TRANSPORT MAPS FOR β-MATRIX MODELS AND UNIVERSALITY

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Abstract. We construct approximate transport maps for non-critical β-matrix models, that is, maps so that the push forward of a non-critical β-matrix model with a given potential is a non-critical β-matrix model with another potential, up to a small error in the total variation distance. One of the main features of our construction is that these maps enjoy regularity estimates which are uniform in the dimension. In addition, we find a very useful asymptotic expansion for such maps which allow us to deduce that local statistics have the same asymptotic behavior for both models.

1. Introduction.

Given a potential $V : \mathbb{R} \to \mathbb{R}$ and $\beta > 0$, we consider the β-matrix model

$$\mathbb{P}_V^N(d\lambda_1, \dots, d\lambda_N) := \frac{1}{Z_V^N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta e^{-N \sum_{i=1}^N V(\lambda_i)}\, d\lambda_1 \dots d\lambda_N, \tag{1.1}$$

where $Z_V^N := \int \prod_{1 \le i<j \le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)}\, d\lambda_1 \dots d\lambda_N$. It is well known (see e.g. [Vil03]) that for any $V, W : \mathbb{R} \to \mathbb{R}$ such that $Z_V^N, Z_{V+W}^N < \infty$ there exist maps $T^N : \mathbb{R}^N \to \mathbb{R}^N$ that transport $\mathbb{P}_V^N$ into $\mathbb{P}_{V+W}^N$, that is, for any bounded measurable function $f : \mathbb{R}^N \to \mathbb{R}$ we have

$$\int f \circ T^N(\lambda_1, \dots, \lambda_N)\, \mathbb{P}_V^N(d\lambda_1, \dots, d\lambda_N) = \int f(\lambda_1, \dots, \lambda_N)\, \mathbb{P}_{V+W}^N(d\lambda_1, \dots, d\lambda_N)\,.$$

However, the dependency in the dimension N of this transport map is in general unclear unless one makes very strong assumptions on the densities [Caf00] that unfortunately are never satisfied in our situation. Hence, it seems extremely difficult to use these maps T^N to understand the relation between the asymptotic of the two models.

The main contribution of this paper is to show that a variant of this approach is indeed possible and provides a very robust and flexible method to compare the asymptotics of local statistics. In the more general context of several-matrices models, it was shown in [GS14] that the maps T^N are asymptotically well approximated by a function of matrices independent of N, but it was left open the question of studying corrections to this limit. In this article we consider one-matrix models, and more precisely their generalization given by β-models, and we construct *approximate* transport maps with a very precise dependence on the dimension. This allows us to compare local fluctuations of the eigenvalues and show

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universality.

Local fluctuations were first studied in the case $\beta = 2$. After the work of Mehta (see [Meh04]) where the sine kernel law was exhibited for the GUE, Pastur and Shcherbina [PS97] proved universality for $V \in C^3$. The Riemann Hilbert approach was then developed in this context in [DKM+99] for analytic potentials. The paper [LL08] provides the most general result for universality in the case $\beta = 2$.

In the case $\beta = 1, 4$, universality was proved in [DG07b] in the bulk and in [DG07a] at the edge for monomials V (see [DG09] for a review). For general one-cut potentials, the first proof of universality was given in [Shc09] in the case $\beta = 1$, whereas [KS10] treated the case $\beta = 4$ in the one-cut case. The multi-cut case was then considered in [Shc12].

The local fluctuations of more general β-ensembles were only derived recently [VV09, RRV11] in the Gaussian case. Universality in the β-ensembles was first addressed in [BEY14a] (in the bulk, $\beta > 0$, $V \in C^4$), then in [BEY14b] (at the edge, $\beta \geq 1$, $V \in C^4$) and [KRV13] (at the edge, $\beta > 0$, V convex polynomial) and finally in [Shc13] (in the bulk, $\beta > 0$, V analytic, multi-cut case included).

The paper [Shc13] and the current one were completed essentially at the same time, and the method developed in [Shc13] is closely related with the one followed in the present work in the sense that it uses a transport map, but still they are very different. Indeed, [Shc13] uses a global transport map $T^N = T_0^{\otimes N}$, independent of the dimension, which maps the probability $\mathbb{P}_V^N$ to an intermediate probability measure $\tilde{\mathbb{P}}_{V+W}^N$ which is absolutely continuous with respect to $\mathbb{P}_{V+W}^N$ but with a non-trivial density, and then shows that this density does not perturb the local fluctuations of local statistics. On the other hand, in the present paper the map T^N is constructed to map $\mathbb{P}_V^N$ to an intermediate probability measure $\tilde{\mathbb{P}}_{V+W}^N$ with density with respect to $\mathbb{P}_{V+W}^N$ going uniformly to one: the dependence of T_N in the dimension N then allows us to retrieve the universality of the local fluctuations.

The main idea of this paper is that we can build a smooth approximate transport map which at the first order is simply a tensor product, and then it has first order corrections of order N^{-1} (see Theorem 1.4). This continues the long standing idea developed in loop equations theory to study the correlation functions of β-matrix models by clever change of variables, see e.g. [Joh98, KS10, AM90, BEMPF12]. On the contrary to loop equations, the change of variable has to be of order one rather than infinitesimal.

The first order term of our map is simply the monotone transport map between the asymptotic equilibrium measures; then, corrections to this first order are constructed so that densities match up to a priori smaller fluctuating terms. As we shall see, our transport map is constructed as the flow of a vector field obtained by approximately solving a linearized Jacobian equation and then making a suitable ansatz on the structure of the vector field. The errors are controlled by deriving bounds on covariances and correlations functions thanks to loop equations, allowing us to obtain a self-contained proof of universality (see Theorem 1.5). Although optimal transportation will never be really used, it will provide us the correct intuition in order to solve this problem (see Sections 2.2 and 2.3).

We notice that this last step could also be performed either by using directly central limit theorems, see e.g. [Shc13, BG13a, Shc12], or local limit laws [BEY14a, BEY14b]. However, our approach has the advantage of being pretty robust and should generalize to many other mean field models. In particular, it is possible to generalize it to the several-matrices models, at least in the perturbative regime considered in [GS14], and to show that non-commutative functions of several GUE matrices close to identity have universal local statistics [FG14].

We now describe our results in detail. Given a potential $V : \mathbb{R} \to \mathbb{R}$, we consider the probability measure (1.1). We assume that V goes to infinity faster than $\log|x|$ (that is $V(x)/\log|x| \to +\infty$ as $|x| \to +\infty$) so that in particular Z_V^N is finite.

We will use μ_V to denote the equilibrium measure, which is obtained as limit of the spectral measure and is characterized as the unique minimizer (among probability measures) of

$$I_V(\mu) := \frac{1}{2}\int \big(V(x) + V(y) - \beta \log|x-y|\big) d\mu(x) d\mu(y)\,. \tag{1.2}$$

We assume hereafter that another smooth potential W is given so that $V+W$ goes to infinity faster than $\beta \log|x|$. We set $V_t := V + tW$, and we shall make the following assumption:

Hypothesis 1.1. *We assume that μ_{V_0} and μ_{V_1} have a connected support and are non-critical, that is, there exists a constant $\bar{c} > 0$ such that, for $t = 0, 1$,*

$$\frac{d\mu_{V_t}}{dx} = S_t(x)\sqrt{(x-a_t)(b_t-x)} \qquad \text{with } S_t \geq \bar{c} \text{ a.e. on } [a_t, b_t].$$

Remark 1.2. The assumption of a connected support could be removed here, following the lines of [Shc13, BG13b]. However, the non-criticality assumption cannot be removed easily, as criticality would result in singularities in the transport map.

Finally, we assume that the eigenvalues stay in a neighborhood of the support $[a_t-\epsilon, b_t+\epsilon]$ with large enough $\mathbb{P}_V^N$-probability, that is with probability greater than $1 - C\,N^{-p}$ for some p large enough. By [BG13b, Lemma 3.1], the latter is fulfilled as soon as:

Hypothesis 1.3. *For $t = 0, 1$, the function*

$$x \mapsto U_{V_t}(x) := V_t(x) - \beta \int d\mu_{V_t}(y) \log|x-y| \tag{1.3}$$

achieves its minimal value on $[a,b]^c$ at its boundary $\{a, b\}$

All these assumptions are verified for instance if V_t is uniformly convex for $t = 0, 1$.

The main goal of this article is to build an approximate transport map between $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$: more precisely, we construct a map $T^N : \mathbb{R}^N \to \mathbb{R}^N$ such that, for any bounded measurable function χ,

$$\left|\int \chi \circ T^N \, d\mathbb{P}_V^N - \int \chi \, d\mathbb{P}_{V+W}^N\right| \leq C\,\frac{(\log N)^3}{N}\|\chi\|_\infty \tag{1.4}$$

for some constant C independent of N, and which has a very precise expansion in the dimension (in the following result, $T_0 : \mathbb{R} \to \mathbb{R}$ is the monotone transport of μ_V onto μ_{V+W}, see Section 4):

Theorem 1.4. *Assume that V, W are of class C^{31} and satisfy Hypotheses 1.1 and 1.3. Then there exists a map $T^N = (T^{N,1}, \ldots, T^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ which satisfies (1.4) and has the form*

$$T^{N,i}(\hat{\lambda}) = T_0(\lambda_i) + \frac{1}{N}T_1^{N,i}(\hat{\lambda}) \quad \forall\, i = 1, \ldots, N, \qquad \hat{\lambda} := (\lambda_1, \ldots, \lambda_N),$$

where $T_0 : \mathbb{R} \to \mathbb{R}$ *and* $T_1^{N,i} : \mathbb{R}^N \to \mathbb{R}$ *are smooth and satisfy uniform (in* N*) regularity estimates. More precisely,* T^N *is of class* C^{23} *and we have the decomposition* $T_1^{N,i} = X_1^{N,i} + \frac{1}{N} X_2^{N,i}$ *where*

$$\sup_{1\le k\le N} \|X_1^{N,k}\|_{L^4(\mathbb{P}_V^N)} \le C \log N, \qquad \|X_2^N\|_{L^2(\mathbb{P}_V^N)} \le C N^{1/2} (\log N)^2, \tag{1.5}$$

for some constant $C > 0$ *independent of* N*. In addition, with probability greater than* $1 - N^{-N/C}$*, we have*

$$\max_{1\le k,k'\le N} |X_1^{N,k}(\hat\lambda) - X_1^{N,k'}(\hat\lambda)| \le C \log N \sqrt{N} |\lambda_k - \lambda_{k'}|. \tag{1.6}$$

As we shall see in Section 5, this result implies universality as follows (compare with [BEY14b, Theorem 2.4]):

Theorem 1.5. *Assume* $V, W \in C^{31}$*, and let* T_0 *be as in Theorem 1.4 above. Denote* $\tilde P_V^N$ *the distribution of the increasingly ordered eigenvalues* λ_i *under* $\mathbb{P}_V^N$*. There exists a constant* $\hat C > 0$*, independent of* N*, such that the following two facts hold true:*

(1) *Let* $M \in (0,\infty)$ *and* $m \in \mathbb{N}$*. Then, for any Lipschitz function* $f : \mathbb{R}^m \to \mathbb{R}$ *supported inside* $[-M, M]^m$*,*

$$\begin{aligned} \Bigg| \int f\big(N(\lambda_{i+1}-\lambda_i), \dots, N(\lambda_{i+m}-\lambda_i)\big)\, d\tilde P_{V+W}^N \\ - \int f\big(T_0'(\lambda_i) N(\lambda_{i+1}-\lambda_i), \dots, T_0'(\lambda_i) N(\lambda_{i+m}-\lambda_i)\big) d\tilde P_V^N \Bigg| \\ \le \hat C \frac{(\log N)^3}{N} \|f\|_\infty + \hat C \left(\sqrt{m} \frac{(\log N)^2}{N^{1/2}} + M \frac{\log N}{N^{1/2}} + \frac{M^2}{N} \right) \|\nabla f\|_\infty. \end{aligned}$$

(2) *Let* a_V *(resp.* a_{V+W}*) denote the smallest point in the support of* μ_V *(resp.* μ_{V+W}*), so that* $\mathrm{supp}(\mu_V) \subset [a_V, \infty)$ *(resp.* $\mathrm{supp}(\mu_{V+W}) \subset [a_{V+W}, \infty)$*), and let* $M \in (0,\infty)$*. Then, for any Lipschitz function* $f : \mathbb{R}^m \to \mathbb{R}$ *supported inside* $[-M, M]^m$*,*

$$\begin{aligned} \Bigg| \int f\big(N^{2/3}(\lambda_1 - a_{V+W}), \dots, N^{2/3}(\lambda_m - a_{V+W})\big)\, d\tilde P_{V+W}^N \\ - \int f\Big(N^{2/3} T_0'(a_V)\big(\lambda_1 - a_V\big), \dots, N^{2/3} T_0'(a_V)\big(\lambda_m - a_V\big)\Big)\, d\tilde P_V^N \Bigg| \\ \le \hat C \frac{(\log N)^3}{N} \|f\|_\infty + \hat C \Big(\sqrt{m} \frac{(\log N)^2}{N^{5/6}} + M \frac{\log N}{N^{5/6}} + \frac{M^2}{N^{4/3}} + \frac{\log N}{N^{1/3}} \Big) \|\nabla f\|_\infty. \end{aligned}$$

The same bound holds around the largest point in the support of μ_V*.*

Remark 1.6. The condition that $V, W \in C^{31}$ in the theorem above is clearly non-optimal (compare with [BEY14a]). For instance, by using Stieltjes transform instead of Fourier transform in some of our estimates, we could reduce the regularity assumptions on V, W to C^{22} by a slightly more cumbersome proof. In addition, by using [BEY14b, Theorem 2.4] we could also weaken our regularity assumptions in Theorem 1.4, as we could use that result to estimate the error terms in Section 3.4. However, the main point of this hypothesis for us is to stress that we do not need analytic potentials, as often required in matrix models theory. Moreover, under this assumption we can provide self-contained and short proofs of Theorems 1.4 and 1.5.

Note that Theorem 1.4 is well suited to prove universality of the spacings distribution in the bulk as stated in Theorem 1.5, but it is not clear how to directly deduce the universality of the rescaled density, see e.g [BEY14a, Theorem 2.5(i)]. Indeed, this corresponds to choosing test functions whose uniform norm blows up like some power of the dimension, so to apply Theorem 1.4 we should have an a priori control on the numbers of eigenvalues inside sets of size of order N^{-1} under both $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$. Notice however that [BEY14a, Theorem 2.5(ii)] requires $\beta \geq 1$, while our results hold for any $\beta > 0$. Also, [KRV13] holds for $\beta > 0$ but only for convex polynomial. In particular, the edge universality proved in Theorem 1.5(2) is new for $\beta \in (0,1)$ and $V \in C^{31}$ which is not a polynomial. In addition our strategy is very robust and flexible. For instance, although we shall not pursue this direction here, it looks likely to us that one could use it prove the universality of the asymptotics of the law of $\{N(\lambda_i - x)\}_{1\leq i\leq m}$ under $\tilde{P}_V^N$ for given i and x, as in [BYY14a, BYY14b]. Indeed, $N(\lambda_i - x)$ can be seen via the approximate transport map to be given by the same kind of difference under the source measure (e.g. the Gaussian one), up to a term coming from the transport map $T_1^{N,i}$. Proving that $T_1^{N,i}$ does not depend on i locally would suffice to prove universality, as the sine-kernel law is translation invariant.

The paper is structured as follows: In Section 2 we describe the general strategy to construct our transport map as the flow of vector fields obtained by approximately solving a linearization of the Monge-Ampère equation (see (2.2)). As we shall explain there, this idea comes from optimal transport theory. In Section 3 we make an ansatz on the structure of an approximate solution to (2.2) and we show that our ansatz actually provides a smooth solution which enjoys very nice regularity estimates that are uniform as $N \to \infty$. In Section 4 we reconstruct the approximate transport map from $\mathbb{P}_V^N$ to $\mathbb{P}_{V+W}^N$ via a flow argument. The estimates proved in this section will be crucial in Section 5 to show universality.

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2. Approximate Monge-Ampère equation

2.1. **Propagating the hypotheses.** The central idea of the paper is to build transport maps as flows, and in fact to build transport maps between $\mathbb{P}_V^N$ and $\mathbb{P}_{V_t}^N$ where $t \mapsto V_t$ is a smooth function so that $V_0 = V$ and $V_1 = V + W$. In order to have a good interpolation between V and $V + W$, it will be convenient to assume that the supports of the two equilibrium measures μ_V and μ_{V+W} (see (1.2)) are the same. This can always be done up to an affine transformation. Indeed, if $L : \mathbb{R} \to \mathbb{R}$ is the affine transformation which maps $[a_1, b_1]$ (the support of μ_{V_1}) onto $[a_0, b_0]$ (the support of μ_{V_0}), we first construct a transport map from $\mathbb{P}_V^N$ to $L_\#^{\otimes N}\mathbb{P}_{V+W}^N = \mathbb{P}_N^{V+\tilde{W}}$ where

$$\tilde{W} = V \circ L^{-1} + W \circ L^{-1} - V, \tag{2.1}$$

and then we simply compose our transport map with $(L^{-1})^{\otimes N}$ to get the desired map from $\mathbb{P}_V^N$ to $\mathbb{P}_{V+W}^N$. Hence, without loss of generality we will hereafter assume that μ_V and μ_{V+W} have the same support. We then consider the interpolation μ_{V_t} with $V_t = V + tW$, $t \in [0, 1]$. We have:

Lemma 2.1. *If Hypotheses 1.1 and 1.3 are fulfilled for $t = 0, 1$, then Hypothesis 1.1 is also fulfilled for all $t \in [0, 1]$. Moreover, we may assume without loss of generality that V goes to infinity as fast as we want up to modify $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$ by a negligible error (in total variation).*

Proof. Let Σ denote the support of μ_V and μ_{V+W}. Following [BG13a, Lemma 5.1], the measure μ_{V_t} is simply given by

$$\mu_{V_t} = (1-t)\mu_V + t\mu_{V+W}.$$

Indeed, μ_V is uniquely determined by the fact that there exists a constant c such that

$$\beta \int \log|x-y| d\mu_V(x) - V \le c$$

with equality on the support of μ_V, and this property extends to linear combinations. As a consequence the support of μ_{V_t} is Σ, and its density is bounded away from zero on Σ. This shows that Hypothesis 1.1 is fulfilled for all $t \in [0, 1]$.

Furthermore, we can modify $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$ outside an open neighborhood of Σ without changing the final result, as eigenvalues will quit this neighborhood only with very small probability under our assumption of non-criticality according to the large deviation estimates developped in [ADG01, BG13a] and culminating in [BG13b] as follows:

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}_N^V \left[\exists\, i \,:\, \lambda_i \in F\right] \le -\frac{\beta}{2} \inf_{x\in F} \tilde{U}_V(x),$$
$$\liminf_{N\to\infty} \frac{1}{N} \ln \mathbb{P}_N^V \left[\exists\, i \,:\, \lambda_i \in \Omega\right] \ge -\frac{\beta}{2} \inf_{x\in \Omega} \tilde{U}_V(x).$$

where $\tilde{U}_V := U_V - \inf U_V$, and U_V is defined as in (1.3). □

Thanks to the above lemma and the discussion immediately before it, we can assume that μ_V and μ_{V+W} have the same support, that W is bounded, and that V goes to infinity faster than x^p for some $p > 0$ large enough.

2.2. **Monge-Ampère equation.** Given the two probability densities $\mathbb{P}_{V_t}^N$ to $\mathbb{P}_{V_s}^N$ as in (1.1) with $0 \le t \le s \le 1$, by optimal transport theory it is well-known that there exists a (convex) function $\phi_{t,s}^N$ such that $\nabla\phi_{t,s}^N$ pushes forward $\mathbb{P}_{V_t}^N$ onto $\mathbb{P}_{V_s}^N$ and which satisfies the Monge-Ampère equation

$$\det(D^2\phi_{t,s}^N) = \frac{\rho_t}{\rho_s(\nabla\phi_{t,s}^N)}, \qquad \rho_\tau := \frac{d\mathbb{P}_{V_\tau}^N}{d\lambda_1 \ldots d\lambda_N}$$

(see for instance [Vil03, Chapters 3 and 4] or the recent survey paper [DPF14] for an account on optimal transport theory and its link to the Monge-Ampère equation).

Because $\phi_{t,t}(x) = |x|^2/2$ (since $\nabla\phi_{t,t}$ is the identity map), we differentiate the above equation with respect to s and set $s = t$ to get

$$\Delta\psi_t^N = c_t^N - \beta \sum_{i<j} \frac{\partial_i\psi_t^N - \partial_j\psi_t^N}{\lambda_i - \lambda_j} + N \sum_i W(\lambda_i) + N \sum_i V_t'(\lambda_i)\partial_i\psi_t^N, \tag{2.2}$$

where $\psi_t^N := \partial_s \phi_{t,s}^N|_{s=t}$ and

$$c_t^N := -N \int \sum_i W(\lambda_i)\, d\mathbb{P}_{V_t}^N = \partial_t \log Z_{V_t}^N .$$

Although this is a formal argument, it suggests to us a way to construct maps $T_{0,t}^N : \mathbb{R}^N \to \mathbb{R}^N$ sending $\mathbb{P}_V^N$ onto $\mathbb{P}_{V_t}^N$: indeed, if $T_{0,t}^N$ sends $\mathbb{P}_V^N$ onto $\mathbb{P}_{V_t}^N$ then $\nabla\phi_{t,s}^N \circ T_{0,t}^N$ sends $\mathbb{P}_V^N$ onto $\mathbb{P}_{V_s}^N$. Hence, we may try to find $T_{0,s}^N$ of the form $T_{0,s}^N = \nabla\phi_{t,s}^N \circ T_{0,t}^N + o(s-t)$. By differentiating this relation with respect to s and setting $s = t$ we obtain $\partial_t T_{0,t}^N = \nabla\psi_t^N(T_{0,t}^N)$.

Thus, to construct a transport map T^N from $\mathbb{P}_V^N$ onto $\mathbb{P}_{V+W}^N$ we could first try to find ψ_t^N by solving (2.2), then solve the ODE $\dot X_t^N = \nabla\psi_t^N(X_t^N)$, and finally set $T^N := X_1^N$. We notice that, in general, T^N is not an optimal transport map for the quadratic cost.

Unfortunately, finding an exact solution of (2.2) enjoying "nice" regularity estimates that are uniform in N seems extremely difficult. So, instead, we make an ansatz on the structure of ψ_t^N (see (3.2) below): the idea is that at first order eigenvalues do not interact, then at order $1/N$ eigenvalues interact at most by pairs, and so on. As we shall see, in order to construct a function which enjoys nice regularity estimates and satisfies (2.2) up to a error that goes to zero as $N \to \infty$, it will be enough to stop the expansion at $1/N$. Actually, while the argument before provides us the right intuition, we notice that there is no need to assume that the vector field generating the flow X_t^N is a gradient, so we will consider general vector fields $\mathbf{Y}_t^N = (\mathbf{Y}_{1,t}^N, \dots, \mathbf{Y}_{N,t}^N) : \mathbb{R}^N \to \mathbb{R}^N$ that approximately solve

$$\text{div}\mathbf{Y}_t^N = c_t^N - \beta \sum_{i<j} \frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N \sum_i W(\lambda_i) + N \sum_i V_t'(\lambda_i)\mathbf{Y}_{i,t}^N, \tag{2.3}$$

We begin by checking that the flow of an approximate solution of (2.3) gives an approximate transport map.

2.3. **Approximate Jacobian equation.** Here we show that if a C^1 vector field $\mathbf{Y}_t^N$ approximately satisfies (2.3), then its flow

$$\dot X_t^N = \mathbf{Y}_t^N(X_t^N), \qquad X_0^N = \text{Id},$$

produces almost a transport map.

More precisely, let $\mathbf{Y}_t^N : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth vector field and denote

$$\mathcal{R}_t^N(\mathbf{Y}^N) := c_t^N - \beta \sum_{i<j} \frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N \sum_i W(\lambda_i) + N \sum_i V_t'(\lambda_i)\mathbf{Y}_{i,t}^N - \text{div}\mathbf{Y}_t^N.$$

Lemma 2.2. *Let $\chi : \mathbb{R}^N \to \mathbb{R}$ be a bounded measurable function, and let X_t^N be the flow of $\mathbf{Y}_t^N$. Then*

$$\left| \int \chi(X_t^N)\, d\mathbb{P}_V^N - \int \chi\, d\mathbb{P}_{V_t}^N \right| \le \|\chi\|_\infty \int_0^t \|\mathcal{R}_s^N(\mathbf{Y}^N)\|_{L^1(\mathbb{P}_{V_s}^N)} ds.$$

Proof. Since $\mathbf{Y}_t^N \in C^1$, by Cauchy-Lipschitz Theorem its flow is a bi-Lipchitz homeomorphism.

If JX_t^N denotes the Jacobian of X_t^N and ρ_t the density of $\mathbb{P}_{V_t}^N$, by the change of variable formula it follows that

$$\int \chi\, d\mathbb{P}_{V_t}^N = \int \chi(X_t^N) JX_t^N \rho_t(X_t^N)\, dx$$

thus

$$\left|\int \chi(X_t^N)\, d\mathbb{P}_V^N - \int \chi\, d\mathbb{P}_{V_t}^N\right| \le \|\chi\|_\infty \int |\rho_0 - JX_t^N \rho_t(X_t^N)|\, dx =: \|\chi\|_\infty A_t \tag{2.4}$$

Using that $\partial_t(JX_t^N) = \mathrm{div}\mathbf{Y}_t^N\, JX_t^N$ and that the derivative of the norm is smaller than the norm of the derivative, we get

$$\begin{aligned}
|\partial_t A_t| &\le \int \left|\partial_t\big(JX_t^N \rho_t(X_t^N)\big)\right| dx \\
&= \int |\mathrm{div}\mathbf{Y}_t^N\, JX_t^N\, \rho_t(X_t^N) + JX_t^N\, (\partial_t\rho_t)(X_t^N) + JX_t^N\, \nabla\rho_t(X_t^N)\cdot \partial_t X_t^N|\, dx \\
&= \int |\mathcal{R}_t^N(\mathbf{Y})|(X_t^N)\, JX_t^N\, \rho_t(X_t^N)\, dx \\
&= \int |\mathcal{R}_t^N(\mathbf{Y})|\, d\mathbb{P}_{V_t}^N.
\end{aligned}$$

Integrating the above estimate in time completes the proof. □

By taking the supremum over all functions χ with $\|\chi\|_\infty \le 1$, the lemma above gives:

Corollary 2.3. *Let X_t^N be the flow of $\mathbf{Y}_t^N$, and set $\hat{\mathbb{P}}_t^N := (X_t^N)_\#\mathbb{P}_V^N$ the image of $\mathbb{P}_V^N$ by X_t^N. Then*

$$\|\hat{\mathbb{P}}_t^N - \mathbb{P}_{V_t}^N\|_{TV} \le \int_0^t \|\mathcal{R}_s^N(\mathbf{Y}^N)\|_{L^1(\mathbb{P}_{V_s}^N)} ds.$$

3. Constructing an approximate solution to (2.2)

Fix $t \in [0,1]$ and define the random measures

$$L_N := \frac{1}{N}\sum_i \delta_{\lambda_i} \quad \text{and} \quad M_N := \sum_i \delta_{\lambda_i} - N\mu_{V_t}. \tag{3.1}$$

As we explained in the previous section, a natural ansatz to find an approximate solution of (2.2) is given by

$$\psi_t^N(\lambda_1,\dots,\lambda_N) := \int \Big[\psi_{0,t}(x) + \frac{1}{N}\psi_{1,t}(x)\Big]\, dM_N(x) + \frac{1}{2N}\iint \psi_{2,t}(x,y)\, dM_N(x)\, dM_N(y), \tag{3.2}$$

for some functions $\psi_{0,t}, \psi_{1,t} : \mathbb{R}\to\mathbb{R}$ and $\psi_{2,t} : \mathbb{R}^2 \to \mathbb{R}$, where (without loss of generality) $\psi_{2,t}(x,y) = \psi_{2,t}(y,x)$.

Since we do not want to use gradient of functions but general vector fields (as this gives us more flexibility), in order to find an ansatz for an approximate solution of (2.3) we compute first the gradient of ψ:

$$\partial_i\psi_t^N = \psi_{0,t}'(\lambda_i) + \frac{1}{N}\psi_{1,t}'(\lambda_i) + \frac{1}{N}\xi_{1,t}^N(\lambda_i, M_N), \qquad \xi_{1,t}^N(x, M_N) := \int \partial_1\psi_{2,t}(x,y)\, dM_N(y).$$

This suggests us the following ansatz for the components of $\mathbf{Y}_t^N$:

$$\mathbf{Y}_{i,t}^N(\lambda_1,\dots,\lambda_N) := \mathbf{y}_{0,t}(\lambda_i) + \frac{1}{N}\mathbf{y}_{1,t}(\lambda_i) + \frac{1}{N}\boldsymbol{\xi}_t(\lambda_i, M_N), \quad \boldsymbol{\xi}_t(x, M_N) := \int \mathbf{z}_t(x,y)\, dM_N(y), \tag{3.3}$$

for some functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t} : \mathbb{R}\to\mathbb{R}$, $\mathbf{z}_t : \mathbb{R}^2 \to \mathbb{R}$.

Here and in the following, given a function of two variables ψ, we write $\psi \in C^{s,v}$ to denote that it is s times continuously differentiable with respect to the first variable and v times with respect to the second.

The aim of this section is to prove the following result:

Proposition 3.1. *Assume $V, W \in C^r$ with $r \geq 31$. Then, there exist $\mathbf{y}_{0,t} \in C^{r-3}$, $\mathbf{y}_{1,t} \in C^{r-9}$, and $\mathbf{z}_t \in C^{s,v}$ for $s+v \leq r-6$, such that*

$$\mathcal{R}_t^N := \left(c_t^N - \beta \sum_{i<j} \frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N \sum_i W(\lambda_i) + N \sum_i V_t'(\lambda_i) \mathbf{Y}_{i,t}^N \right) - \operatorname{div} \mathbf{Y}_t^N$$

satisfies

$$\|\mathcal{R}_t^N\|_{L^1(\mathbb{P}_{V_t}^N)} \leq C \, \frac{(\log N)^3}{N}$$

for some positive constant C independent of $t \in [0,1]$.

The proof of this proposition is pretty involved, and will take the rest of the section.

3.1. Finding an equation for $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$.

Using (3.3) we compute

$$\operatorname{div} \mathbf{Y}_t^N = N \int \mathbf{y}_{0,t}'(x) \, dL_N(x) + \int \mathbf{y}_{1,t}'(\lambda) \, dL_N(x) + \int \partial_1 \boldsymbol{\xi}_t(x, M_N) \, dL_N(x) + \boldsymbol{\eta}(L_N),$$

where, given a measure ν, we set

$$\boldsymbol{\eta}(\nu) := \int \partial_2 \mathbf{z}_t(y,y) \, d\nu(y).$$

Therefore, recalling that $M_N = N(L_N - \mu_{V_t})$, we get

$$\begin{aligned}
\mathcal{R}_t^N = &-\frac{\beta N^2}{2} \iint \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x-y} \, dL_N(x) \, dL_N(y) + N^2 \int (V_t'(x) \, \mathbf{y}_{0,t}(x) + W(x)) \, dL_N(x) \\
&- \frac{\beta N}{2} \iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y} \, dL_N(x) \, dL_N(y) + N \int V_t'(x) \, \mathbf{y}_{1,t}(x) \, dL_N(x) \\
&- \frac{\beta N}{2} \iint \frac{\boldsymbol{\xi}_t(x, M_N) - \boldsymbol{\xi}_t(y, M_N)}{x-y} \, dL_N(x) \, dL_N(y) + N \int V_t'(x) \, \boldsymbol{\xi}_t(x, M_N) \, dL_N(x) \\
&- \frac{1}{N} \boldsymbol{\eta}(M_N) - N\Big(1 - \frac{\beta}{2}\Big) \int \mathbf{y}_{0,t}'(x) \, dL_N(x) \\
&- \Big(1 - \frac{\beta}{2}\Big) \int \mathbf{y}_{1,t}'(x) \, dL_N(x) - \Big(1 - \frac{\beta}{2}\Big) \int \partial_1 \boldsymbol{\xi}_t(x, M_N) \, dL_N(x) - \boldsymbol{\eta}(\mu_{V_t}) + \tilde{c}_t^N,
\end{aligned}$$

where $\tilde{c}_t^N$ is a constant and we use the convention that, when we integrate a function of the form $\frac{f(x)-f(y)}{x-y}$ with respect to $L_N \otimes L_N$, the diagonal terms give $f'(x)$.

We now observe that L_N converges towards μ_{V_t} as $N \to \infty$ [dMPS95], see also [AG97, AGZ10] for the corresponding large deviation principle, and the latter minimizes I_{V_t} (see (1.2)). Hence, considering $\mu_\varepsilon := (x + \varepsilon f)_{\#} \mu_{V_t}$ and writing that $I_{V_t}(\mu_\varepsilon) \geq I_{V_t}(\mu_{V_t})$, by taking the derivative with respect to ε at $\varepsilon = 0$ we get

$$\int V_t'(x) f(x) \, d\mu_{V_t}(x) = \frac{\beta}{2} \iint \frac{f(x) - f(y)}{x-y} \, d\mu_{V_t}(x) \, d\mu_{V_t}(y) \tag{3.4}$$

for all smooth bounded functions $f : \mathbb{R} \to \mathbb{R}$.

This fact allows us to recenter L_N by μ_{V_t} in the formula above: more precisely, if we set

$$\Xi_t f(x) := -\beta \int \frac{f(x)-f(y)}{x-y}\, d\mu_{V_t}(y) + V_t'(x) f(x), \tag{3.5}$$

then

$$\begin{aligned} N^2 \int V_t'(x)\, f(x)\, dL_N(x) &- \frac{\beta N^2}{2} \iint \frac{f(x)-f(y)}{x-y}\, dL_N(x)\, dL_N(y) \\ &= N \int \Xi_t f(x)\, dM_N(x) - \frac{\beta}{2} \iint \frac{f(x)-f(y)}{x-y}\, dM_N(x)\, dM_N(y) \end{aligned}$$

Applying this identity to $f = \mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \boldsymbol{\xi}_t(\cdot, M_N)$ and recalling the definition of $\boldsymbol{\xi}_t(\cdot, M_N)$ (see (3.3)), we find

$$\begin{aligned} \mathcal{R}_t^N = N &\int [\Xi_t \mathbf{y}_{0,t} + W](x)\, dM_N(x) \\ &+ \int \left(\Xi_t \mathbf{y}_{1,t}(x) + \Big(\frac{\beta}{2} - 1\Big) \Big[\mathbf{y}_{0,t}'(x) + \int \partial_1 \mathbf{z}_t(z,x) d\mu_{V_t}(z) \Big] \right) dM_N(x) \\ &+ \iint dM_N(x)\, dM_N(y) \left(\Xi_t \mathbf{z}_t(\cdot, y)[x] - \frac{\beta}{2} \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x-y} \right) + C_t^N + E_N, \end{aligned}$$

where

$$\Xi_t \mathbf{z}_t(\cdot, y)[x] = -\beta \int \frac{\mathbf{z}_t(x,y) - \mathbf{z}_t(\tilde{x}, y)}{x - \tilde{x}}\, d\mu_{V_t}(\tilde{x}) + V_t'(x)\, \mathbf{z}_t(x,y),$$

C_t^N is a deterministic term, and E_N is a remainder that we will prove to be negligible:

$$\begin{aligned} E_N := &-\frac{1}{N} \int \partial_2 \mathbf{z}_t(x,x)\, dM_N(x) - \frac{1}{N}\Big(1 - \frac{\beta}{2}\Big) \int \mathbf{y}_{1,t}'(x)\, dM_N(x) \\ &- \frac{1}{N}\Big(1 - \frac{\beta}{2}\Big) \iint \partial_1 \mathbf{z}_t(x,y)\, dM_N(x)\, dM_N(y) \\ &- \frac{\beta}{2N} \iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y) \\ &- \frac{\beta}{2N} \iiint \frac{\mathbf{z}_t(x,y) - \mathbf{z}_t(\tilde{x}, y)}{x - \tilde{x}}\, dM_N(x)\, dM_N(y)\, dM_N(\tilde{x}). \end{aligned} \tag{3.6}$$

Hence, for $\mathcal{R}_t^N$ to be small we want to impose

$$\begin{aligned} \Xi_t \mathbf{y}_{0,t} &= -W + c, \\ \Xi_t \mathbf{z}_t(\cdot, y)[x] &= -\frac{\beta}{2} \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x-y} + c(y), \\ \Xi_t \mathbf{y}_{1,t} &= -\Big(\frac{\beta}{2} - 1\Big) \Big[\mathbf{y}_{0,t}' + \int \partial_1 \mathbf{z}_t(z, \cdot)\, d\mu_{V_t}(z) \Big] + c', \end{aligned} \tag{3.7}$$

where c, c' are some constant to be fixed later, and $c(y)$ does not depend on x.

3.2. **Inverting the operator Ξ.** We now prove a key lemma, that will allow us to find the desired functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$. Recall that, for a function $h : \mathbb{R} \to \mathbb{R}$, $\|h\|_{C^j(\mathbb{R})} := \sum_{r=0}^{j} \|h^{(r)}\|_{L^\infty(\mathbb{R})}$, where $h^{(r)}$ denotes the r-th derivative of h.

Lemma 3.2. *Given $V : \mathbb{R} \to \mathbb{R}$, assume that μ_V has support given by $[a, b]$ and that*

$$\frac{d\mu_V}{dx}(x) = S(x)\sqrt{(x-a)(b-x)}$$

with $S(x) \geq \bar{c} > 0$ a.e. on $[a, b]$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a C^k function and assume that V is of class C^p. Set

$$\Xi f(x) := -\beta \int \frac{f(x) - f(y)}{x - y} d\mu_V(x) + V'(x)f(x)$$

Then there exists a unique constant c_g such that the equation

$$\Xi f(x) = g(x) + c_g$$

has a solution of class $C^{(k-2)\wedge(p-3)}$. More precisely, for $j \leq (k-2) \wedge (p-3)$ there is a finite constant C_j such that

$$\|f\|_{C^j(\mathbb{R})} \leq C_j \|g\|_{C^{j+2}(\mathbb{R})}. \tag{3.8}$$

Moreover f (and its derivatives) behaves like $(g(x)+c_g)/V'(x)$ (and its corresponding derivatives) when $|x| \to +\infty$.

This solution will be denoted by $\Xi^{-1}g$.

Note that $Lf(x) = \Xi f'(x)$ can be seen as the asymptotics of the infinitesimal generator of the Dyson Brownian motion taken in the set where the spectral measure approximates μ_V. This operator is central in our approach, as much as the Dyson Brownian motion is central to prove universality in e.g. [ESYY12, BEY14a, BEY14b].

Proof. As a consequence of (3.4), we have

$$\beta\, PV \int \frac{1}{x-y}\, d\mu_V(y) = V'(x) \qquad \text{on the support of } \mu_V. \tag{3.9}$$

Therefore the equation $\Xi f(x) = g(x) + c_g$ on the support of μ_V amounts to

$$\beta\, PV \int \frac{f(y)}{x-y}\, d\mu_V(y) = g(x) + c_g \qquad \forall\, x \in [a, b]. \tag{3.10}$$

Let us write

$$d(x) := d\mu_V/dx = S(x)\sqrt{(x-a)(b-x)}$$

with S positive inside the support $[a, b]$. We claim that $S \in C^{p-3}([a, b])$.

Indeed, by (3.4) with $f(x) = (z-x)^{-1}$ for $z \in [a, b]^c$, we find that the Stieltjes transform $G(z) = \int (z-y)^{-1}\, d\mu_V(y)$ satisfies, for z outside $[a, b]$,

$$\frac{\beta}{2} G(z)^2 = G(z)V'(\Re(z)) + F(z), \qquad \text{with } F(z) = \int \frac{V'(y) - V'(\Re(z))}{z-y}\, d\mu_V(y)\,.$$

Solving this quadratic equation so that $G \to 0$ as $|z| \to \infty$ yields

$$G(z) = \frac{1}{\beta}\Big(V'(\Re(z)) - \sqrt{[V'(\Re(z))]^2 + 2\beta F(z)}\Big). \tag{3.11}$$

Notice that $V'(\Re(z))^2 + 2\beta F(z)$ becomes real as z goes to the real axis, and negative inside $[a, b]$. Hence, since $-\pi^{-1}\Im G(z)$ converges to the density of μ_V as z goes to the real axis (see e.g [AGZ10, Theorem 2.4.3]), we get

$$-S(x)^2(x-a)(b-x) = (\beta\pi)^{-2}\left[V'(x)^2 + 2\beta F(x)\right]. \tag{3.12}$$

This implies in particular that $\{a, b\}$ are the two points of the real line where $V'(x)^2+2\beta F(x)$ vanishes. Moreover $F(x) = -\int\int_0^1 V''(\alpha y+(1-\alpha)x)\, d\alpha\, d\mu_V(y)$ is of class C^{p-2} on $\mathbb{R}$ (recall that $V \in C^p$ by assumption), therefore $(V')^2 + 2\beta F \in C^{p-2}(\mathbb{R})$. Since we assumed that S does not vanish in $[a, b]$, from (3.12) we deduce that S is of class C^{p-3} on $[a, b]$.

To solve (3.10) we apply Tricomi's formula [Tri57, formula 12, p.181] and we find that, for $x \in [a, b]$,

$$\beta f(x)\sqrt{(x-a)(b-x)}d(x) = PV \int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}(g(y)+c_g)\, dy + c_2 := h(x)$$

for some constant c_2, hence

$$\begin{aligned} h(x) &= \beta f(x)(x-a)(b-x)S(x) \\ &= PV \int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}(g(y)+c_g)\, dy + c_2 \\ &= PV \int_a^b \sqrt{(y-a)(b-y)}\frac{g(y)-g(x)}{y-x}\, dy + (g(x)+c_g) PV \int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}\, dy + c_2 \\ &= \int_a^b \sqrt{(y-a)(b-y)}\frac{g(y)-g(x)}{y-x}\, dy - \pi\left(x - \tfrac{a+b}{2}\right)(g(x)+c_g) + c_2, \end{aligned}$$

where we used that, for $x \in [a, b]$,

$$PV \int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}\, dy = -\pi\Big(x - \frac{a+b}{2}\Big).$$

Set

$$h_0(x) = \int_a^b \sqrt{(y-a)(b-y)}\,\frac{g(y)-g(x)}{y-x}\, dy.$$

Then h_0 is of class C^{k-1} (recall that g is of class C^k). We next choose c_g and c_2 such that h vanishes at a and b (notice that this choice uniquely identifies c_g).

We note that $f \in C^{(k-2)\wedge(p-3)}([a, b])$. Moreover, we can bound its derivatives in terms of the derivatives of h_0, g and S: if we assume $j \le p-3$, we find that there exists a constant C_j, which depends only on the derivatives of S, such that

$$\|f^{(j)}\|_{L^\infty([a,b])} \le C_j \max_{p\le j}\Big(\|h_0^{(p+1)}\|_{L^\infty([a,b])} + \|g^{(p+1)}\|_{L^\infty([a,b])}\Big) \le C_j \max_{p\le j+2} \|g^{(p)}\|_{L^\infty([a,b])}.$$

Let us define

$$k(x) := \beta\, PV \int \frac{f(y)}{x-y} d\mu_V(y) - g(x) - c_g \qquad \forall\, x \in \mathbb{R}.$$

By (3.10) we see that $k \equiv 0$ on $[a, b]$. To ensure that $\Xi f = g + c_g$ also outside the support of μ_V we want

$$f(x)\bigg(\beta\, PV \int \frac{1}{x-y}\, d\mu_V(y) - V'(x)\bigg) = k(x) \qquad \forall\, x \in [a, b]^c.$$

Let us consider the function $\ell : \mathbb{R} \to \mathbb{R}$ defined as

$$\ell(x) := \beta\, PV \int \frac{1}{x-y}\, d\mu_V(y) - V'(x). \tag{3.13}$$

Notice that, thanks to (3.11), $\ell(x) = \beta G(x) - V'(x) = -\beta\sqrt{[V'(x)]^2 + 2\beta F(x)}$. Hence, comparing this expression with (3.12) and recalling that $S \ge \bar{c} > 0$ in $[a, b]$, we deduce that $[V'(x)]^2 + 2\beta F(x)$ is smooth and has simple zeroes both at a and b, therefore $[V'(x)]^2 + 2\beta F(x) > 0$ in $[a-\epsilon, b+\epsilon]\setminus[a, b]$ for some $\epsilon > 0$.

This shows that ℓ does not vanish in $[a-\epsilon, b+\epsilon]\setminus[a,b]$. Recalling that we can freely modify V outside $[a-\epsilon, b+\epsilon]$ (see proof of Lemma 2.1), we can actually assume that ℓ vanishes at $\{a,b\}$ and does not vanish in the whole $[a,b]^c$.

We claim that ℓ is Hölder 1/2 at the boundary points, and in fact is equivalent to a square root there. Indeed, it is immediate to check that ℓ is of class C^{p-1} except possibly at the boundary points $\{a,b\}$. Moreover

$$PV\int \frac{1}{x-y}\,d\mu_V(y) = S(a)\int_a^b \frac{1}{x-y}\sqrt{(y-a)(b-y)}\,dy + \int_a^b \frac{y-a}{x-y}\left(\int_0^1 S'(\alpha a+(1-\alpha)y)d\alpha\right)\sqrt{(y-a)(b-y)}\,dy.$$

The first term can be computed exactly and we have, for some $c\neq 0$,

$$\int_a^b \frac{1}{x-y}\sqrt{(y-a)(b-y)}\,dy = c\,(b-a)\left(\frac{x-\frac{a+b}{2}}{b-a} - \sqrt{\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 - \frac{1}{4}}\right) \tag{3.14}$$

which is Hölder 1/2, and in fact behaves as a square root at the boundary points. On the other hand, since S is of class C^{p-3} on $[a,b]$ with $p\geq 4$, the second function is differentiable, with derivative at a given by

$$\int_a^b \frac{1}{a-y}\left(\int_0^1 S'(\alpha a+(1-\alpha)y)d\alpha\right)\sqrt{(y-a)(b-y)}\,dy,$$

which is a convergent integral. The claim follows.

Thus, for x outside the support of μ_V we can set

$$f(x) := \ell(x)^{-1}k(x).$$

With this choice $\Xi f = g + c_g$ and f is of class $C^{(k-2)\wedge(p-3)}$ on $\mathbb{R}\setminus\{a,b\}$.

We now want to show that f is of class $C^{(k-2)\wedge(p-3)}$ on the whole $\mathbb{R}$. For this we need to check the continuity of f and its derivatives at the boundary points, say at a (the case of b being similar). We take hereafter $r\leq (k-2)\wedge(p-3)$, so that f has r derivatives inside $[a,b]$ according to the above considerations.

Let us first deduce the continuity of f at a. We write, with $f(a^+) = \lim_{x\downarrow a} f(x)$,

$$k(x) = f(a^+)\ell(x) + k_1(x)$$

with

$$k_1(x) := \beta\left(PV\int \frac{f(y)}{x-y}\,d\mu_V(y) - PV\int \frac{f(a^+)}{x-y}\,d\mu_V(y)\right) + g(x) + c_g + f(a^+)V_t'(x).$$

Notice that since $f=\ell^{-1}k$ outside $[a,b]$, if we can show that $\ell^{-1}(x)k_1(x)\to 0$ as $x\uparrow a$ then we would get $f(a^-) = f(a^+)$, proving the desired continuity.

To prove it we first notice that k_1 vanishes at a (since both k and ℓ vanish inside $[a,b]$), hence

$$\begin{aligned} k_1(x) &= \beta\left(PV\int \frac{f(y)-f(a^+)}{x-y}\,d\mu_V(y) - PV\int \frac{f(y)-f(a^+)}{a-y}\,d\mu_V(y)\right) + \tilde g(x) - \tilde g(a) \\ &= \beta(a-x)\,PV\int \frac{f(y)-f(a^+)}{(x-y)(a-y)}\,d\mu_V(y) + \tilde g(x) - \tilde g(a), \end{aligned}$$

with $\tilde g := g + f(a^+)V' \in C^1$. Assume $1 \le (k-2)\wedge(p-3)$. Since f is of class C^1 inside $[a,b]$ we have $|f(y)-f(a^+)| \le C|y-a|$, from which we deduce that $|k_1(x)| \le C|x-a|$ for $x \le a$.

Hence $\ell^{-1}(x)k_1(x) \to 0$ as $x \uparrow a$ (recall that ℓ behaves as a square root near a), which proves that

$$\lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x)$$

and shows the continuity of f at a.

We now consider the next derivative: we write

$$k(x) = \big[f(a) + f'(a^+)(x-a)\big]\ell(x) + k_2(x)$$

with

$$\begin{aligned} k_2(x) := \beta(a-x)\, PV \int \frac{f(y)-f(a^+)-(y-a)f'(a^+)}{(x-y)(a-y)}\, d\mu_V(y) \\ + \tilde g(x) - \tilde g(a) + f'(a^+)(x-a)V_t'(x). \end{aligned}$$

Since $k = \ell \equiv 0$ on $[a,b]$ we have $k_2(a) = k_2'(a^+) = k_2'(a^-) = 0$. Hence, since f is of class C^2 on $[a,b]$ we see that $|k_2(x)| \le C|x-a|^2$ for $x \le a$, therefore $k_2(x)/\ell(x)$ is of order $|x-a|^{3/2}$, thus

$$f(x) = f(a) + f'(a^+)(x-a) + O(|x-a|^{3/2}) \qquad \text{for } x \le a,$$

which shows that f has also a continuous derivative.

We obtain the continuity of the next derivatives similarly. Moreover, away from the boundary point the j-th derivative of f outside $[a,b]$ is of the same order than that of g/V', while near the boundary points it is governed by the derivatives of g nearby, therefore

$$\|f^{(j)}\|_{L^\infty([a,b]^c)} \le C_j' \max_{r\le j+2} \|g^{(r)}\|_{L^\infty(\mathbb{R})}\,. \tag{3.15}$$

Finally, it is clear that f behaves like $(g(x)+c_g)/V'(x)$ when x goes to infinity. □

3.3. Defining the functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$.

To define the functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$ according to (3.7), notice that Lemma 2.1 shows that the hypothesis of Lemma 3.2 are fulfilled. Hence, as a consequence of Lemma 3.2 we find the following result (recall that $\psi \in C^{s,v}$ means that ψ is s times continuously differentiable with respect to the first variable and v times with respect to the second).

Lemma 3.3. *Let $r \ge 8$. If $W, V \in C^r$, we can choose $\mathbf{y}_{0,t}$ of class C^{r-3}, $\mathbf{z}_t \in C^{s,v}$ for $s+v \le r-6$, and $\mathbf{y}_{1,t} \in C^{r-9}$. Moreover, these functions (and their derivatives) go to zero at infinity like $1/V'$ (and its corresponding derivatives).*

Proof. Note that V_t is of class C^r as both V, W are. By Lemma 3.2 we have $\mathbf{y}_{0,t} = \Xi_t^{-1}W \in C^{r-3}$. For $\mathbf{z}_t$, we can rewrite

$$\begin{aligned} \Xi_t \mathbf{z}_t(\cdot,y)[x] &= -\frac{\beta}{2}\int_0^1 \mathbf{y}_{0,t}'(\alpha x + (1-\alpha)y)\, d\alpha + c(y) \\ &= -\frac{\beta}{2}\int_0^1 [\mathbf{y}_{0,t}'(\alpha x + (1-\alpha)y) + c_\alpha(y)]\, d\alpha \end{aligned}$$

where we choose $c_\alpha(y)$ to be the unique constant provided by Lemma 3.2 which ensures that $\Xi_t^{-1}[\mathbf{y}_{0,t}'(\alpha x + (1-\alpha)y) + c_\alpha(y)]$ is smooth. This gives that $c(y) = \int_0^1 c_\alpha(y)\, d\alpha$. Since Ξ_t^{-1} is a linear integral operator, we have

$$\mathbf{z}_t(x,y) = -\frac{\beta}{2}\int_0^1 \Xi_t^{-1}[\mathbf{y}'_{0,t}(\alpha\cdot+(1-\alpha)y)](x)\,d\alpha\,.$$

As the variable y is only a translation, it is not difficult to check that $\mathbf{z}_t \in C^{s,v}$ for any $s+v \le r-5$. It follows that

$$-\Big(\frac{\beta}{2}-1\Big)\Big[\mathbf{y}'_{0,t} + \int \partial_1 \mathbf{z}_t(z,\cdot)\,d\mu_{V_t}(z)\Big] + c'$$

is of class C^{r-6} and therefore by Lemma 3.2 we can choose $\mathbf{y}_{1,t} \in C^{r-8}$, as desired.

The decay at infinity is finally again a consequence of Lemma 3.2. □

3.4. **Getting rid of the random error term E_N.** We show that the $L^1_{\mathbb{P}^N_{V_t}}$-norm of the error term E_N defined in (3.6) goes to zero. This could be easily derived from [BEY14a], but we here provide a self-contained proof. To this end, we first make some general consideration on the growth of variances.

Following e.g. [MMS14, Theorem 1.6], up to assume that V_t goes sufficiently fast at infinity (which we did, see Lemma 2.1), it is easy to show that there exists a constant $\tau_0 > 0$ so that for all $\tau \ge \tau_0$,

$$\mathbb{P}^N_{V_t}\bigg(D(L_N,\mu_{V_t}) \ge \tau\sqrt{\frac{\log N}{N}}\bigg) \le e^{-c\tau^2 N\log N}\,,$$

where D is the 1-Wasserstein distance

$$D(\mu,\nu) := \sup_{\|f'\|_\infty\le 1}\bigg|\int f(d\mu - d\nu)\bigg|.$$

Since $M_N = N(L_N - \mu_{V_t})$ we get

$$D(L_N,\mu_{V_t}) = \frac{1}{N}\sup_{\|f'\|_\infty\le 1}\bigg|\int f\,dM_N\bigg|, \tag{3.16}$$

hence, for $\tau\ge\tau_0$,

$$\mathbb{P}^N_{V_t}\bigg(\sup_{\|f\|_{\mathrm{Lip}}\le 1}\bigg|\int f(x)\,dM_N(x)\bigg| \ge \tau\sqrt{N\log N}\bigg) \le e^{-c\tau^2 N\log N}. \tag{3.17}$$

This already shows that, if f is sufficiently smooth, $\int f(x,y)dM_N(x)\,dM_N(y)$ is of order at most $N\log N$. More precisely

$$\int f(x,y)\,dM_N(x)\,dM_N(y) = \int \hat f(\zeta,\xi)\bigg(\int e^{i\zeta x}dM_N(x)\int e^{i\xi x}dM_N(x)\bigg)\,d\xi\,d\zeta,$$

so that with probability greater than $1-e^{-c\tau_0^2 N\log N}$ we have

$$\bigg|\int f(x,y)\,dM_N(x)\,dM_N(y)\bigg| \le \tau_0^2\,N\log N\int |\hat f(\zeta,\xi)|\,|\zeta|\,|\xi|\,d\zeta\,d\xi\,. \tag{3.18}$$

To improve this estimate, we shall use loop equations as well as Lemma 3.2. Given a function g and a measure ν, we use the notation $\nu(g) := \int g\,d\nu$.

Lemma 3.4. *Let g be a smooth function. Then, if $\tilde{M}_N = NL_N - N\mathbb{E}_{V_t}[L_N]$, there exists a finite constant C such that for all $t \in [0,1]$,*

$$\sigma^{(1)}_{N,t}(g) := \left| \int M_N(g)\, d\mathbb{P}^N_{V_t} \right| \le C\, m_t(g) =: B^1_{N,t}(g)$$

$$\sigma^{(2)}_{N,t}(g) := \int \Big(\tilde{M}_N(g)\Big)^2 d\mathbb{P}^N_{V_t} \le C\Big(m_t(g)^2 + m_t(g)\|g\|_\infty + \|\Xi_t^{-1} g\|_\infty \|g'\|_\infty\Big) =: B^2_{N,t}(g)$$

$$\sigma^{(4)}_{N,t}(g) := \int \Big(\tilde{M}_N(g)\Big)^4 \, d\mathbb{P}^N_{V_t}$$

$$\le C\Big(\|\Xi_t^{-1} g\|_\infty \|g'\|_\infty \sigma^{(2)}_{N,t}(g) + \|g\|^3_\infty m_t(g) + m_t(g)^2 \sigma^{(2)}_{N,t}(g) + m_t(g)^4\Big) =: B^4_{N,t}(g),$$

where

$$m_t(g) := \left|1 - \frac{\beta}{2}\right| \|(\Xi_t^{-1} g)'\|_\infty + \frac{\beta}{2} \log N \int |\hat{\Xi}_t^{-1} g|(\xi)\, |\xi|^3 \, d\xi.$$

Proof. First observe that, by integration by parts, for any C^1 function f

$$\int \Big(N \sum_i V'(\lambda_i) f(\lambda_i) - \beta \sum_{i<j} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \Big) d\mathbb{P}^N_{V_t} = \int \sum_i f'(\lambda_i)\, d\mathbb{P}^N_{V_t} \tag{3.19}$$

which we can rewrite as the first loop equation

$$\int M_N(\Xi_t f)\, d\mathbb{P}^N_{V_t} = \int \Big[\Big(1 - \frac{\beta}{2}\Big) L_N(f') + \frac{\beta}{2N} \int \frac{f(x) - f(y)}{x - y} dM_N(x) dM_N(y) \Big] \, d\mathbb{P}^N_{V_t}\,. \tag{3.20}$$

We denote

$$F_N(g) := \Big(1 - \frac{\beta}{2}\Big) L_N((\Xi_t^{-1} g)') + \frac{\beta}{2N} \int \frac{\Xi_t^{-1} g(x) - \Xi_t^{-1} g(y)}{x - y}\, dM_N(x)\, dM_N(y)$$

so that taking $f := \Xi_t^{-1} g$ in (3.20) we deduce

$$\int M_N(g)\, d\mathbb{P}^N_{V_t} = \int F_N(g)\, d\mathbb{P}^N_{V_t}.$$

To bound the right hand side above, we notice that $\Xi_t^{-1} g$ goes to zero at infinity like $1/V'$ (see Lemma 3.2). Hence we can write its Fourier transform and get

$$\int \frac{\Xi_t^{-1} g(x) - \Xi_t^{-1} g(y)}{x - y}\, dM_N(x)\, dM_N(y)$$

$$= i \int d\xi\, \xi\, \hat{\Xi}_t^{-1} g(\xi) \int_0^1 d\alpha \int e^{i\alpha\xi x}\, dM_N(x) \int e^{i(1-\alpha)\xi y}\, dM_N(y),$$

so that we deduce (recall (3.16))

$$\sup_{D(L_N, \mu_{V_t}) \le \tau_0 \sqrt{\log N / N}} F_N(g) \le (1 + \tau_0^2)\, m_t(g).$$

On the other hand, as the mass of M_N is always bounded by $2N$, we deduce that $F_N(g)$ is bounded everywhere by $N m_t(g)$. Since the set $\{D(L_N, \mu_{V_t}) \ge \tau_0 \sqrt{N \log N}\}$ has small probability (see (3.17)), we conclude that

$$\left| \int M_N(g)\, d\mathbb{P}^N_{V_t} \right| \le N e^{-c\tau_0^2 N \log N} m_t(g) + (1 + \tau_0^2) m_t(g) \le C\, m_t(g), \tag{3.21}$$

which proves our first bound.

Before proving the next estimates, let us make a simple remark: using the definition of M_N and $\tilde{M}_N$ it is easy to check that, for any function g,

$$\left|M_N(g) - \tilde{M}_N(g)\right| = \left|\int M_N(g)\, d\mathbb{P}^N_{V_t}\right|. \tag{3.22}$$

To get estimates on the covariance we obtain the second loop equation by changing $V(x)$ into $V(x) + \delta\, g(x)$ in (3.19) and differentiating with respect to δ at $\delta = 0$. This gives

$$\begin{aligned}\int M_N(\Xi_t f)\tilde{M}_N(g)\, d\mathbb{P}^N_{V_t} &= \int L_N(fg')\, d\mathbb{P}^N_{V_t} \\ &+ \int \Big[\Big(1-\frac{\beta}{2}\Big)L_N(f') + \frac{\beta}{2N}\int \frac{f(x)-f(y)}{x-y} dM_N(x)dM_N(y)\Big]\tilde{M}_N(g)\, d\mathbb{P}^N_{V_t}.\end{aligned} \tag{3.23}$$

We now notice that $M_N(\Xi_t f) - \tilde{M}_N(\Xi_t f)$ is deterministic and $\int \tilde{M}_N(g)\, d\mathbb{P}^N_{V_t} = 0$, hence the left hand side is equal to

$$\int \tilde{M}_N(\Xi_t f)\tilde{M}_N(g)\, d\mathbb{P}^N_{V_t}.$$

We take $f := \Xi_t^{-1} g$ and we argue similarly to above (that is, splitting the estimate depending whether $D(L_N, \mu_{V_t}) \geq \tau_0\sqrt{N\log N}$ or not, and use that $|\tilde{M}_N(g)| \leq N\|g\|_\infty$) to deduce that $\sigma^{(2)}_{N,t}(g) := \int |\tilde{M}_N(f)|^2 d\mathbb{P}^N_{V_t}$ satisfies

$$\begin{aligned}\sigma^{(2)}_{N,t}(g) &\leq \|g'\Xi_t^{-1}g\|_\infty + \int |F_N(g)||\tilde{M}_N(g)| d\mathbb{P}^N_{V_t} \\ &\leq \|\Xi_t^{-1}g\|_\infty\, \|g'\|_\infty + N^2 e^{-c\tau_0^2 N\log N}\|g\|_\infty m_t(g) + C\, m_t(g)\int |\tilde{M}_N(g)| d\mathbb{P}^N_{V_t} \\ &= \|\Xi_t^{-1}g\|_\infty \|g'\|_\infty + N^2 e^{-c\tau_0^2 N\log N} m_t(g)\|g\|_\infty + C\, m_t(g)\sigma^{(2)}_{N,t}(g)^{1/2}.\end{aligned} \tag{3.24}$$

Solving this quadratic inequality yields

$$\sigma^{(2)}_{N,t}(g) \leq C\Big[m_t(g)^2 + m_t(g)\|g\|_\infty + \|\Xi_t^{-1}g\|_\infty\|g'\|_\infty\Big]$$

for some finite constant C.

We finally turn to the fourth moment. If we make an infinitesimal change of potential $V(x)$ into $V(x) + \delta_1\, g_2(x) + \delta_2\, g_3(x)$ and differentiate at $\delta_1 = \delta_2 = 0$ into (3.23) we get, denoting $g = g_1$,

(3.25)

$$\begin{aligned}&\int M_N(\Xi_t f)\tilde{M}_N(g_1)\tilde{M}_N(g_2)\tilde{M}_N(g_3)\, d\mathbb{P}^N_{V_t} = \int \Big[\sum_\sigma L_N(fg'_{\sigma(1)})\tilde{M}_N(g_{\sigma(2)})\tilde{M}_N(g_{\sigma(3)})\Big]\, d\mathbb{P}^N_{V_t} + \\ &\int \Big[\Big(1-\frac{\beta}{2}\Big)L_N(f') + \frac{\beta}{2N}\int \frac{f(x)-f(y)}{x-y}\, dM_N(x)\, dM_N(y)\Big] M_N(g_1)\tilde{M}_N(g_2)\tilde{M}_N(g_3)\, d\mathbb{P}^N_{V_t},\end{aligned}$$

where we sum over all permutations σ of $\{1,2,3\}$. Taking $\Xi_t f = g_1 = g_2 = g_3 = g$, eqrefeq:difference MN tilde, (3.21), and Cauchy-Schwarz inequality give

$$\sigma^{(4)}_N(g) \leq C\Big[\|g'\Xi_t^{-1}g\|_\infty \sigma^{(2)}_{N,t}(g) + \|g\|^3_\infty m_t(g) + m_t(g)\sigma^{(4)}_{N,t}(g)^{3/4} + m_t(g)^2\sigma^{(2)}_{N,t}(g)\Big],$$

which implies

$$\sigma_N^{(4)}(g) \le C\Big[\|g'\Xi_t^{-1}g\|_\infty \sigma_{N,t}^{(2)}(g) + \|g\|_\infty^3 m_t(g) + m_t(g)^2\sigma_{N,t}^{(2)}(g) + m_t(g)^4\Big].$$

□

Applying the above result with $g = e^{i\lambda\cdot}$ we deduce the following:

Corollary 3.5. *Assume that $V, W \in C^r$ with $r \ge 8$. Then there exists a finite constant C such that, for all $\lambda \in \mathbb{R}$ and all $t \in [0,1]$,*

$$\int |M_N(e^{i\lambda\cdot})|^2 d\mathbb{P}_{V_t}^N \le C[\log N(1+|\lambda|^7)]^2\,, \tag{3.26}$$

$$\int |M_N(e^{i\lambda\cdot})|^4 d\mathbb{P}_{V_t}^N \le C[\log N(1+|\lambda|^7)]^4\,. \tag{3.27}$$

Proof. In the case $g(x) = e^{i\lambda x}$ we estimate the norms of $\Xi_t^{-1}g$ by using Lemma 3.2 and we get a finite constant C such that

$$\|\Xi_t^{-1}g\|_\infty \le C|\lambda|^2, \qquad \|\Xi_t^{-1}g'\|_\infty \le C|\lambda|^3,$$

whereas, since $\Xi_t^{-1}g$ goes fast to zero at infinity (as $1/V'$), for $j \le r-3$ we have

$$|\hat{\Xi}_t^{-1}g|(\xi) \le C\frac{\|\Xi_t^{-1}g\|_{C^j}}{1+|\xi|^j} \le C'\frac{\|g\|_{C^{j+2}}}{1+|\xi|^j} \le C'\frac{1+|\lambda|^{j+2}}{1+|\xi|^j}.$$

Hence, by taking $j=5$ we deduce that there exists a finite constant C' such that

$$\begin{aligned}
m_t(g) &\le C\log N\left(|\lambda|^3 + 1 + \int d\xi\,\frac{1+|\lambda|^7}{1+|\xi|^5}|\xi|^3\right) = C'\log N\left(1+|\lambda|^7\right),\\
B_{N,t}^1(g) &\le C'\log N\left(1+|\lambda|^7\right),\\
B_{N,t}^2(g) &\le C'(\log N)^2\left(1+|\lambda|^7\right)^2,\\
B_{N,t}^4(g) &\le C'(\log N)^4\left(1+|\lambda|^7\right)^4.
\end{aligned}$$

Finally, for $k=2,4$, using (3.22) and (3.21) we have

$$\int |M_N(e^{i\lambda\cdot})|^k\, d\mathbb{P}_{V_t}^N \le 2^{k-1}\left(\int |\tilde{M}_N(e^{i\lambda\cdot})|^k\, d\mathbb{P}_{V_t}^N + \left(B_{N,t}^1(g)\right)^k\right)$$

from which the result follows.

□

We can now estimate E_N.

The linear term can be handled in the same way as we shall do now for the quadratic and cubic terms (which are actually more delicate), so we just focus on them.

We have two quadratic terms in M_N which sum up into

$$E_N^1 = -\frac{1}{N}\Big(1-\frac{\beta}{2}\Big)\iint \partial_1\mathbf{z}_t(x,y)\,dM_N(x)\,dM_N(y) - \frac{\beta}{2N}\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\,dM_N(x)\,dM_N(y).$$

Writing

$$\frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y} = \int_0^1 \mathbf{y}_{1,t}'(\alpha x+(1-\alpha)y)\,d\alpha = \int_0^1\left(\int \widehat{\mathbf{y}_{1,t}'}(\xi)e^{i(\alpha x+(1-\alpha)y)\xi}d\xi\right)d\alpha$$

we see that

$$\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\,dM_N(x)\,dM_N(y)=\int d\xi\,\widehat{\mathbf{y}_{1,t}'}(\xi)\int_0^1 d\alpha\,M_N(e^{i\alpha\xi\cdot})M_N(e^{i(1-\alpha)\xi\cdot}),$$

so using (3.26) we get

$$\int |E_N^1|\,d\mathbb{P}_{V_t}^N\le C\,\frac{(\log N)^2}{N}\Bigg(\int d\xi\,|\hat{\mathbf{y}}_{1,t}|(\xi)\,|\xi|\left(1+|\xi|^7\right)^2 \\ +\iint d\xi\,d\zeta\,|\hat{\mathbf{z}}_t|(\xi,\zeta)\,|\xi|\left(1+|\xi|^7\right)\left(1+|\zeta|^7\right)\Bigg).$$

It is easy to see that the right hand side is finite if $\mathbf{y}_{1,t}$ and $\mathbf{z}_t$ are smooth enough (recall that these functions and their derivatives decay fast at infinity). More precisely, to ensure that

$$|\hat{\mathbf{y}}_{1,t}|(\xi)\,|\xi|\left(1+|\xi|^7\right)^2\le\frac{C}{1+|\xi|^2}\in L^1(\mathbb{R})$$

and

$$|\hat{\mathbf{z}}_t|(\xi,\zeta)\,|\xi|\left(1+|\xi|^7\right)\left(1+|\zeta|^7\right)\le\frac{C}{1+|\xi|^3+|\zeta|^3}\in L^1(\mathbb{R}^2),$$

we need $\mathbf{y}_{1,t}\in C^{17}$ and $\mathbf{z}_t\in C^{11,7}\cap C^{8,10}$, so (recalling Lemma 3.3) $V,W\in C^{26}$ is enough to guarantee that the right hand side is finite.

Using (3.26), (3.27), and Hölder inequality, we can similarly bound the expectation of the cubic term

$$\begin{aligned} E_N^2&=\frac{\beta}{2N}\iiint\frac{\mathbf{z}_t(x,y)-\mathbf{z}_t(\tilde{x},y)}{x-\tilde{x}}\,dM_N(x)\,dM_N(y)\,dM_N(\tilde{x})\\ &=i\frac{\beta}{2N}\iint d\xi\,d\zeta\,\widehat{\partial_1\mathbf{z}_t}(\xi,\zeta)\int_0^1 d\alpha\,M_N(e^{i\alpha\xi\cdot})M_N(e^{i(1-\alpha)\xi\cdot})M_N(e^{i\zeta\cdot})\end{aligned}$$

to get

$$\int |E_N^2|\,d\mathbb{P}_{V_t}^N\le C\,\frac{(\log N)^3}{N}\iint d\xi\,d\zeta\,|\hat{\mathbf{z}}_t(\xi,\zeta)|\,|\xi|\left(1+|\xi|^7\right)^2\left(1+|\zeta|^7\right).$$

Again the right hand side is finite if $\mathbf{z}_t\in C^{18,7}\cap C^{15,10}$, which is ensured by Lemma 3.3 if V,W are of class C^{31}.

3.5. **Control on the deterministic term C_t^N.** By what we proved above we have

$$\int |\mathcal{R}_t^N-C_t^N|\,d\mathbb{P}_{V_t}^N\le C\,\frac{(\log N)^3}{N},$$

thus, in particular,

$$\left|C_t^N-\mathbb{E}[\mathcal{R}_t^N]\right|\le C\,\frac{(\log N)^3}{N}.$$

Notice now that, by construction,

$$\mathcal{R}_t^N=-\mathcal{L}\mathbf{Y}_t^N+N\sum_i W(\lambda_i)+c_t^N$$

with $c_t^N=-\mathbb{E}[N\sum_i W(\lambda_i)]$ and

$$\mathcal{L}\mathbf{Y}:=\mathrm{div}\mathbf{Y}+\beta\sum_{i<j}\frac{\mathbf{Y}^i-\mathbf{Y}^j}{\lambda_i-\lambda_j}-N\sum_i V'(\lambda_i)\mathbf{Y}^i.$$

Also, an integration by parts shows that, under P_V^N, $\mathbb{E}[\mathcal{L}\mathbf{Y}] = 0$ for any vector field $\mathbf{Y}$. This implies that $\mathbb{E}[\mathcal{R}_t^N] = 0$, therefore $|C_t^N| \le C\,\frac{(\log N)^3}{N}$.

This concludes the proof of Proposition 3.1.

4. Reconstructing the transport map via the flow

In this section we study the properties of the flow generated by the vector field $\mathbf{Y}_t^N$ defined in (3.3). As we shall see, we will need to assume that $W, V \in C^r$ with $r \ge 16$.

We consider the flow of $\mathbf{Y}_t^N$ given by

$$X_t^N : \mathbb{R}^N \to \mathbb{R}^N, \qquad \dot{X}_t^N = \mathbf{Y}_t^N(X_t^N).$$

Recalling the form of $\mathbf{Y}_t^N$ (see (3.3)) it is natural to expect that we can give an expansion for X_t^N. More precisely, let us define the flow of $\mathbf{y}_{0,t}$,

$$X_{0,t} : \mathbb{R} \to \mathbb{R}, \qquad \dot{X}_{0,t} = \mathbf{y}_{0,t}(X_{0,t}), \quad X_{0,0}(\lambda) = \lambda, \tag{4.1}$$

and let $X_{1,t}^N = (X_{1,t}^{N,1}, \dots, X_{1,t}^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ be the solution of the linear ODE

$$\begin{aligned}\dot{X}_{1,t}^{N,k}(\lambda_1, \dots, \lambda_N) &= \mathbf{y}_{0,t}'(X_{0,t}(\lambda_k)) \cdot X_{1,t}^{N,k}(\lambda_1, \dots, \lambda_N) + \mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) \\ &\quad + \int \mathbf{z}_t(X_{0,t}(\lambda_k), y)\, dM_N^{X_{0,t}}(y) \\ &\quad + \frac{1}{N}\sum_{j=1}^N \partial_2 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big) \cdot X_{1,t}^{N,j}(\lambda_1, \dots, \lambda_N)\end{aligned} \tag{4.2}$$

with the initial condition $X_{1,t}^N = 0$, where $M_N^{X_{0,t}}$ is defined as

$$\int f(y)\, dM_N^{X_{0,t}}(y) := \sum_{i=1}^N \left[f(X_{0,t}(\lambda_i)) - \int f\, d\mu_{V_t} \right] \qquad \forall\, f \in C_c(\mathbb{R}).$$

If we set

$$X_{0,t}^N(\lambda_1, \dots, \lambda_N) := \big(X_{0,t}(\lambda_1), \dots, X_{0,t}(\lambda_N)\big),$$

then the following result holds.

Lemma 4.1. *Assume that $W, V \in C^r$ with $r \ge 16$. Then the flow $X_t^N = (X_t^{N,1}, \dots, X_t^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^{r-9} and the following properties hold: Let $X_{0,t}$ and $X_{1,t}^N$ be as in (4.1) and (4.2) above, and define $X_{2,t}^N : \mathbb{R}^N \to \mathbb{R}^N$ via the identity*

$$X_t^N = X_{0,t}^N + \frac{1}{N} X_{1,t}^N + \frac{1}{N^2} X_{2,t}^N\,.$$

Then

$$\sup_{1 \le k \le N} \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} \le C \log N, \qquad \|X_{2,t}^N\|_{L^2(\mathbb{P}_V^N)} \le C N^{1/2} (\log N)^2, \tag{4.3}$$

where

$$\|X_{i,t}^N\|_{L^2(\mathbb{P}_V^N)} = \left(\int |X_{i,t}^N|^2 d\mathbb{P}_V^N \right)^{1/2}, \qquad |X_{i,t}^N| := \sqrt{\sum_{j=1,\dots,N} |X_{i,t}^{N,j}|^2}, \quad i = 0, 1, 2.$$

In addition, there exists a constant $C > 0$ such that, with probability greater than $1 - N^{-N/C}$,

$$\max_{1\le k,k'\le N} |X_{1,t}^{N,k}(\lambda_1,\dots,\lambda_N) - X_{1,t}^{N,k'}(\lambda_1,\dots,\lambda_N)| \le C \log N\sqrt{N}|\lambda_k - \lambda_{k'}|. \tag{4.4}$$

Proof. Since $\mathbf{Y}_t^N \in C^{r-9}$ (see Lemma 3.3) it follows by Cauchy-Lipschitz theory that X_t^N is of class C^{r-9}.

Using the notation $\hat\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^N$ and

$$X_t^{N,k,\sigma}(\hat\lambda) := X_{0,t}(\lambda_k) + \sigma\frac{X_{1,t}^{N,k}}{N}(\hat\lambda) + \sigma\frac{X_{2,t}^{N,k}}{N^2}(\hat\lambda) = (1-\sigma)X_{0,t}(\lambda_k) + \sigma X_t^{N,k}(\hat\lambda)$$

and defining the measure $M_N^{X_t^{N,s}}$ as

$$\int f(y)\,dM_N^{X_t^{N,s}}(y) = \sum_{i=1}^N \left[f\big((1-s)X_{0,t}(\lambda_i) + sX_t^{N,i}(\hat\lambda)\big) - \int f\,d\mu_{V_t}\right] \qquad \forall f \in C_c(\mathbb{R}). \tag{4.5}$$

by a Taylor expansion we get an ODE for $X_{2,t}^N$:

(4.6)

$$\begin{aligned}
\dot X_{2,t}^{N,k}(\hat\lambda) &= \int_0^1 \mathbf{y}_{0,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big)\,ds\cdot X_{2,t}^{N,k}(\hat\lambda)\\
&\quad + N\int_0^1 \Big[\mathbf{y}_{0,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big) - \mathbf{y}_{0,t}'\Big(X_{0,t}(\lambda_k)\Big)\Big]\,ds\cdot X_{1,t}^{N,k}(\hat\lambda)\\
&\quad + \int_0^1 \mathbf{y}_{1,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big)\,ds\cdot\Big(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\Big)\\
&\quad + \int_0^1 \Big[\int \partial_1\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), y\Big)\,dM_N^{X_t^{N,s}}(y)\\
&\qquad\qquad - \int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\,dM_N^{X_{0,t}}(y)\Big]\,ds\cdot\Big(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\Big)\\
&\quad + \int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\,dM_N^{X_{0,t}}(y)\cdot\Big(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\Big)\\
&\quad + \sum_{j=1}^N \int_0^1 \Big[\partial_2\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), X_t^{N,j,s}(\hat\lambda)\Big) - \partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big)\Big]\,ds\cdot X_{1,t}^{N,j}(\hat\lambda)\\
&\quad + \sum_{j=1}^N \int_0^1 \Big[\partial_2\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), X_t^{N,j,s}(\hat\lambda)\Big)\Big]\,ds\cdot\frac{X_{2,t}^{N,j}(\hat\lambda)}{N},
\end{aligned}$$

with the initial condition $X_{2,0}^{N,k} = 0$. Using that

$$\|\mathbf{y}_{0,t}\|_{C^{r-2}(\mathbb{R})} \le C$$

(see Lemma 3.3) we obtain

$$\|X_{0,t}\|_{C^{r-2}(\mathbb{R})} \le C. \tag{4.7}$$

We now start to control $X_{1,t}^N$. First, simply by using that M_N has mass bounded by $2N$ we obtain the rough bound $|X_{1,t}^{N,k}| \le C\,N$. Inserting this bound into (4.6) one easily obtain the bound $|X_{2,t}^{N,k}| \le C\,N^2$. We now prove finer estimates.

By (3.17) together with the fact that $X_{0,t}$ and $x \mapsto \mathbf{z}_t(y,x)$ are Lipschitz (uniformly in y), it follows that there exists a finite constant C such that, with probability greater than $1 - N^{-N/C}$,

$$\left\| \int \mathbf{z}_t(\cdot,\lambda)\, dM_N^{X_{0,t}}(\lambda) \right\|_\infty \le C \log N \sqrt{N}. \tag{4.8}$$

Hence it follows easily from (4.2) that

$$\max_k \|X_{1,t}^{N,k}\|_\infty \le C \log N \sqrt{N} \tag{4.9}$$

outside a set of probability bounded by $N^{-N/C}$.

In order to control $X_{2,t}^N$ we first estimate $X_{1,t}^N$ in $L^4(\mathbb{P}_V^N)$: using (4.2) again, we get

$$\begin{aligned} &\frac{d}{dt}\Big(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} \Big) \\ &\qquad \le C \Bigg(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} + 1 + \left\| \int \mathbf{z}_t(X_{0,t}(\lambda_k), y)\, dM_N^{X_{0,t}}(y) \right\|_{L^4(\mathbb{P}_V^N)} \Bigg). \end{aligned} \tag{4.10}$$

To bound $X_{1,t}^N$ in $L^4(\mathbb{P}_V^N)$ and then to be able to estimate $X_{2,t}^N$ in $L^2(\mathbb{P}_V^N)$, we will use the following estimates:

Lemma 4.2. *For any $k = 1, \ldots, N$,*

$$\left\| \int \mathbf{z}_t(X_{0,t}(\lambda_k), y)\, dM_N^{X_{0,t}}(y) \right\|_{L^4(\mathbb{P}_V^N)} \le C \log N, \tag{4.11}$$

$$\left\| \int \partial_1 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y) \right\|_{L^4(\mathbb{P}_V^N)} \le C \log N. \tag{4.12}$$

Proof. We write the Fourier decomposition of $\eta_t(x,y) := \mathbf{z}_t(X_{0,t}(x), X_{0,t}(y))$ to get

$$\int \eta_t(x,y)\, dM_N(y) = \int \hat{\eta}_t(x,\xi) \int e^{i\xi y}\, dM_N(y)\, d\xi\,.$$

Since $\mathbf{z}_t \in C^{u,v}$ for $u + v \le r - 6$ and $X_{0,t} \in C^{r-2}$ (see (4.7)), we deduce that

$$|\hat{\eta}_t(x,\xi)| \le \frac{C}{1 + |\xi|^{r-6}},$$

so that using (3.27) we get

$$\begin{aligned} \left\| \sup_x \left| \int \eta_t(x,y)\, dM_N(y) \right| \right\|_{L^4(\mathbb{P}_V^N)} &\le \int \Big\| \hat{\eta}_t(\cdot,\xi) \Big\|_\infty \left\| \int e^{i\xi y} dM_N(y) \right\|_{L^4(\mathbb{P}_V^N)} d\xi \\ &\le C \log N \int \Big\| \hat{\eta}_t(\cdot,\xi) \Big\|_\infty \left(1 + |\xi|^7\right)\, d\xi \\ &\le C \log N, \end{aligned}$$

provided $r > 14$. The same arguments work for $\partial_1 \mathbf{z}_t$ provided $r > 15$. Since by assumption $r \ge 16$, this concludes the proof. □

Inserting (4.11) into (4.10) we get

$$\|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} \leq C \log N \qquad \forall\, k = 1, \ldots, N, \tag{4.13}$$

which proves the first part of (4.3).

We now bound the time derivative of the L^2 norm of $X_{2,t}^N$: using that M_N has mass bounded by $2N$, in (4.6) we can easily estimate

$$\left| N \int_0^1 \Big[\mathbf{y}_{0,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big) - \mathbf{y}_{0,t}'\Big(X_{0,t}(\lambda_k)\Big)\Big]\, ds \cdot X_{1,t}^{N,k}(\hat\lambda) \right| \leq C |X_{1,t}^{N,k}|^2 + \frac{C}{N} |X_{1,t}^{N,k}|\, |X_{2,t}^{N,k}|,$$

$$\int_0^1 \left| \int \partial_1 \mathbf{z}_t \Big(X_t^{N,k,s}(\hat\lambda), y\Big)\, dM_N^{X_t^{N,s}}(y) - \int \partial_1 \mathbf{z}_t \Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y) \right| ds$$
$$\leq C |X_{1,t}^{N,k}| + \frac{C}{N} |X_{2,t}^{N,k}| + \frac{C}{N} \sum_j \Big(|X_{1,t}^{N,j}| + \frac{1}{N} |X_{2,t}^{N,j}| \Big),$$

$$\sum_{j=1}^N \int_0^1 \left| \partial_2 \mathbf{z}_t \Big(X_t^{N,k,s}(\hat\lambda), X_t^{N,j,s}(\hat\lambda)\Big) - \partial_2 \mathbf{z}_t \Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big) \right| ds\, |X_{1,t}^{N,j}|$$
$$\leq \frac{C}{N} \sum_j \Big(|X_{1,t}^{N,j}|^2 + \frac{1}{N} |X_{2,t}^{N,j}|\, |X_{1,t}^{N,j}| \Big),$$

hence

$$\begin{aligned}
\frac{d}{dt} \|X_{2,t}^N\|_{L^2(\mathbb{P}_V^N)}^2 &= 2 \int \sum_k X_{2,t}^{N,k} \cdot \dot X_{2,t}^{N,k}\, d\mathbb{P}_V^N \\
&\leq C \int \sum_k |X_{2,t}^{N,k}|^2\, d\mathbb{P}_V^N + C \int \sum_k |X_{1,t}^{N,k}|^2 |X_{2,t}^{N,k}| d\mathbb{P}_V^N \\
&+ \frac{C}{N} \int \sum_k |X_{1,t}^{N,k}| |X_{2,t}^{N,k}|^2 d\mathbb{P}_V^N + C \int \sum_k |X_{1,t}^{N,k}| |X_{2,t}^{N,k}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N^2} \int \sum_k |X_{2,t}^{N,k}|^3\, d\mathbb{P}_V^N + \frac{C}{N} \int \sum_{k,j} |X_{1,t}^{N,j}|\, |X_{1,t}^{N,k}|\, |X_{2,t}^{N,k}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N^3} \int \sum_{k,j} |X_{2,t}^{N,k}|^2\, |X_{2,t}^{N,j}|\, d\mathbb{P}_V^N \\
&+ \sum_k \int X_{2,t}^{N,k} \cdot \int_0^1 \Big[\int \partial_1 \mathbf{z}_t \Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y) \Big]\, ds \cdot X_{1,t}^{N,k}\, d\mathbb{P}_V^N \\
&+ \frac{C}{N} \int \sum_{k,j} |X_{1,t}^{N,j}|^2\, |X_{2,t}^{N,k}|\, d\mathbb{P}_V^N + \frac{C}{N^2} \int \sum_{k,j} |X_{2,t}^{N,k}|\, |X_{2,t}^{N,j}|\, |X_{1,t}^{N,j}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N} \int \sum_{k,j} |X_{2,t}^{N,k}|\, |X_{2,t}^{N,j}|\, d\mathbb{P}_V^N.
\end{aligned}$$

Using the trivial bounds $|X_{1,t}^{N,k}| \le C\,N$ and $|X_{2,t}^{N,k}| \le C\,N^2$, (4.12), and elementary inequalities such as, for instance,

$$\sum_{k,j} |X_{1,t}^{N,j}|\,|X_{1,t}^{N,k}|\,|X_{2,t}^{N,k}| \le \sum_{k,j}\Big(|X_{1,t}^{N,j}|^4 + |X_{1,t}^{N,k}|^4 + |X_{2,t}^{N,k}|^2\Big),$$

we obtain

$$\begin{aligned}\frac{d}{dt}\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} \le C\bigg(&\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + \int\sum_k |X_{1,t}^{N,k}|^4\,d\mathbb{P}_V^N \\ &+ \int\sum_k |X_{1,t}^{N,k}|^2\,d\mathbb{P}_V^N + \sum_k \log N\,\|X_{2,t}^{N,k}\|_{L^2(\mathbb{P}_V^N)}\|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)}\bigg).\end{aligned} \tag{4.14}$$

We now observe that, thanks to (4.13), that the last term is bounded by

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + (\log N)^2\sum_k \|X_{1,t}^{N,k}\|^2_{L^4(\mathbb{P}_V^N)} \le \|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + C\,N(\log N)^4.$$

Hence, using that $\|X_{1,t}^{N,k}\|_{L^2(\mathbb{P}_V^N)} \le \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)}$ and (4.13) again, the right hand side of (4.14) can be bounded by $C\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + C\,N(\log N)^4$ and a Gronwall argument gives

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} \le C\,N(\log N)^4,$$

thus

$$\|X_{2,t}^N\|_{L^2(\mathbb{P}_V^N)} \le C\,N^{1/2}(\log N)^2,$$

concluding the proof of (4.3).

We now prove (4.4): using (4.2) we have

$$\begin{aligned}&|\dot X_{1,t}^{N,k}(\hat\lambda) - \dot X_{1,t}^{N,k'}(\hat\lambda)| \\ &\le |\mathbf{y}_{0,t}'(X_{0,t}(\lambda_k)) - \mathbf{y}_{0,t}'(X_{0,t}(\lambda_{k'}))|\,|X_{1,t}^{N,k}(\hat\lambda)| \\ &+ |\mathbf{y}_{0,t}'(X_{0,t}(\lambda_{k'}))|\,|X_{1,t}^{N,k}(\hat\lambda) - X_{1,t}^{N,k'}(\hat\lambda)| + |\mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) - \mathbf{y}_{1,t}(X_{0,t}(\lambda_{k'}))| \\ &+ \left|\int\Big(\mathbf{z}_t(X_{0,t}(\lambda_k),y) - \mathbf{z}_t(X_{0,t}(\lambda_{k'}),y)\Big)\,dM_N^{X_{0,t}}(y)\right| \\ &+ \frac{1}{N}\sum_{j=1}^N\int_0^1\Big|\partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_k),X_{0,t}(\lambda_j)\Big) - \partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_{k'}),X_{0,t}(\lambda_j)\Big)\Big|\,ds\,|X_{1,t}^{N,j}(\hat\lambda)|.\end{aligned}$$

Using that $|X_{0,t}(\lambda_k) - X_{0,t}(\lambda_{k'})| \le C|\lambda_k - \lambda_{k'}|$, the bound (4.9), the Lipschitz regularity of $\mathbf{y}_{0,t}'$, $\mathbf{y}_{1,t}$, $\mathbf{z}_t$, and $\partial_2\mathbf{z}_t$, and the fact that

$$\left\|\int \partial_1\mathbf{z}_t(\cdot,\lambda)\,dM_N^{X_{0,t}}(\lambda)\right\|_\infty \le C\,\log N\sqrt{N}$$

with probability greater than $1 - N^{-N/C}$ (see (3.17)), we get

$$|\dot X_{1,t}^{N,k}(\hat\lambda) - \dot X_{1,t}^{N,k'}(\hat\lambda)| \le C|X_{1,t}^{N,k}(\hat\lambda) - X_{1,t}^{N,k'}(\hat\lambda)| + C\log N\sqrt{N}|\lambda_k - \lambda_{k'}|$$

outside a set of probability less than $N^{-N/C}$, so (4.4) follows from Gronwall. □

5. Transport and universality

In this section we prove Theorem 1.5 on universality using the regularity properties of the approximate transport maps obtained in the previous sections.

Proof of Theorem 1.5. Let us first remark that the map T_0 from Theorem 1.4 coincides with $X_{0,1}$, where $X_{0,t}$ is the flow defined in (4.1). Also, notice that $X_1^N : \mathbb{R}^N \to \mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_{V+W}^N$ (see Lemma 2.2 and Proposition 3.1). Set $\hat{X}_1^N := X_{0,1}^N + \frac{1}{N}X_{1,1}^N$, with $X_{0,t}^N$ and $X_{1,t}^N$ as in Lemma 4.1. Since $X_1^N - \hat{X}_1^N = \frac{1}{N^2}X_{2,1}^N$, recalling (4.3) and using Hölder inequality to control the L^1 norm with the L^2 norm, we see that

$$\begin{aligned}\left|\int g(\hat{X}_1^N)\,d\mathbb{P}_V^N - \int g(X_1^N)\,d\mathbb{P}_V^N\right| &\leq \|\nabla g\|_\infty \frac{1}{N^2}\int |X_{2,1}^N|\,d\mathbb{P}_V^N \\ &\leq \|\nabla g\|_\infty \frac{1}{N^2}\|X_{2,1}^N\|_{L^2(\mathbb{P}_V)} \\ &\leq C\|\nabla g\|_\infty \frac{(\log N)^2}{N^{3/2}}.\end{aligned} \tag{5.1}$$

This implies that also $\hat{X}_1^N : \mathbb{R}^N \to \mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_{V+W}^N$. In addition, we see that $\hat{X}_1^N$ preserves the order of the λ_i with large probability. Indeed, first of all $X_{0,t} : \mathbb{R} \to \mathbb{R}$ is the flow of $\mathbf{y}_{0,t}$ which is Lipschitz with some constant L, hence differentiating (4.1) we get

$$\frac{d}{dt}\left|X'_{0,t}\right| \leq \left|\mathbf{y}'_{0,t}(X_{0,t})\right|\left|X'_{0,t}\right| \leq L\left|X'_{0,t}\right|, \qquad X'_{0,0} = 1,$$

so Gronwall's inequality gives the bound

$$e^{-Lt} \leq \left|X'_{0,t}\right| \leq e^{Lt}$$

Since $X'_{0,0} = 1$, it follows by continuity that $X'_{0,t}$ must remain positive for all time and it satisfies

$$e^{-Lt} \leq X'_{0,t} \leq e^{Lt}, \tag{5.2}$$

from which we deduce that

$$e^{-Lt}\big(\lambda_j - \lambda_i\big) \leq X_{0,t}(\lambda_j) - X_{0,t}(\lambda_i) \leq e^{Lt}\big(\lambda_j - \lambda_i\big), \qquad \forall\,\lambda_i < \lambda_j.$$

In particular,

$$e^{-L}\big(\lambda_j - \lambda_i\big) \leq X_{0,1}(\lambda_j) - X_{0,1}(\lambda_i) \leq e^{L}\big(\lambda_j - \lambda_i\big).$$

Hence, using the notation $\hat{\lambda} = (\lambda_1, \ldots, \lambda_N)$, since

$$\left|\frac{1}{N}X_{1,t}^{N,j}(\hat{\lambda}) - \frac{1}{N}X_{1,t}^{N,i}(\hat{\lambda})\right| \leq C\,\frac{\log N}{\sqrt{N}}|\lambda_i - \lambda_j|$$

with probability greater than $1 - N^{-N/C}$ (see (4.4)), we get

$$\frac{1}{C}\big(\lambda_j - \lambda_i\big) \leq \hat{X}_1^{N,j}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda}) \leq C\big(\lambda_j - \lambda_i\big)$$

with probability greater than $1 - N^{-N/C}$.

We now make the following observation: the ordered measures $\tilde{P}_V^N$ and $\tilde{P}_{V+W}^N$ are obtained as the image of $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$ via the map $R : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$[R(x_1, \ldots, x_N)]_i := \min_{\sharp J = i}\max_{j \in J} x_j.$$

Notice that this map is 1-Lipschitz for the sup norm.

Hence, if g is a function of m-variables we have $\|\nabla(g\circ R)\|_\infty \le \sqrt{m}\|\nabla g\|_\infty$, so by Lemma 2.2, Proposition 3.1, and (5.1), we get

$$\left|\int g\circ R(\hat X_1^N)\, d\mathbb{P}_V^N - \int g\circ R\, d\mathbb{P}_{V+W}^N\right| \le C\frac{(\log N)^3}{N}\,\|g\|_\infty + C\sqrt{m}\,\frac{(\log N)^2}{N^{3/2}}\,\|\nabla g\|_\infty.$$

Since $\hat X_1^N$ preserves the order with probability greater than $1-N^{-N/C}$, we can replace $g\circ R(N\hat X_1^N)$ with $g(N\hat X_1^N\circ R)$ up to a very small error bounded by $\|g\|_\infty N^{-N/C}$. Hence, since $R_\#\mathbb{P}_V^N = \tilde P_V^N$ and $R_\#\mathbb{P}_{V+W}^N = \tilde P_{V+W}^N$, we deduce that, for any Lipschitz function $f:\mathbb{R}^m\to\mathbb{R}$,

$$\begin{aligned}\Bigg|\int f\big(N(\lambda_{i+1}-\lambda_i),\dots,N(\lambda_{i+m}-\lambda_i)\big)\, d\tilde P_{V+W}^N \\ -\int f\Big(N\big(\hat X_1^{N,i+1}(\hat\lambda)-\hat X_1^{N,i}(\hat\lambda)\big),\dots,N\big(\hat X_1^{N,i+m}(\hat\lambda)-\hat X_1^{N,i}(\hat\lambda)\big)\Big)\, d\tilde P_V^N\Bigg| \\ \le C\,\frac{(\log N)^3}{N}\,\|f\|_\infty + C\,\sqrt{m}\,\frac{(\log N)^2}{N^{1/2}}\,\|\nabla f\|_\infty.\end{aligned}$$

Recalling that

$$\hat X_1^{N,j}(\hat\lambda) = X_{0,1}(\lambda_j) + \frac{1}{N}X_{1,1}^{N,j}(\hat\lambda),$$

we observe that, as $X_{0,1}$ is of class C^2,

$$X_{0,1}(\lambda_{i+k}) - X_{0,1}(\lambda_i) = X_{0,1}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i) + O(|\lambda_{i+k}-\lambda_i|^2).$$

Also, by (4.4) we deduce that, out of a set of probability bounded by $N^{-N/C}$,

$$|X_{1,1}^{N,i+k}(\hat\lambda) - X_{1,1}^{N,i}(\hat\lambda)| \le C\,\log N\sqrt{N}\,|\lambda_{i+k}-\lambda_i|\,. \tag{5.3}$$

As $X_{0,1}'(\lambda_i)\ge e^{-L}$ (see (5.2)) we conclude that

$$\frac{1}{N}\,|X_{1,1}^{N,i+k}(\hat\lambda) - X_{1,1}^{N,i}(\hat\lambda)| \le C\,|X_{0,1}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i)|\,\frac{\log N}{N^{1/2}}$$

and

$$O(|\lambda_{i+k}-\lambda_i|^2) = O\big(\big|X_{0,1}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i)\big|^2\big)$$

hence with probability greater than $1-N^{-N/C}$ it holds

$$\hat X_1^{N,i+k}(\hat\lambda) - \hat X_1^{N,i}(\hat\lambda) = X_{0,t}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i)\left[1+O\left(\frac{\log N}{N^{1/2}}\right)+O\big(\big|X_{0,1}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i)\big|\big)\right].$$

Since we assume f supported in $[-M,M]^m$, the domain of integration is restricted to $\hat\lambda$ such that $\{NX_{0,t}'(\lambda_i)\,(\lambda_i-\lambda_{i+k})\}_{1\le k\le m}$ is bounded by $2M$ for N large enough, therefore

$$\hat X_1^{N,i+k}(\hat\lambda) - \hat X_1^{N,i}(\hat\lambda) = X_{0,t}'(\lambda_i)\,(\lambda_{i+k}-\lambda_i) + O\left(2M\frac{\log N}{N^{3/2}}\right) + O\left(\frac{4M^2}{N^2}\right),$$

from which the first bound follows easily.

For the second point we observe that $a_{V+W} = X_{0,1}(a_V)$ and, arguing as before,

$$\Big| \int f\big(N^{2/3}(\lambda_1 - a_{V+W}), \ldots, N^{2/3}(\lambda_m - a_{V+W})\big)\, d\tilde{P}^N_{V+W} - \int f\Big(N^{2/3}\big(\hat{X}_1^{N,1}(\hat\lambda) - X_{0,1}(a_V)\big), \ldots, N^{2/3}\big(\hat{X}_1^{N,m}(\hat\lambda) - X_{0,1}(a_V)\big)\Big)\, d\tilde{P}^N_V \Big| \le C\,\frac{(\log N)^3}{N}\,\|f\|_\infty + C\sqrt{m}\frac{(\log N)^2}{N^{5/6}}\,\|\nabla f\|_\infty.$$

Since, by (4.3),

$$\hat{X}_1^{N,i}(\lambda) = X_{0,1}(\lambda_i) + O_{L^4(\mathbb{P}^N_V)}\left(\frac{\log N}{N}\right) = X_{0,1}(a_V) + X'_{0,1}(a_V)\,(\lambda_i - a_V) + O\big(|\lambda_i - a_V|^2\big) + O_{L^4(\mathbb{P}^N_V)}\left(\frac{\log N}{N}\right),$$

we conclude as we did for the first point. □

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