i) dx.$$

Proof of Claim. After translating, we can assume wlog $p_i = o$. Let $A_i = \{x: f(x) > h_{s_i}(x)\}$ be an open set. We have

$$\int_{\mathbb{R}^n} (f - f'_i) dx = \int_A (f - h_{s_i}) dx.$$

As h_{s_i} is a p - plane, we also have

$$\int_{\mathbb{R}^n} (M_{\lambda, p}^*(f, f) - M_{\lambda, p}^*(f'_i, f'_i)) dx \geq \int_{\mathbb{R}^n} (M_{\lambda, p}^*(f, f) - M_{\lambda, p}^*(h_{s_i}, h_{s_i})) dx = \int_{\mathbb{R}^n} (M_{\lambda, p}^*(f, f) - h_{s_i}) dx.$$

By the previous claim we have $h_{s_i}(o) = f(o)$, so we further get

$$\int_{(1-\lambda)A} (M_{\lambda, p}^*(f, f) - h_{s_i}) dx \geq (1-\lambda)^n \int_A (M_{\lambda, p}(h_{s_i}(o), f(x)) - h_{s_i}((1-\lambda)x)) dx$$

It is enough to argue that for i sufficiently large and for every $x \in A$ we have

$$M_{\lambda, p}(h_{s_i}(o), f(x)) - h_{s_i}((1-\lambda)x) \geq (1+c)(1-\lambda)^{-n}(f(x) - h_{s_i}(x))$$

By the previous claim, when i is sufficiently large, we have $h_{s_i}(o) > 8^{-1}mf(x)$. Denoting $a = h_{s_i}(o)$, $b = h_{s_i}(x)$ and $\mu = f(x) - h_{s_i}(x)$, we have $a > 8^{-1}m(\mu + b) > 0$ and the above inequality can be rewritten as

$$(1+c)(1-\lambda)^{-n}\mu \leq M_{\lambda, p}(a, b+\mu) - M_{\lambda, p}(a, b).$$

Consider now the case $p \in (-1/n, 0)$. In this case both $a, b > 0$. If we let $r(x) = M_{\lambda, p}(a, x)$, then $r'(x) = (1-\lambda) \left(\lambda(a/x)^p + (1-\lambda) \right)^{\frac{1-p}{p}}$. Hence, there exist $0 \leq \nu \leq \mu$ such that

$$M_{\lambda, p}(a, b+\mu) - M_{\lambda, p}(a, b) = \mu r'(b+\nu) = \mu(1-\lambda) \left(\lambda(a/(b+\nu))^p + (1-\lambda) \right)^{\frac{1-p}{p}}.$$

By the conditions on a, b, μ , we have

$$M_{\lambda, p}(a, b+\mu) - M_{\lambda, p}(a, b) \geq \mu(1-\lambda) \left(\lambda(m/10)^p + (1-\lambda) \right)^{\frac{1-p}{p}} \geq (1+c)(1-\lambda)^{-n}\mu.$$

This concludes the proof in the case $p \in (-1/n, 0)$. Consider now the case $p = 0$. In this case both $a, b > 0$. If we let $r(x) = M_{\lambda, p}(a, x)$, then $r'(x) = (1-\lambda)a^\lambda x^{-\lambda}$. Hence, there exist $0 \leq \nu \leq \mu$ such that

$$M_{\lambda, p}(a, b+\mu) - M_{\lambda, p}(a, b) = \mu r'(b+\nu) = \mu(1-\lambda)a^\lambda(b+\nu)^{-\lambda}.$$

By the conditions on a, b, μ , we have

$$M_{\lambda,p}(a, b + \mu) - M_{\lambda,p}(a, b) \geq \mu(1 - \lambda)(m/10)^\lambda \geq (1 + c)(1 - \lambda)^{-n}\mu.$$

This concludes the case $p = 0$. Consider now the case $p \in (0, 1]$. The important observation is that $A \cap (1 - \lambda)A = \emptyset$.

Indeed, assume for a contradiction there exists $x \in A \cap (1 - \lambda)A$. By the previous claim we have $\text{co}_p(f)(o) > 8^{-1}m \text{co}_p(f)(x) > 0$ and $\text{co}_p(f)(o) > 8^{-1}m \text{co}_p(f)((1 - \lambda)x) > 0$. However, as $\text{co}_p(f)$ is p -concave, we deduce that

$$\text{co}_p(f)((1 - \lambda)x) \geq (\lambda(\text{co}_p(f)(o))^p + (1 - \lambda)(\text{co}_p(f)(x))^p)^{1/p} > \lambda^{1/p} \text{co}_p(f)(o).$$

This gives the desired contradiction. We conclude the observation.

As $M_{\lambda,p}^*(f, f) \geq f$, we have

$$\int_A (M_{\lambda,p}^*(f, f) - h_{s_i}) dx \geq \int_A (f - h_{s_i}) dx.$$

Combining this last inequality with the first three centered inequalities and the fact that $A \cap (1 - \lambda)A = \emptyset$, it is enough to show that for i sufficiently large and for every $x \in A$, we have

$$M_{\lambda,p}(h_{s_i}(o), f(x)) - h_{s_i}((1 - \lambda)x) \geq c(1 - \lambda)^{-n}(f(x) - h_{s_i}(x))$$

By the previous claim, when i is sufficiently large, we have $h_{s_i}(o) > 8^{-1}mf(x)$. Denoting $a = h_{s_i}(o)$, $b = h_{s_i}(x)$ and $\mu = f(x) - h_{s_i}(x)$, we have $a > 8^{-1}m(\mu + b) > 0$ and the above inequality can be rewritten as

$$c(1 - \lambda)^{-n}\mu \leq M_{\lambda,p}(a, b + \mu) - M_{\lambda,p}(a, b).$$

We can assume wlog both $a, b > 0$. If we let $r(x) = M_{\lambda,p}(a, x)$, then $r'(x) = (1 - \lambda) \left(\lambda(a/x)^p + (1 - \lambda) \right)^{\frac{1-p}{p}}$. Hence, there exist $0 \leq \nu \leq \mu$ such that

$$M_{\lambda,p}(a, b + \mu) - M_{\lambda,p}(a, b) = \mu r'(b + \nu) = \mu(1 - \lambda) \left(\lambda(a/(b + \nu))^p + (1 - \lambda) \right)^{\frac{1-p}{p}}.$$

By the conditions on a, b, μ , we have

$$M_{\lambda,p}(a, b + \mu) - M_{\lambda,p}(a, b) \geq \mu(1 - \lambda)^p \geq c(1 - \lambda)^{-n}\mu.$$

This concludes the proof in the case $p \in (0, 1]$. □

Now redefine $c \in (a, b)$ such that $\text{co}_p(cv) > 2^{-1}m \text{co}_p(bv)$. For $s \in \mathbb{R}_-$ construct the p -plane $h_s = h_{y+z_i+s y, d-sc < v, y >}$. Redefine $s_i \in \mathbb{R}_-$ and $p_i \in \overline{[bv, v] + B(o, 1/i)}$ such that $h_{s_i}(p_i) = f(p_i)$ and for every $p \in \overline{[bv, v] + B(o, 1/i)}$, we have $h_{s_i}(p) \geq f(p)$.

By an argument identical to the one in the case $p \in (0, 1)$, the following holds.

Remark 9.28. $f'_i = \min(f, h_{s_i})$ is continuous and $f' \neq f$

Claim 9.29. If $p > 1$, i is sufficiently large, then $\int_{\mathbb{R}^n} (M_{\lambda,p}^*(f, f) - M_{\lambda,p}^*(f'_i, f'_i)) dx \geq (1 + c) \int_{\mathbb{R}^n} (f - f'_i) dx$

The last remark and the last claim combine to give the desired contradiction in the case $p > 1$. The previous remark and the previous claim combine to give the desired contradiction in the case $p \in (-1/n, 1]$. The conclusion of the proposition now follows. □

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