UNIFORM BOUNDEDNESS FOR FINITE MORSE INDEX SOLUTIONS TO SUPERCRITICAL SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider finite Morse index solutions to semilinear elliptic questions, and we investigate their smoothness. It is well-known that:
- For $n = 2$, there exist Morse index 1 solutions whose L^∞ norm goes to infinity.
- For $n \geq 3$, uniform boundedness holds in the subcritical case for power-type nonlinearities, while for critical nonlinearities the boundedness of the Morse index does not prevent blow-up in L^∞.
In this paper, we investigate the case of general supercritical nonlinearities inside convex domains, and we prove an interior a priori L^∞ bound for finite Morse index solution in the sharp dimensional range $3 \leq n \leq 9$. As a corollary, we obtain uniform bounds for finite Morse index solutions to the Gelfand problem constructed via the continuity method.

1. Introduction

Given $\Omega \subset \mathbb{R}^n$ a bounded domain, and $f : \mathbb{R} \to \mathbb{R}$ a nonnegative C^1 function, we consider a solution $u : \Omega \to \mathbb{R}$ to the following semilinear equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Note that, by the nonnegativity of f and the maximum principle, $u > 0$ inside Ω (unless $u \equiv 0$).

Set $F(t) := \int_0^t f(s)\, ds$. Then (1.1) corresponds to the Euler-Lagrange equation for the energy functional

$$\mathcal{E}[u] := \int_\Omega \Big(\frac{|\nabla u|^2}{2} - F(u) \Big)\, dx.$$

Consider the second variation of $\mathcal{E}$, that is,

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \mathcal{E}[u + \epsilon\xi] = \int_\Omega \Big(|\nabla \xi|^2 - f'(u)\xi^2 \Big)\, dx.$$

Given a subdomain $\Omega' \subseteq \Omega$ and $k \in \mathbb{N}$, u is said to have *finite Morse index* k in Ω', and we write $\operatorname{ind}(u, \Omega') = k$, if k is the maximal dimension of a subspace $X_k \subset C_c^1(\Omega')$ such that, for any $\xi \in X_k \setminus \{0\}$,

$$Q_u(\xi) := \int_{\Omega'} \Big(|\nabla \xi|^2 - f'(u)\xi^2 \Big)\, dx < 0.$$

Also, u is said to be *stable* in Ω' if $\operatorname{ind}(u, \Omega') = 0$ (that is, $Q_u(\xi) \geq 0$ for all $\xi \in C_c^1(\Omega')$).

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1.1. **Finite Morse index vs uniform boundedness.** The idea of using a bound on the Morse index to characterize the uniform boundedness of a solution to a semilinear elliptic equation was first introduced in the seminal paper [1]. In this work, as well in several other subsequent papers (see for instance [27, 21]), the authors considered (variants of) the subcritical case, namely

$$f(t) \simeq (\alpha+t)^{\frac{n+2}{n-2}-\epsilon}, \qquad \alpha, \epsilon > 0,$$

and they proved that the boundedness of solutions is equivalent to the boundedness of the Morse index.[1]

In the critical case, namely

$$f(u) = (\alpha+u)^{\frac{n+2}{n-2}}, \qquad \alpha > 0,$$

the finiteness of the Morse index does not imply the boundedness of the solutions. Indeed it is not difficult to check that the functions

$$u_{\alpha,\mu}(x) = \left(\left(\frac{\mu\sqrt{n(n-2)}}{\mu^2+|x|^2}\right)^{\frac{n-2}{2}} - \alpha\right)_+, \ \mu > 0 \tag{1.2}$$

are solutions with Morse index 1. In particular, choosing $\alpha_\mu := \left(\frac{\mu\sqrt{n(n-2)}}{1+\mu^2}\right)^{\frac{n-2}{2}}$ so that $u_{\alpha_\mu,\mu} = 0$ on ∂B_1, and letting $\mu \to 0$, one can construct a family Morse index 1 solutions in B_1 whose L^∞ norm goes to infinity (see, e.g., [8]).

The supercritical case, instead, is much less understood. The special case where f is a polynomial or an exponential function has been studied in [16] and [11], respectively. There, the uniform boundedness of solutions is obtained by proving suitable Liouville-type results. Unfortunately, this approach does not seem suitable for general nonlinearities.

We finally mention a recent result [15], where the authors investigate the regularity and symmetry properties of finite Morse index solutions.

1.2. **Main result: finite Morse index solutions with supercritical nonlinearities are uniformly bounded.** Very recently, in [9] the authors investigated the properties of stable solutions for *all* nonlinearities, and they proved uniform boundedness when $3 \le n \le 9$,[2] and interior $W^{1,2}$ estimates in all dimensions.

In this paper, we exploit these result to develop a series of new tools for finite Morse index solutions (cf. Section 1.3.3 below) that allow us to prove a universal L^∞ bound for solutions to (1.1) when f grows superlinearly in a suitably quantified way.[3] As common in these problems, we assume that $f(0) > 0$ (actually, we quantify this assumption by asking that $f(0) \ge c_0 > 0$, so to better emphasize the dependences in our L^∞ bound). This assumption is particularly natural in the superlinear case, since the Derrick-Pohozaev identity prevents the existence of nontrivial solutions (see [17, Theorem 1, Page 515]).

[1] Note however that this result does not cover the full subcritical case: as shown in [22, Lemma 5 and Theorem 1(iii))], for $f_\lambda(t) = \lambda(1+t)^p$, $\lambda > 0$ and $p < \frac{n+2}{n-2}$, there exists a family u_λ of solutions with Morse index 1 with $\|u_\lambda\|_{L^\infty(B_1)} \to \infty$ as $\lambda \to 0$. In other words, in the subcritical case, both upper and lower bounds are needed on f in order to show the equivalence between boundedness in L^∞ and boundedness of the Morse index.

[2] In the stable case, the case $n = 2$ is a consequence of the case $n = 3$ by noticing that a stable solution in two dimensions is also stable in three dimensions (by looking at it as a function constant in the third variable). This is why the results in [9] hold for $n \le 9$. This is not the case anymore when $\mathrm{ind}(u) > 0$, and indeed a change of behavior of finite Morse index solutions between dimension $n = 2$ and dimension $3 \le n \le 9$ was already observed in [22]. In particular, as [22, Fig. 1 b, pag. 245] shows, there exists a curve of two-dimensional solutions with Morse index 1 that blows-up in L^∞.

[3] Our quantitative superlinarity assumption (1.4) already appeared in the paper [26] (see also [13]), where the author proved the uniqueness of solutions to (1.3) for small values of λ.

Actually, because of applications to the Gelfand problem described in Section 1.4 below, it will be convenient to prove a more robust result that establishes a uniform bound whenever the nonlinearity is of the form λf, where $\lambda \in [0, \hat{\lambda}]$ for some fixed $\hat{\lambda}$. Also, for the sake of generality, it is interesting to observe how the bound depends on f. So, instead of considering a fixed nonlinearity f, we assume that f belongs to a locally compact C^1 family. As shown in Section 1.3.2 below, this assumption can be considerably weakened if f is assumed to be convex.

Theorem 1.1. *Let $3 \le n \le 9$, $\Omega \subset \mathbb{R}^n$ a bounded convex domain, and $c_0 > 0$. Consider*

$$\mathcal{K} \subset \{h \in C^1(\mathbb{R}) :\ h \ge 0,\ h' \ge 0, h(0) \ge c_0\},$$

and assume that $\mathcal{K}$ is compact for the $C^1_{\mathrm{loc}}(\mathbb{R})$ topology. Let $\hat{\lambda} > 0$, and let $u \in C^2(\Omega)$ solve

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for some $f \in \mathcal{K}$ and $\lambda \in [0, \hat{\lambda}]$. Finally, assume that $\mathrm{ind}(u, \Omega) \le k$ and that there exist $\epsilon, t_0 > 0$ such that

$$f(t)t \ge \left(\frac{2n}{n-2} + \epsilon\right) F(t) \quad \text{for all } \ t \ge t_0, \tag{1.4}$$

where $F(t) := \int_0^t f(s)\, ds$. Then

$$\|u\|_{L^\infty(\Omega)} \le C(n, k, \mathcal{K}, \hat{\lambda}, \epsilon, t_0, \Omega).$$

Remark 1.2. We observe that, as a consequence of (1.4), it holds

$$f(t) \ge c_1 t^{\frac{n+2}{n-2}+\epsilon} \qquad \forall\, t \ge 0, \tag{1.5}$$

with $c_1 := f(0) t_0^{-\frac{n+2}{n-2}-\epsilon}$.

Indeed, (1.4) can be rewritten as

$$F'(t) \ge \frac{\left(\frac{2n}{n-2} + \epsilon\right)}{t} F(t) \quad \text{for all } \ t \ge t_0,$$

so it follows from Grönwall inequality that

$$F(t) \ge F(t_0) \left(\frac{t}{t_0}\right)^{\frac{2n}{n-2}+\epsilon}.$$

Inserting this information in (1.4), we get

$$f(t) \ge \left(\frac{2n}{n-2} + \epsilon\right) F(t_0) t_0^{-\frac{2n}{n-2}-\epsilon} t^{\frac{n+2}{n-2}+\epsilon} \quad \text{for all } \ t \ge t_0.$$

Also, since f is increasing we have $F(t_0) \ge f(0) t_0$, and therefore

$$f(t) \ge \begin{cases} f(0)\left(\frac{2n}{n-2} + \epsilon\right) t_0^{-\frac{n+2}{n-2}-\epsilon} t^{\frac{n+2}{n-2}+\epsilon} & \text{for } t \ge t_0 \\ f(0) & \text{for } \ 0 \le t < t_0, \end{cases}$$

which implies (1.5).

Remark 1.3. As mentioned before, the dimensional range $3 \le n \le 9$ follows from [9, Theorem 1.2], since boundedness of stable solutions for all nonlinearities is true only under this assumption. However, for some particular choices of nonlinearities (e.g., $f(u) = (1 + u)^p$ for suitable values of p), we believe that our ideas and techniques could be applied also in higher dimension (cf. [12]).

1.3. **About Theorem 1.1: extensions and tools used in the proof.** We first discuss some possible extensions and generalizations of Theorem 1.1, and then we briefly present the three key ingredients behind its proof.

1.3.1. *On the convexity of* Ω. The convexity assumption on Ω in Theorem 1.1 allows us:
- to focus only on interior regularity, since the regularity near the boundary is handled via the moving plane method, see Lemma 2.8 below;
- to apply the classical Derrick-Pohozaev argument on convex domains, see the argument after (3.9).

We believe that a nontrivial modification of our techniques could be used to analyze the boundary behavior inside general smooth domains. However our proof strongly relies on the Derrick-Pohozaev argument, and this requires Ω to be at least star-shaped (see, e.g., [17, Theorem 1, Page 515]). Hence, it looks likely to us that by combining the ideas developed in this paper with the boundary regularity from [9], one should be able to extend Theorem 1.1 to (sufficiently smooth) star-shaped domains.

1.3.2. *A result for convex nonlinearities.* The assumption that the nonlinearity f belongs to a family $\mathcal{K}$ that is compact for the $C^1_{\mathrm{loc}}(\mathbb{R})$ topology can be removed, if one assumes the nonlinearities to be convex and to be dominated by a fixed continuous nonnegative function $g:\mathbb{R}\to\mathbb{R}$. More precisely, if f is a convex function such that $0\le f\le g$ and (1.4) holds, then $\|u\|_{L^\infty(\Omega)}\le C(n,k,g,\hat{\lambda},\epsilon,t_0,\Omega)$.

Indeed, the compactness assumption in C^1_{loc} is used only to apply Proposition 2.3. If all the nonlinearities are convex, then the bound $0\le f\le g$ guarantees compactness in C^0_{loc}. Therefore, one only needs to check that Proposition 2.3 holds if f_j are convex functions satisfying $f_j\to f_\infty$ in $C^0_{\mathrm{loc}}(\mathbb{R})$. This can be done by suitable adapting the notion of stability for convex functions, defining

$$Q_u(\xi):=\int_{\Omega'}\Big(|\nabla\xi|^2-f_-'(u)\xi^2\Big)\,dx\qquad\text{with } f_-'(t):=\lim_{\tau\to0^+}\frac{f(t)-f(t-\tau)}{\tau}=\sup_{\tau>0}\frac{f(t)-f(t-\tau)}{\tau}.$$

Indeed, with this definition, the results from [9] still apply. In addition, the following lower semicontinuity property holds:

$$t_j\to t,\quad f_j\to f\text{ in }C^0_{\mathrm{loc}}(\mathbb{R})\qquad\Rightarrow\qquad f_-'(t)\le\liminf_{j\to\infty}(f_j)_-'(t_j),$$

and this allows one to show that upper bounds on the Morse index are preserved. We leave the details to the interested reader.

1.3.3. *Main tools.* As mentioned before, the proof of Theorem 1.1 is based on a series of new important results for finite Morse index solutions. These are:
(i) A general stability result for bounded Morse index solutions stating that, for $3\le n\le 9$, these families are weakly compact in $W^{1,2}$ and they converge in C^2_{loc} outside finitely many points (see Proposition 2.3). This result relies on the smoothness of stable solutions for $n\le 9$ obtained in [9], and on a slight improvement of it proved in Appendix A.
(ii) A uniform $W^{1,2}$ integrability estimate for finite Morse index solutions (see Proposition 2.6). This result depends both on the supercriticality assumption (1.4) and on the interior $W^{1,2}$ estimates for stable solutions, cf. [9].
(iii) A ε-regularity theorem for finite Morse index solutions stating that, if the $W^{1,2}$ norm of a solution inside a ball B_r decays sufficiently fast for $r\in[\varepsilon,1]$ with $\varepsilon\ll1$, then it decays all the way to the origin (see Proposition 2.7).
It is worth observing that while (i) needs the dimensional restriction $n\le 9$, both (ii) and (iii) hold in every dimension. Besides playing a crucial role in proving Theorem 1.1, we believe that these results have their own interest.

1.4. **An application to the Gelfand problem associated to analytic supercritical nonlinearities.** Given $f : \mathbb{R} \to \mathbb{R}$ nonnegative and increasing, and $\lambda \geq 0$, the so-called Gelfand problem for f consists in studying the nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem has a long history: it was first presented by Barenblatt in a volume edited by Gelfand [18], and a series of authors studied it later, in particular in the range where u is stable; we refer the interested reader to [2, 3, 4, 14, 6] for a complete account on this topic.

In this paper we want to study the solution curve associated to the Gelfand problem: we look for a continuous curve $\lambda\colon [0,\infty) \to [0,\infty)$ with $\lambda(0) = 0$, and for a one-parameter family of solutions $\{u_s\}_{s\geq 0}$, such that

$$\begin{cases} -\Delta u_s = \lambda(s) f(u_s) & \text{in } \Omega, \\ u_s = 0 & \text{on } \partial\Omega. \end{cases}$$

When $\Omega = B_1$, the cases $f(t) = (1+\alpha t)^\beta$, $\alpha, \beta > 0$, and $f(t) = e^t$, have been fully understood in [22] via ODE methods. In particular, when $3 \leq n \leq 9$, the authors proved that there are infinitely many turning points in the solution curve $s \mapsto (\lambda(s), \|u_s\|_{L^\infty(\Omega)},)$ for suitable values of β. Later, similar phenomena were observed for special functions f or in low dimensional domains with suitable symmetries (see, e.g., [24, 10, 20, 11, 23] and the reference therein).

Assume now that Ω is a convex set of class C^3, let $C^1_0(\overline{\Omega})$ denote the Banach space of C^1 functions on $\overline{\Omega}$ that vanish on $\partial\Omega$, and consider the following open subset of $C^1_0(\overline{\Omega})$ endowed with the C^1 topology:

$$\mathcal{O} := \{u \in C^1_0(\overline{\Omega}) \,:\, u > 0 \text{ in } \Omega,\, \partial_\nu u|_{\partial\Omega} < 0\}, \tag{1.6}$$

where ν denotes the outer unit normal to $\partial\Omega$. Following [10], assume the map

$$\mathcal{O} \ni u \mapsto f(u) \in C^0(\overline{\Omega})$$

to be real analytic (as noted in [10], this is the case for instance if f is analytic). Then, thanks to our Theorem 1.1, one can apply the global analytic bifurcation theory developed in [5, Section 2.1] to show the existence of a piecewise analytic continuous curve $[0,\infty) \ni s \mapsto (\lambda(s), u_s)$, with $(\lambda(0), u_0) = (0, 0)$, such that both $\|u_s\|_{L^\infty(\Omega)}$ and $\operatorname{ind}(u_s, \Omega)$ tend to infinity as $s \to \infty$. Moreover there exists a sequence $(\lambda(s_i), u_{s_i})$ such that $\|u_{s_i}\|_{L^\infty(\Omega)} \to \infty$ and each point of this sequence is either a bifurcation or a turning point. This is a complete statement:

Theorem 1.4. *Let $3 \leq n \leq 9$, $\Omega \subset \mathbb{R}^n$ a bounded convex domain of class C^3, and $f > 0$ an increasing analytic function satisfying* (1.4). *Let*

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R}_+ \times C^1_0(\overline{\Omega})\colon -\Delta u - \lambda f(u) = 0 \textit{ and } -\Delta - \lambda f'(u) \textit{ is invertible with bounded inverse}\}.$$

Then there exist two continuous mappings

$$h_1\colon \mathbb{R}_+ \to \mathbb{R}_+, \qquad h_2\colon \mathbb{R}_+ \to C^1_0(\overline{\Omega}),$$

so that, denoting $h = (h_1, h_2)$, we have:

(i) $h\colon \mathbb{R}_+ \to \overline{\mathcal{S}}$ *and* $\lim_{s\to\infty} \operatorname{ind}(h_2(s), \Omega) = \infty$.
(ii) *h is injective on $h^{-1}(\mathcal{S})$ with $h_1'(s) \neq 0$, and real analytic at all points $s \in h^{-1}(\mathcal{S})$.*
(iii) *The set $h^{-1}(\overline{\mathcal{S}} \setminus \mathcal{S})$ consists of isolated values.*
(iv) *For every point $s_0 \in h^{-1}(\overline{\mathcal{S}} \setminus \mathcal{S})$ there exists an injective and continuous reparameterization $s = \gamma(\sigma)$, $\sigma \in [-1, 1]$, such that $s_0 = \gamma(0)$ and $h \circ \gamma$ is a real analytic function whose derivatives might only vanish at* 0.

(v) There are infinitely many values of $s > 0$ where $h(s) \in \overline{\mathcal{S}} \setminus \mathcal{S}$ is either a bifurcation or a turning point. Namely, either in every neighborhood of $h(s)$ there exists a solution of (1.1) which is not in the image of h (and then $h(s)$ is a bifurcation point), or the previous case do not happen but h_1 is not locally injective (and then $h(s)$ is a turning point).

Proof. Let $\mathcal{O}$ be as in (1.6), and define the analytic map

$$\mathcal{F}\colon \mathbb{R}_+ \times \mathcal{O} \to C^1_0(\overline{\Omega}), \qquad \mathcal{F}(\lambda, u) := -u + \lambda \mathcal{A}(u),$$

where $\mathcal{A}(u) := (-\Delta)^{-1}[f(u)]$, and $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet Laplacian in Ω. Arguing exactly as in the proof of [10, Theorem 1] (see also the remark after the statement of the theorem), the result follows from [5, Section 2.1]. $\square$

Remark 1.5. It was pointed out in [10, Remark 4] that, when Ω is a C^3 strongly convex domain with certain symmetries, a careful modification of [25] gives that, the image of h is a smooth curve with only infinitely many turning points but not bifurcation points. Also, this property in generic in a neighborhood of such domains. We expect a similar result to hold also in our setting.

1.5. **Structure of the paper.** The paper is organized as follows. In Section 2 we present a series of results on finite Morse index solutions, which will be crucial for proving Theorem 1.1. Then, in Section 3 we prove Theorem 1.1. Finally, in a first appendix, we show that [9, Theorem 1.2] holds also for $W^{1,2}$ stable solution that are C^2 outside one point. This result is used in the proof of Proposition 2.3. Then, in a second appendix, we describe how the method in [9] implies uniform boundedness of solutions whenever the spectrum of $-\Delta - f'(u)$ is bounded from below.

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2. Technical tools on finite Morse index solutions

Let us fix some notation. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x)$ the Euclidean ball centered at x with radius r. The center is usually omitted when x is the origin. By αB we mean the ball with the same center as B but α times its radius. We write constants as positive real numbers $C(\cdot)$, with the parentheses including all the parameters on which the constants depend. We note that $C(\cdot)$ may vary between appearances, even within a chain of inequalities. Sometimes we use C_n, c_n to emphasize that a constant depends only on the dimension.

The goal of this section is to prove several new important results on finite Morse index solutions that will be used in the next section to prove Theorem 1.1. First, we need to introduce a notion of weak solution with bounded Morse index.

Definition 2.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, and let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative. We say that $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{U})$ is a weak solution of $-\Delta u = f(u)$ in $\mathcal{U}$ if $f(u) \in L^1_{\mathrm{loc}}(\mathcal{U})$ and*

$$\int_{\mathcal{U}} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathcal{U}} f(u)\, \varphi \, dx \qquad \forall\, \varphi \in C^1_c(\mathcal{U}).$$

Assume in addition that f is of class C^1. Then we say that u has finite Morse index $k \in \mathbb{N}$ *in $\mathcal{U}$, and we write* $\mathrm{ind}(u, \mathcal{U}) = k$, *if $f'(u) \in L^1_{\mathrm{loc}}(\mathcal{U})$ and k is the maximal dimension of a subspace $X_k \subset C^1_c(\mathcal{U})$ such that*

$$Q_u(\xi) := \int_{\mathcal{U}} \Big(|\nabla \xi|^2 - f'(u)\xi^2 \Big)\, dx < 0 \qquad \forall\, \xi \in X_k \setminus \{0\}.$$

As we shall see below, whenever $f' \geq 0$ it is possible to prove an a priori bound on the $L^1_{\rm loc}$ norm of $f'(u)$ in terms of the Morse index. Then, by Fatou's Lemma, this a priori bound holds for all weak solutions that are limits of smooth solutions (see Proposition 2.3 below).

Lemma 2.2. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative, increasing, and of class C^1, and let $u \in W^{1,2}_{\rm loc}(\mathcal{U})$ be a weak solution of $-\Delta u = f(u)$ in $\mathcal{U}$ with $\mathrm{ind}(u, \mathcal{U}) \leq k$. Then:*

(i) *If $\{\mathcal{U}_i\}_{i=1}^{k+1}$ is a disjoint family of open subsets of $\mathcal{U}$, then u is stable in at least one set $\mathcal{U}_i$.*
(ii) *The following uniform bound holds:*

$$\int_{B_r(\bar{x})} f'(u)\,dx \leq C_n(1+k)^{\frac{2}{n}} r^{n-2} \qquad \forall\, B_{2r(\bar{x})} \subset \mathcal{U}. \tag{2.1}$$

Proof. To prove (i) we note that, if by contradiction u was unstable inside each set $\mathcal{U}_i$, then there would exist functions $\xi_i \in C^1_c(\mathcal{U}_i)$ such that

$$\int_{\mathcal{U}} \left(|\nabla \xi_i|^2 - f'(u)\xi_i^2\right)\,dx < 0.$$

Since the functions $\{\xi_i\}_{i=1}^{k+1}$ have disjoint support, this implies that

$$\int_{\mathcal{U}} \left(|\nabla \xi|^2 - f'(u)\xi^2\right)\,dx < 0 \qquad \forall\, \xi \in \mathrm{Span}(\xi_1, \ldots, \xi_{k+1}) \setminus \{0\},$$

therefore $\mathrm{ind}(u, \mathcal{U}) \geq k+1$, a contradiction.

We now prove (ii), following the ideas in [19, Theorem 5.9]. Given an open set $\mathcal{O}$ and a pair of sets E, $F \subset \mathcal{O}$, the *p-capacity between E and F inside* $\mathcal{O}$ for $p > 1$ is defined as

$$\mathrm{Cap}_p(E,\, F,\, \mathcal{O}) = \inf\{\|v\|^p_{W^{1,\,p}(\mathcal{O})} :\ v \in \Delta(E,\, F)\},$$

where $\Delta(E,\, F)$ denotes the class of all functions $v \in W^{1,\,p}(\mathcal{O})$ that are continuous in $\mathcal{O}$ and satisfy $v = 1$ on E, and $v = 0$ on F. In particular, if $A = B_\rho(z) \setminus B_{\rho/2}(z)$ is an annulus such that $2A := B_{2\rho}(z) \setminus B_{\rho/4}(z)$ is contained inside $B_{2r}(\bar{x})$, to control $\mathrm{Cap}_2(A, \partial(2A), B_{2r}(\bar{x}))$ we can choose the function $v_{z,\rho}(x) := \min\left\{1, (4\rho^{-1}|x-z| - 1)_+, (2 - \rho^{-1}|x-z|)_+\right\}$ to obtain

$$\mathrm{Cap}_2(A,\, \partial(2A),\, B_{2r}(\bar{x})) \leq \|v_{z,\rho}\|_{W^{1,2}} = C(n)|A|^{1-\frac{2}{n}}. \tag{2.2}$$

Let us now consider the metric space $X := \overline{B_{2r}(\bar{x})}$ endowed with the Euclidean metric. In this space, we call "X-annuli" sets of the form $\big(B_\rho(z) \setminus B_{\rho/2}(z)\big) \cap \overline{B_{2r}(\bar{x})}$ for some $z \in \overline{B_{2r}(\bar{x})}$.

Define the measure on X given by $\sigma := \chi_{B_r(\bar{x})} f'(u)dx$, and let $\kappa \in \mathbb{N}$ be a large constant to be fixed later. Since σ has no atoms, we can apply [19, Theorem 1.1] to deduce the existence of a family of Euclidean annuli $\{A_i := B_{\rho_i}(z_i) \setminus B_{\rho_i/2}(z_i)\}_{i=1}^{\kappa}$, with $z_i \in B_{2r}(\bar{x})$, such that

$$\int_{B_r(\bar{x})} f'(u)\,dx \leq C_0(n)\,\kappa \int_{B_r(\bar{x}) \cap A_i} f'(u)\,dx \qquad \forall\, i = 1, \ldots, k+1 \tag{2.3}$$

and $\{(2A_i) \cap B_{2r}(\bar{x})\}_{i=1}^{\kappa}$ are pairwise disjoint.

With no loss of generality we can assume that $B_r(\bar{x}) \cap A_i \neq \emptyset$ (otherwise (2.3) would imply that $\int_{B_r(\bar{x})} f'(u)\,dx = 0$ and the result would be trivially true). Let us split these annuli into two families: if $\rho_i < r/4$ then we say that $i \in \mathcal{I}_1$, otherwise we say that $i \in \mathcal{I}_2$.

Note that, since $B_r(\bar{x}) \cap A_i \neq \emptyset$, for $i \in \mathcal{I}_2$ it holds $(2A_i) \cap B_{2r}(\bar{x}) \geq c_n r^n$ for some dimensional constant $c_n > 0$. Hence, since the sets $\{(2A_i) \cap B_{2r}(\bar{x})\}_{i \in \mathcal{I}_2}$ are disjoint, we deduce that $\#\mathcal{I}_2 \leq N_n$ for some dimensional constant $N_n \geq 1$.

On the other hand, when $i \in \mathcal{I}_1$, since $B_r(\bar{x}) \cap A_i \neq \emptyset$ and $\rho_i < r/4$ it follows that $(2A_i) \cap B_{2r}(\bar{x}) = 2A_i$, hence the sets $\{2A_i\}_{i \in \mathcal{I}_1}$ are pairwise disjoint. Also, it follows from (2.2) that

$$\mathrm{Cap}_2(A_i,\, \partial(2A_i),\, B_{2r}(\bar{x})) \leq C(n)|A_i|^{1-\frac{2}{n}}. \tag{2.4}$$

Now, fix $\kappa := N_n + 2(k+1)$ so that $\#\mathcal{I}_1 \geq 2(k+1)$. Since the sets $\{2A_i\}_{i \in \mathcal{I}_1}$ are pairwise disjoint and contained inside $B_{2r}(\bar{x})$, there exists a subset of indices $\mathcal{I}_1' \subset \mathcal{I}_1$ such that $\#\mathcal{I}_1' \geq k+1$ and

$$|2A_i| \leq \frac{1}{k+1}|B_{2r}| \qquad \forall\, i \in \mathcal{I}_1',$$

that combined with (2.4) gives

$$\mathrm{Cap}_2(A_i,\, \partial(2A_i),\, B_{2r}(\bar{x})) \leq C_1(n)(1+k)^{\frac{2}{n}-1} r^{n-2} \qquad \forall\, i \in \mathcal{I}_1'. \tag{2.5}$$

Now, assume by contradiction that (2.1) does not hold with $C_n = 4C_0C_1(N_n+1)$, namely

$$\int_{B_r(\bar{x})} f'(u) > C_n(1+k)^{\frac{2}{n}} r^{n-2}, \tag{2.6}$$

where C_0 and C_1 are as in (2.3) and (2.5). Then, since $2(N_n+1)(k+1) \geq \kappa$, combining (2.6), (2.5), and (2.3), we get

$$\mathrm{Cap}_2(A_i,\, \partial(2A_i),\, B_{2r}(\bar{x})) < (2C_0\kappa)^{-1} \int_{B_r(\bar{x})} f'(u)\, dx \leq \frac{1}{2} \int_{A_i} f'(u)\, dx \qquad \forall\, i \in \mathcal{I}_1'.$$

Choose functions $\xi_i \in C_c^1(2A_i)$ that almost minimize the capacity $\mathrm{Cap}_2(A_i,\, \partial(2A_i),\, B_{2r}(\bar{x}))$, so that

$$\int_{B_{2r}(\bar{x})} |\nabla \xi_i|^2\, dx \leq \frac{2}{3} \int_{A_i} f'(u)\, dx \leq \frac{2}{3} \int_{B_{2r}(\bar{x})} f'(u)\xi_i^2\, dx \qquad \forall\, i \in \mathcal{I}_1'.$$

Since the sets $\{2A_i\}_{i \in \mathcal{I}_1'}$ are pairwise disjoint and $\#\mathcal{I}_1' \geq k+1$, we conclude that $\{\xi_i\}_{i \in \mathcal{I}_1'}$ spans a $(k+1)$-dimensional subspace of $C_c^1(B_{2r}(\bar{x}))$ where the stability inequality fails. This contradicts $\mathrm{ind}(u, B_{2r}(\bar{x})) \leq k$ and concludes the proof. □

We now prove a crucial convergence result for weak $W^{1,2}$ limits of smooth solutions with bounded Morse index. Note that, a consequence of Proposition 2.3 below, limit of smooth solutions with bounded Morse index are still smooth. However the result does not provide any uniform bound on the sequence u_j. In particular, it could be that $\|u_j\|_{L^\infty} \to \infty$, as the example provided by (1.2) shows.

Proposition 2.3. *Let $n \leq 9$, $\mathcal{U} \subset \mathbb{R}^n$ an open set, and $u_j \in C^2(\mathcal{U})$ a sequence of functions satisfying*

$$-\Delta u_j = f_j(u_j) \qquad \text{in } \mathcal{U}$$

with $f_j : \mathbb{R} \to \mathbb{R}$ nonnegative, increasing, and of class C^1. Assume that

$$\mathrm{ind}(u_j, \mathcal{U}) \leq k \text{ for some } k \in \mathbb{N}, \qquad \sup_j \|u_j\|_{W^{1,2}(\mathcal{U})} < +\infty, \qquad f_j \to f_\infty \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}).$$

Then there exist a subsequence $u_{j(m)}$ and a discrete set $\Sigma_\infty \subset \mathcal{U}$, with $\#\Sigma_\infty \leq k$, such that

$$u_{j(m)} \rightharpoonup u_\infty \text{ in } W^{1,2}(\mathcal{U}), \qquad u_{j(m)} \to u_\infty \text{ in } C^2_{\mathrm{loc}}(\mathcal{U} \setminus \Sigma_\infty),$$

and u_∞ satisfies

$$-\Delta u_\infty = f_\infty(u_\infty) \text{ in } \mathcal{U}, \qquad f'_\infty(u_\infty) \in L^1_{\mathrm{loc}}(\mathcal{U}), \qquad \mathrm{ind}(u_\infty, \Omega) \leq k, \qquad u_\infty \in C^2(\mathcal{U}).$$

Proof. Given $x \in \mathcal{U}$, for any j we denote by $r_{j,x}$ the largest radius where u_j is stable around x:

$$r_{j,x} := \sup\{r \in [0, \operatorname{dist}(x, \partial\mathcal{U})) : \operatorname{ind}(u_j, B_r(x)) = 0\},$$

Then, we define

$$r_{\infty,x} := \limsup_{j\to\infty} r_{j,x}, \qquad \Sigma_\infty := \{x \in \mathcal{U} : r_{\infty,x} = 0\}.$$

We claim that Σ_∞ is a discrete set of cardinality at most k.

Indeed, suppose by contradiction that Σ_∞ contains $k+1$ points $x_1, \ldots, x_{k+1}$, and fix

$$0 < r < \min\left\{\min_{1\le i\le k+1} \operatorname{dist}(x_i, \partial\mathcal{U})), \frac{1}{2}\min_{1\le i,l\le k+1} |x_i - x_l|\right\}.$$

Since $r_{j,x_i} \to 0$ as $j \to \infty$ (because $x_i \in \Sigma_\infty$), for j large enough u_j is unstable inside each of the balls $\{B_r(x_i)\}_{i=1}^{k+1}$. However, since these balls are disjoint (by the choice of r), Lemma 2.2(i) provides the desired contradiction.

Consider now a family of compact sets $\{K_\ell\}_{\ell\in\mathbb{N}}$ such that $\mathcal{U} \setminus \Sigma_\infty = \cup_\ell K_\ell$, and for any ℓ consider the covering of K_ℓ given by $\{B_{r_{\infty/2,x}}(x)\}_{x\in K_\ell}$. By compactness, there exists a finite set of points $\{x_i\}_{i\in\mathcal{I}_\ell} \subset K_\ell$ such that $K_\ell \subset \cup_{i\in\mathcal{I}_\ell} B_{r_\infty/2,x_i}(x_i)$. Note that, since each set $\mathcal{I}_\ell$ is finite, for each $\ell \in \mathbb{N}$ we can choose a subsequence $j_\ell(m)$ such that

$$r_{\infty,x_i} = \lim_{m\to\infty} r_{j_\ell(m),x_i} \qquad \forall\, i \in \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_\ell.$$

Then, by a diagonal argument we can find a subsequence $j(m)$, independent of ℓ, such that

$$r_{\infty,x_i} = \lim_{m\to\infty} r_{j(m),x_i} \qquad \forall\, i \in \cup_{\ell\in\mathbb{N}}\mathcal{I}_\ell.$$

Since the functions $u_{j(m)}$ are uniformly bounded in $W^{1,2}(\mathcal{U})$, up to extracting a further subsequence, there exists a weak limit in $W^{1,2}(\mathcal{U})$ that we denote by u_∞. We now want to show that u_∞ satisfies all the desired properties.

First of all, for each $\ell \in \mathbb{N}$ we define the open set

$$\mathcal{O}_\ell := \cup_{i\in\mathcal{I}_\ell} B_{r_\infty/2,x_i}(x_i) \supset K_\ell.$$

Since $u_{j(m)}$ is stable on $B_{r_{j(m)},x_i}(x_i)$ and $r_{j(m),x_i} \to r_{\infty,x_i}$ as $m \to \infty$, it follows by [9, Theorem 1.2] and elliptic regularity[4] that

$$\|u_{j(m)}\|_{C^{2,\alpha}(\mathcal{O}_\ell)} \le C_{\ell,\alpha} \qquad \forall\, m \gg 1, \forall\, \alpha \in (0,1),$$

which implies that $u_{j(m)} \to u_\infty$ in $C^2(\mathcal{O}_\ell)$. Since $\cup_\ell \mathcal{O}_\ell = \mathcal{U} \setminus \Sigma_\infty$, this proves the convergence in $C^2_{\mathrm{loc}}(\mathcal{U} \setminus \Sigma_\infty)$.

To show that u_∞ solves the desired equation, by the $C^2_{\mathrm{loc}}(\mathcal{U}\setminus\Sigma_\infty)$ convergence it follows immediately that

$$-\Delta u_\infty = f_\infty(u_\infty) \qquad \text{in } \mathcal{U} \setminus \Sigma_\infty.$$

Then, since $u_\infty \in W^{1,2}(\mathcal{U})$ and Σ_∞ consists of finitely many points (hence it has zero $W^{1,2}$-capacity), the equation $-\Delta u_\infty = f_\infty(u_\infty)$ must hold inside the whole domain $\mathcal{U}$.

We now note that, thanks to Lemma 2.2(ii),

$$\int_{B_r(\bar{x})} f'_{j(m)}(u_{j(m)}) \le C_n(1+k)^{\frac{2}{n}} r^{n-2} \qquad \forall\, B_{2r(\bar{x})} \subset \mathcal{U}.$$

[4]Recall that $n \le 9$, and note that f_j are uniformly C^1 on compact set since they converge to f_∞.

Since $f'_{j(m)}(u_{j(m)})$ are nonnegative and converge pointwise to $f'_\infty(u_\infty)$ inside $\mathcal{U}\setminus\Sigma_\infty$ (and so a.e.), Fatou's Lemma implies that

$$\int_{B_r(\bar x)} f'_\infty(u_\infty) \le C_n(1+k)^{\frac{2}{n}} r^{n-2} \qquad \forall\, B_{2r(\bar x)}\subset\mathcal{U},$$

thus $f'_\infty(u_\infty)\in L^1_{\rm loc}(\mathcal{U})$.

Now, to prove the bound on the index, assume by contradiction that there exists a $k+1$ dimensional subspace $X'\subset C^1_c(\mathcal{U})$ where

$$Q_\infty(\xi) := \int_{\mathcal{U}} \left(|\nabla\xi|^2 - f'_\infty(u_\infty)\xi^2\right)\,dx < 0 \qquad \forall\,\xi\in X'\setminus\{0\}.$$

We claim that also $Q_{j(m)}$ is strictly negative on $X'\setminus\{0\}$ for m sufficiently large. Indeed, if not, by homogeneity there exists a sequence $\xi_m\in X'\setminus\{0\}$, with $\|\xi_m\|_{C^1}=1$, such that $Q_{j(m)}(\xi_m)\ge 0$. Since $X'\subset C^1_c(\mathcal{U})$ is finite dimensional, all functions ξ_m live in a fixed compact set and, up to a subsequence, they converge in $C^1_c(\mathcal{U})$ to a limiting function $\xi_\infty\in X'$ with $\|\xi_\infty\|_{C^1}=1$. In particular,

$$\int_{\mathcal{U}} |\nabla\xi_m|^2\,dx \to \int_{\mathcal{U}} |\nabla\xi_\infty|^2\,dx \qquad \text{as } m\to\infty.$$

Also, since $f'_{j(m)}(u_{j(m)})\xi_m^2$ are nonnegative and converge pointwise to $f'_\infty(u_\infty)\xi_\infty^2$ inside $\mathcal{U}\setminus\Sigma_\infty$ (and so a.e.), Fatou's Lemma implies that

$$\liminf_{m\to\infty}\int_{\mathcal{U}} f'_{j(m)}(u_{j(m)})\xi_m^2 \ge \int_{\mathcal{U}} f'_\infty(u_\infty)\xi_\infty^2\,dx.$$

Combining these two facts, we deduce that

$$0\le \limsup_{m\to\infty} Q_{j(m)}(\xi_m) \le Q_\infty(\xi_\infty),$$

a contradiction since $\xi_\infty\in X'\setminus\{0\}$. Hence $Q_{j(m)}$ is strictly negative on $X'\setminus\{0\}$ for m sufficiently large, which is impossible since $\mathrm{ind}(u_{j(m)},\mathcal{U})\le k$. This contradiction proves that $\mathrm{ind}(u_\infty,\mathcal{U})\le k$.

Finally, to prove that $u_\infty\in C^2(\mathcal{U})$, we recall that finite Morse index solutions are locally stable (see for instance [14, Proposition 1.5.1] or [12, Proposition 2.1]). Hence, we can apply Proposition A.1 and elliptic regularity around each of the points in Σ_∞ to deduce that $u_\infty\in C^2(\mathcal{U})$. □

Our next goal is to show a uniform $W^{1,2}$ integrability estimate for finite index solutions. It is for this result that the growth assumption on f plays a crucial role. Before stating and proving it, we first recall the following simple estimate that can be found, for instance, in [9, Lemma A.1].

Lemma 2.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative function, and let $v\in C^2$ solve $-\Delta v = f(v)$ inside $B_r(\bar x)$. Then*

$$\int_{B_{r/2}(\bar x)} f(v)\,dx \le C_n r^{-2}\int_{B_r(\bar x)} |v|\,dx.$$

We begin by proving a uniform $W^{1,2}$ integrability estimate for stable solutions, that will be used below to address the general case.

Proposition 2.5. *Let $f\in C^1$ be nonnegative, and let $u\in C^2$ be a nonnegative stable solution to $-\Delta u = f(u)$ in $B_r(\bar x)$ for some $r\in(0,1]$. Assume that f satisfies (1.5) for some $c_1>0$. Then there exists $\delta=\delta(n,\epsilon)>0$ such that*

$$\int_{B_\rho(\bar x)} |\nabla u|^2\,dx \le C(c_1,n)\,\rho^\delta \qquad \textit{for all } 0<\rho<\frac{r}{4}. \tag{2.7}$$

Proof. With no loss of generality we can assume $\bar{x} = 0$.

By Hölder inequality, (1.5), and Lemma 2.4, for any ball $B_{2\rho}(z) \subset B_r$ it holds

$$\left(\rho^{-n}\int_{B_\rho(z)} u\,dx\right)^{\frac{n+2}{n-2}+\epsilon} \le C(n)\rho^{-n}\int_{B_\rho(z)} u^{\frac{n+2}{n-2}+\epsilon}\,dx \le C(n,c_1)\rho^{-n}\int_{B_\rho(z)} f(u)\,dx \le C_0\rho^{-2-n}\int_{B_{2\rho}(z)} u\,dx, \quad (2.8)$$

where $C_0 = C_0(n, c_1)$. Let $\delta = \delta(n, \epsilon) > 0$ be small enough so that

$$\left(\frac{n-2}{2}-\delta\right)\left(\frac{n+2}{n-2}+\epsilon\right)-2 \ge \frac{n-2}{2}-\delta, \quad (2.9)$$

and define

$$G(z,\rho) := \max\left\{1, \gamma \sup_{B_s(y)\subset B_\rho(z)} s^{-\frac{n+2}{2}-\delta}\int_{B_s(y)} u\,dx\right\},$$

where $\gamma \in (0,1)$ is a small constant to be fixed later. Then, thanks to (2.8) and (2.9), whenever $B_{2\rho}(z) \subset B_r$ we have

$$G(z,\rho) \le G(z,\rho)^{\frac{n+2}{n-2}+\epsilon} \le 1 + \gamma^{\frac{n+2}{n-2}+\epsilon}\sup_{B_s(y)\subset B_\rho(z)}\left(s^{-\frac{n+2}{2}-\delta}\int_{B_s(y)} u\,dx\right)^{\frac{n+2}{n-2}+\epsilon} \le 1 + C_0\gamma^{\frac{n+2}{n-2}+\epsilon}\sup_{B_s(y)\subset B_\rho(z)} s^{-\frac{n+2}{2}-\delta}\int_{B_{2s}(y)} u\,dx \le 1 + C_0\gamma^{\frac{4}{n-2}+\epsilon}G(z,2\rho). \quad (2.10)$$

We now claim that $G(0, r/2)$ is uniformly bounded.

To show this, consider the quantity

$$Q := \sup_{z\in B_r,\rho\le r-|z|} G(z,\rho/2).$$

Note that, since u is of class C^2, Q is a finite constant. Also, we can assume that $Q > 2$ (otherwise there is nothing to prove). Consider now $z \in B_r$ and $\rho \le r - |z|$ such that $G(z, \rho/2) \ge Q/2$. Since $Q/2 > 1$, it follows from the definition of G that there exists $B_s(y) \subset B_{\rho/2}(z)$ such that $\gamma s^{-\frac{n+2}{2}-\delta}\int_{B_s(y)} u \ge Q/3$. We can now cover $B_s(y)$ with N_n balls $\{B_{s/4}(y_k)\}_{k=1}^{N_n}$ with $y_k \in B_s(y) \subset B_{\rho/2}(z)$, where N_n is a dimensional constant, and observe that

$$\frac{Q}{3} \le \gamma s^{-\frac{n+2}{2}-\delta}\int_{B_s(y)} u\,dx \le \gamma s^{-\frac{n+2}{2}-\delta}\sum_{k=1}^{N_n}\int_{B_{s/4}(y_k)} u\,dx \le \gamma(s/4)^{-\frac{n+2}{2}-\delta}\sum_{k=1}^{N_n}\int_{B_{s/4}(y_k)} u\,dx \le \sum_{k=1}^{N} G(y_k, s/4). \quad (2.11)$$

Note now that, since $s \le \rho/2$ and $y_k \in B_{\rho/2}(z)$,

$$|y_k| + s \le |z| + \frac{\rho}{2} + \frac{\rho}{2} \le |z| + \rho \le r.$$

In particular $B_{s/2}(y_k) \subset B_r$, and it follows by (2.10) and the definition of Q that

$$G(y_k, s/4) \le 1 + C_0\gamma^{\frac{4}{n-2}+\epsilon}G(y_k, s/2) \le 1 + C_0\gamma^{\frac{4}{n-2}+\epsilon}Q.$$

Combining this bound with (2.11), this yields

$$\frac{Q}{3} \leq N_n \left(1 + C_0 \gamma^{\frac{4}{n-2}+\epsilon} Q\right) \leq N_n \left(1 + C_0 \gamma^{\frac{4}{n-2}} Q\right),$$

and by choosing γ small enough (depending only on C_0 and the dimension), we conclude that $Q \leq 4N_n$, and therefore

$$\gamma \sup_{s \leq r/2} s^{-\frac{n+2}{2}-\delta} \int_{B_s(0)} u\, dx \leq G(0, r/2) \leq Q \leq 4N_n, \tag{2.12}$$

as desired.

Recall now that, by [9, Theorem 1.2], if u is stable on a ball B then

$$\|u\|_{W^{1,2}\left(\frac{1}{2}B\right)} \leq C(n)\,(\,\mathrm{diam}\,(B))^{-\frac{n+2}{2}} \|u\|_{L^1(B)}.$$

Combining this estimate with (2.12), we obtain (2.7). □

We next improve this result to solutions with finite Morse index.

Proposition 2.6. *Let $f \in C^1$ be nonnegative, and let $u \in C^2$ be a nonnegative solution to $-\Delta u = f(u)$ in $B_r(\bar{x})$ for some $r > 0$. Assume that* $\mathrm{ind}(u, B_r(\bar{x})) \leq k$ *for some $k \in \mathbb{N}$, and that f satisfies* (1.5) *for some $c_1 > 0$. Then*

$$\int_{B_\rho(\bar{x})} |\nabla u|^2\, dx \leq C(c_1, n, \epsilon)\, k\, \rho^\delta \qquad \text{for all } \rho \in (0, r/4),$$

where $\delta = \delta(n, \epsilon) > 0$ is as in Proposition 2.5.

Proof. Let $M > 1$ be a fixed constant[5], define the set $Q^0 := B_\rho(\bar{x})$, and consider the covering of Q^0 given by $\left\{B_{M^{-1}\rho}(z)\right\}_{z \in Q^0}$. By Besicovitch Covering Theorem, there exist a dimensional constant $N_n \in \mathbb{N}$ and a subfamily of balls $\left\{B_\ell^0\right\}_{\ell \in \mathfrak{I}_0} \subset \left\{B_{M^{-1}\rho}(z)\right\}_{z \in Q^0}$ such that

$$1 \leq \sum_{\ell \in \mathfrak{I}_0} \chi_{B_\ell^0}(y) \leq N_n \qquad \text{for all } y \in Q^0. \tag{2.13}$$

In particular, since these balls have radius $M^{-1}\rho$ and are contained inside $B_{2\rho}(\bar{x})$, it follows that

$$\#\mathfrak{I}_0\, |B_{M^{-1}\rho}| \leq N_n |B_{2\rho}| \qquad \Rightarrow \qquad \#\mathfrak{I}_0 \leq 2^n M^n N_n.$$

Moreover, since each point $y \in B_\rho(\bar{x})$ is covered by at most N_n balls of radius $M^{-1}\rho$, then the same is true if we double the radius of the balls: more precisely, there exists a dimensional constant $N_n' \in \mathbb{N}$ such that[6]

$$1 \leq \sum_{\ell \in \mathfrak{I}_0} \chi_{2B_\ell^0}(y) \leq N_n' \qquad \forall\, y \in Q^0. \tag{2.14}$$

Let us split $\left\{2B_\ell^0\right\}_{\ell \in \mathfrak{I}_0}$ into N_n' subfamilies of balls, where the balls of each subfamily are disjoint. As $\mathrm{ind}(u_s) \leq k$, we can apply Lemma 2.2(i) to each subfamily. Then we deduce that, except for at most

[5]One can choose M to be any constant larger than 1, for instance $M = 2$. However, for notational convenience we prefer to use the notation M instead of fixing its value, as we believe that the estimates become easier to follow.

[6]A simple way to see this is to note that, as a consequence of (2.13), we can split $\left\{B_\ell^0\right\}_{\ell \in \mathfrak{I}_0}$ into N_n subfamilies of balls, where the balls of each subfamily are disjoint. This implies that the centers of the balls of each subfamily are at mutual distance at least $2M^{-1}\rho$. Then, if we double the radius, the overlapping for each of these subfamilies is bounded by a dimensional constant $C_n \geq 1$.

$N_n'k$ balls, say $B^0_1,\dots,B^0_{k_0}$ with $k_0\le N_n'k$, the function u is stable inside each ball $\{2B^0_\ell\}_{\ell\in\mathcal{I}_0\setminus\{1,\dots,k_0\}}$. Thus by Proposition 2.5, we have

$$\int_{B^0_\ell}|\nabla u|^2\,dx\le C(c_1,n)M^{-\delta}\rho^\delta\qquad\forall\,\mathcal{I}_0\setminus\{1,\dots,k_0\}.$$

Now, we consider the set $Q^1:=\bigcup_{1\le\ell\le k_0}B^0_\ell$ and the covering $\{B_{M^{-2}\rho}(z)\}_{z\in Q^1}$. Again by Besicovitch Covering Theorem, there exists a subfamily of balls $\{B^1_\ell\}_{\ell\in\mathcal{I}_1}\subset\{B_{M^{-2}\rho}(z)\}_{z\in Q^1}$ such that

$$1\le\sum_{\ell\in\mathcal{I}_1}\chi_{B^1_\ell}(y)\le N_n\qquad\text{for all } y\in Q^1. \tag{2.15}$$

Also, since these balls are contained inside $\bigcup_{1\le\ell\le k_0}2B^0_\ell$, it follows that (recall that $k_0\le N_n'k$)

$$\#\mathcal{I}_1\,|B_{M^{-2}\rho}|\le k_0N_n|B_{2M^{-1}\rho}|\qquad\Rightarrow\qquad\#\mathcal{I}_1\le 2^nM^nk_0N_n\le 2^nM^nN_n'N_nk.$$

Furthermore, as before,

$$1\le\sum_{\ell\in\mathcal{I}_1}\chi_{2B^1_\ell}(y)\le N_n'\qquad\forall\,y\in Q^1.$$

Hence (up to renaming the indices) u is stable inside each ball $\{2B^1_\ell\}_{\ell\in\mathcal{I}_1\setminus\{1,\dots,k_1\}}$ with $k_1\le N_n'k$, and therefore

$$\int_{B^1_\ell}|\nabla u|^2\,dx\le C(c_1,n)M^{-2\delta}\rho^\delta\qquad\forall\,\ell\in\mathcal{I}_1\setminus\{1,\dots,k_1\}.$$

To continue this construction, define

$$Q^2:=\bigcup_{1\le\ell\le k_2}B^2_\ell.$$

Then, we can apply the very same argument used for Q^1 to find a family of balls $\{B^2_\ell\}_{\ell\in\mathcal{I}_2}$, with $\#\mathcal{I}_2\le 2^nM^nk_1N_n\le 2^nM^nN_n'N_nk$, such that

$$\int_{B^2_\ell}|\nabla u|^2\,dx\le C(c_1,n)M^{-3\delta}\rho^\delta\qquad\forall\,\ell\in\mathcal{I}_2\setminus\{1,\dots,k_2\},\qquad\text{with } k_2\le N_n'k.$$

Iterating this construction, we obtain that the family of balls $\{B^j_\ell\}_{\ell\in\mathcal{I}_j\setminus\{1,\dots,k_j\},\,j\in\mathbb{N}}$ covers $Q^0\setminus K$, with $K:=\cap_{j\in\mathbb{N}}Q^j$,[7] and

$$\int_{B^j_\ell}|\nabla u|^2\,dx\le C(c_1,n)M^{-j\delta}\rho^\delta\quad\forall\,\ell\in\mathcal{I}_j\setminus\{1,\dots,k_j\},\qquad\#\mathcal{I}_j\le 2^nM^nN_n'N_nk,\qquad k_j\le N_n'k.$$

Since K has measure zero (because $|Q^j|\le k_j|B_{M^{-j}\rho}|\le N_n'k|B_{M^{-j}\rho}|\to 0$ as $j\to\infty$), we have

$$\begin{aligned}\int_{Q^0}|\nabla u|^2\,dx=\int_{Q^0\setminus K}|\nabla u|^2\,dx&\le\sum_{j=0}^\infty\sum_{\ell\in\mathcal{I}_j\setminus\{1,\dots,k_j\}}\int_{B^j_\ell}|\nabla u|^2\,dx\\&\le C(c_1,n)\Big(\sum_{j=0}^\infty\#\mathcal{I}_jM^{-j\delta}\Big)\rho^\delta\le C(c_1,n,\delta)\,k\,\rho^\delta.\end{aligned}$$

Recalling that $\delta=\delta(n,\epsilon)$, this concludes the lemma. □

[7] Here one could note that, since u is smooth, every ball sufficiently small is stable and therefore Q^j is empty for j large enough, hence $K=\emptyset$. However this information is not needed, and this proof also applies to weak solutions with bounded index.

The next result is a powerful ε-regularity theorem which shows the following: given $\gamma \in (0,1)$, if the $W^{1,2}$ norm of a solution in a ball B_r decays like $r^{n-2+2\gamma}$ for $r \in [\varepsilon, 1]$ with ε small enough, then it decays all the way to the origin.

Proposition 2.7. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function, let $u \in C^2(B_1)$ solve $-\Delta u = f(u)$ for some increasing function $f : \mathbb{R} \to \mathbb{R}$ of class C^1 satisfying $0 \le f \le g$ and (1.4), and assume that $\mathrm{ind}(u, B_1) \le k$ and $\int_{B_1} |u| \le M$. Then, for any $\gamma \in (0,1)$ there exists $m_0 = m_0(n, g, \epsilon, t_0, k, M, \gamma) \in \mathbb{N}$ such that the following holds:*
Suppose that

$$\int_{B_r} |\nabla u|^2 \, dx \le r^{n-2+2\gamma} \qquad \forall\, r \in [2^{-m_0}, 1].$$

Then

$$\int_{B_r} |\nabla u|^2 \, dx \le r^{n-2+2\gamma} \qquad \forall\, r \in [0, 1] \tag{2.16}$$

and $|u(0)| \le M + C(n, \gamma)$.

Proof. We begin with the proof of (2.16). For that, it suffices to prove the following implication: if

$$\int_{B_r} |\nabla u|^2 \, dx \le r^{n-2+2\gamma} \qquad \forall\, r \in [2^{-m}, 1]$$

for some $m \ge m_0$, then

$$\int_{B_r} |\nabla u|^2 \, dx \le r^{n-2+2\gamma} \qquad \forall\, r \in [2^{-(m+1)}, 1].$$

Indeed, iterating this result with $m = m_0, m_0 + 1, \ldots$, the result follows.

To prove the implication above, we argue by contradiction. If it was false, we could find a sequence of functions $u_j \in C^2(B_1)$, and $f_j \in C^1(\mathbb{R})$ satisfying (1.4), such that

$$-\Delta u_j = f_j(u_j), \qquad f_j \text{ increasing}, \qquad 0 \le f_j \le g, \qquad \mathrm{ind}(u_j, B_1) \le k, \qquad \int_{B_1} |u_j| \le M,$$

and a sequence $m_j \to \infty$, such that

$$\int_{B_r} |\nabla u_j|^2 \, dx \le r^{n-2+2\gamma} \qquad \forall\, r \in [2^{-m_j}, 1] \tag{2.17}$$

but

$$\int_{B_{r_j}} |\nabla u_j|^2 \, dx \ge r^{n-2+2\gamma} \qquad \text{for some } r_j \in [2^{-(m_j+1)}, 2^{-m_j}]. \tag{2.18}$$

We introduce the notation $A_r := B_r \setminus B_{r/2}$.

We first note that, as a consequence of (2.17) and the bound $\int_{B_1} |u_j| \le M$, it follows that

$$\fint_{A_{2^{-(m_j+1)}}} |u_j| \, dx \le M + C(n, \gamma). \tag{2.19}$$

Indeed, thanks to (2.17), for any $2^{-m_j} \le s \le r \le 1$ we have

$$\begin{aligned}\left|\fint_{\partial B_r}|u_j| - \fint_{\partial B_s}|u_j|\right| &\le \fint_{\partial B_1}|u_j(ry)-u_j(sy)|\,dy \le \int_s^r\left(\fint_{\partial B_1}|\nabla u_j(\tau y)|\,dy\right)d\tau\\ &= \int_{B_r\setminus B_s}\frac{|\nabla u_j(x)|}{|x|^{n-1}}\,dx \le \sum_{\ell=1}^{m_j}\int_{A_{2^{-\ell}}}\frac{|\nabla u_j(x)|}{|x|^{n-1}}\,dx \le \sum_{\ell=1}^{m_j}2^{\ell(n-1)}\int_{A_{2^{-\ell}}}|\nabla u_j|\,dx\\ &= C(n)\sum_{\ell=1}^{m_j}2^{-\ell}\fint_{A_{2^{-\ell}}}|\nabla u_j|\,dx \le C(n)\sum_{\ell=1}^{m_j}2^{-\ell}\left(\fint_{A_{2^{-\ell}}}|\nabla u_j|^2\,dx\right)^{1/2}\\ &\le C(n)\sum_{\ell=1}^{m_j}2^{-\ell}2^{-\ell(\gamma-1)} = C(n)\sum_{\ell=1}^{m_j}2^{-\ell\gamma} \le C(n,\gamma),\end{aligned} \tag{2.20}$$

therefore

$$\left|\fint_{A_1}|u_j|\,dx - \fint_{A_{2^{-(m_j+1)}}}|u_j|\,dx\right| \le C(n,\gamma),$$

and (2.19) follows.

To simplify the notation, we set $r_j := 2^{-m_j}$, and we define

$$a_j := \fint_{A_{2r_j}} u_j\,dx, \qquad w_j(x) := r_j^{-\gamma}[u_j(r_jx)-a_j],$$

so that

$$\fint_{A_2} w_j = 0, \qquad -\Delta w_j = h_j(w_j), \qquad h_j(t) := r_j^{2-\gamma}f_j(a_j + r_j^{\gamma}t). \tag{2.21}$$

Note that $\mathrm{ind}(w_j, B_{2^{m_j}}) \le k$ and $0 \le h_j \le r_j^{2-\gamma}g(a_j + r_j^{\gamma}t)$, so it follows from (2.19) that $h_j \to 0$ in C^1_{loc}. Also (2.17) and (2.18) imply that

$$\int_{B_{2^\ell}}|\nabla w_j|^2dx \le 2^{\ell(n-2+2\gamma)} \qquad \forall\, 0 \le \ell \le m_j \tag{2.22}$$

and

$$\int_{B_1}|\nabla w_j|^2dx \ge 2^{-(n-2+2\gamma)}. \tag{2.23}$$

Thus, thanks to Proposition 2.3 and a diagonal argument we deduce that, up to a subsequence,

$$w_j \rightharpoonup w_\infty \text{ in } W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n), \qquad w_j \to w_\infty \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\Sigma_\infty),$$

where Σ_∞ has cardinality at most k, and w_∞ satisfies (by (2.22) and (2.21))

$$-\Delta w_\infty = 0 \quad \text{in } \mathbb{R}^n, \qquad \fint_{A_2} w_\infty = 0, \qquad \int_{B_{2^\ell}}|\nabla w_\infty|^2dx \le 2^{\ell(n-2+2\gamma)} \qquad \forall\,\ell \ge 0.$$

Then it follows from Liouville Theorem for harmonic functions that $w_\infty \equiv 0$,[8] and therefore

$$w_j \rightharpoonup 0 \text{ in } W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n), \qquad w_j \to 0 \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\Sigma_\infty), \qquad \#\Sigma_\infty \le k.$$

We now want to get a contradiction with (2.23).

[8]Indeed, by Liouville Theorem w_∞ must be a harmonic polynomial, and the bound $\int_{B_{2^\ell}}|\nabla w_\infty|^2dx \le 2^{\ell(n-2+2\gamma)}$ for $\ell \ge 0$ implies that w_∞ must be constant (recall that $\gamma < 1$). Finally, since $\int_{A_2} w_\infty = 0$ we deduce that $w_\infty \equiv 0$.

Consider the annuli

$$B_2 \setminus B_1, \qquad B_3 \setminus B_2, \qquad \dots, \qquad B_{k+2} \setminus B_{k+1}.$$

Since $\#\Sigma_\infty \le k$, there exists $\hat{i} \in \{1, \dots, k+1\}$ such that $(B_{\hat{i}+1} \setminus B_{\hat{i}}) \cap \Sigma_\infty = \emptyset$. In particular, if we fix $\varphi \in C_c^\infty(B_{\hat{i}+3/4})$ nonnegative such that $\varphi|_{B_{\hat{i}+1/4}} = 1$, then $w_j \to 0$ in C^2 on $\{\nabla\varphi \neq 0\}$.

Now we first test the equation for w_j (see (2.21)) with $w_j\varphi$ to get

$$\begin{aligned} r_j^{2-\gamma} \int_{B_{\hat{i}+1}} f_j\big(a_j + r_j^\gamma w_j\big) w_j \varphi \, dx &= \int_{B_{\hat{i}+1}} -w_j \Delta w_j \varphi \, dx \\ &= \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \varphi + w_j \nabla w_j \cdot \nabla \varphi \, dx \\ &= \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \varphi \, dx + o(1), \end{aligned} \tag{2.24}$$

where $o(1)$ denotes a quantity that goes to 0 as $j \to \infty$, and the last equality follows from the C^2 convergence of w_j to 0 on the set $\{\nabla\varphi \neq 0\}$.

Similarly, testing (2.21) with $(\nabla w_j \cdot x)\varphi$, we obtain

$$\begin{aligned} \int_{B_{\hat{i}+1}} r_j^{2-\gamma} f_j\big(a_j + r_j^\gamma w_j\big)(\nabla w_j \cdot x)\varphi \, dx &= \int_{B_{\hat{i}+1}} -\Delta w_j (\nabla w_j \cdot x)\varphi \, dx \\ &= \int_{B_{\hat{i}+1}} D^2 w_j \nabla w_j \cdot x\varphi + |\nabla w_j|^2 \varphi + (x \cdot \nabla w_j)\nabla w_j \cdot \nabla\varphi \, dx \\ &= \Big(1 - \frac{n}{2}\Big) \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \varphi \, dx - \frac{1}{2} \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \nabla\varphi \cdot x \, dx + o(1) \\ &= \Big(1 - \frac{n}{2}\Big) \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \varphi \, dx + o(1). \end{aligned}$$

Also, if we define $F_j(t) := \int_0^t f_j(\tau)\, d\tau$, then we can rewrite the first term above as follows:

$$\begin{aligned} &\int_{B_{\hat{i}+1}} r_j^{2-\gamma} f_j\left(a_j + r_j^\gamma w_j\right)(\nabla w_j \cdot x)\varphi \, dx \\ &= \int_{B_{\hat{i}+1}} r_j^{2-2\gamma} \nabla \left[F_j\left(a_j + r_j^\gamma w_j\right) - F_j(a_j)\right] \cdot x \, \varphi \, dx \\ &= -n \int_{B_{\hat{i}+1}} r_j^{2-2\gamma} \left[F_j\left(a_j + r_j^\gamma w_j\right) - F_j(a_j)\right] \varphi \, dx + \int_{B_{\hat{i}+1}} r_j^{2-\gamma} \left[F_j\left(a_j + r_j^\gamma w_j\right) - F_j(a_j)\right] x \cdot \nabla\varphi \, dx \\ &= -n \int_{B_{\hat{i}+1}} r_j^{2-2\gamma} \left[F_j\left(a_j + r_j^\gamma w_j\right) - F_j(a_j)\right] \varphi \, dx + o(1), \end{aligned}$$

and we eventually get

$$r_j^{2-2\gamma} \int_{B_{\hat{i}+1}} \left[F_j\left(a_j + r_j^\gamma w_j\right) - F_j(a_j)\right] \varphi \, dx = \frac{n-2}{2n} \int_{B_{\hat{i}+1}} |\nabla w_j|^2 \varphi \, dx + o(1). \tag{2.25}$$

Now, given a constant $N > 0$, we define the set

$$S_N := \left\{ x \in B_{\hat{i}+1} \colon r_j^\gamma |w_j(x)| \le N \right\}.$$

Since w_j is uniformly bounded in $W^{1,2}(B_{\hat{i}+1})$ and a_j is uniformly bounded, for any $N > 0$ fixed we have

$$r_j^{2-\gamma}\int_{S_N}\left[F_j\left(a_j+r_j^\gamma w_j\right)-F_j(a_j)\right]\varphi\,dx\to 0,\qquad r_j^{2-\frac{\gamma}{2}}\int_{S_N}f_j\left(a_j+r_j^\gamma w_j\right)w_j\varphi\,dx\to 0,$$

so it follows from (2.24) and (2.25) that

$$r_j^{2-\gamma}\int_{B_{\hat{i}+1}\setminus S_N}f_j\big(a_j+r_j^\gamma w_j\big)w_j\varphi\,dx=\int_{B_{\hat{i}+1}}|\nabla w_j|^2\varphi\,dx+o(1)\tag{2.26}$$

and

$$r_j^{2-2\gamma}\int_{B_{\hat{i}+1}\setminus S_N}\left[F_j\left(a_j+r_j^\gamma w_j\right)-F_j(a_j)\right]\varphi\,dx=\frac{n-2}{2n}\int_{B_{\hat{i}+1}}|\nabla w_j|^2\varphi\,dx+o(1).\tag{2.27}$$

Note now that, thanks to (1.4), the fact that a_j is uniformly bounded (see (2.19)), and that $0\le f_j\le g$, there exists a large constant $N=N(M,n,\gamma,t_0,\epsilon)$ such that, for all j,

$$\left(\frac{2n}{n-2}+\frac{\epsilon}{2}\right)\big[F_j(t+a_j)-F_j(a_j)\big]\le f_j(t+a_j)t\qquad\forall\,t\ge N.$$

Combining this inequality with (2.26) and (2.27), we get

$$\left(\frac{2n}{n-2}+\frac{\epsilon}{2}\right)\frac{n-2}{2n}\int_{B_{\hat{i}+1}}|\nabla w_j|^2\varphi\le\int_{B_{\hat{i}+1}}|\nabla w_j|^2\varphi+o(1),$$

or equivalently

$$\frac{(n-2)\epsilon}{4n}\int_{B_{\hat{i}+1}}|\nabla w_j|^2\varphi\le o(1).$$

This contradicts (2.23) and concludes the proof of (2.16).

Now, to prove that bound on $|u(0)|$, we observe that (2.16) allows us to deduce the validity of (2.20) for all $0\le s\le r\le 1$. In particular this implies that

$$\left|\fint_{A_1}|u_j|\,dx-|u(0)|\right|\le C(n,\gamma),$$

so $|u(0)|\le M+C(n,\gamma)$ as desired. □

Finally, we conclude this section with a useful consequence of the moving plane method.

Lemma 2.8. *Let $\Omega\subset\mathbb{R}^n$ be a bounded convex domain, and let $u\in C^2(\Omega)$ solve (1.1) for some increasing positive function $f:\mathbb{R}\to\mathbb{R}$ of class C^1. Then there exists $\rho_0=\rho_0(\Omega)\in(0,1)$ such that*

$$\max_\Omega u=\max_{\Omega_0}u,$$

where $\Omega_0:=\{x\in\Omega\colon\ \mathrm{dist}\,(x,\partial\Omega)>\rho_0\}$.

Proof. Recall that, since $f\ge 0$, the maximum principle implies that $u>0$ (unless $u\equiv 0$, in which case the result is trivially true). Then, since Ω is bounded and convex, the result follows by the classical moving plane method (see also the footnote inside [9, Proof of Corollary 1.4] for more details). □

3. Uniform finite Morse index: Proof of Theorem 1.1

Let us assume, by contradiction, that there exists a sequence of C^2 solutions u_j

$$\begin{cases} -\Delta u_j = \lambda_j f_j(u_j) & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathrm{ind}(u_j, \Omega) \le k$, the functions $f_j : \mathbb{R} \to \mathbb{R}$ satisfy all the assumptions in the statement of the theorem, $0 \le \lambda_j \le \hat{\lambda}$, but $\|u_j\|_{L^\infty(\Omega)} \to \infty$ as $j \to \infty$. Since $f_j \in \mathcal{K}$ which is a compact family, up to a subsequence $f_j \to f_\infty$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ and $\lambda_j \to \lambda_\infty \in [0, \hat{\lambda}]$. Define

$$\hat{f}(t) := \sup_j f_j(t) \qquad \forall\, t \in \mathbb{R}, \tag{3.1}$$

so that $0 \le f_j \le \hat{f}$ for all j. Since the functions f_j are locally uniformly Lipschitz (by the C^1_{loc} compactness), $\hat{f}$ is a continuous function.

We distinguish two cases, depending on the value of λ_∞.

3.1. The case $\lambda_\infty > 0$.

Since $\lambda_j \to \lambda_\infty$, it follows from Remark 1.2 that, for j large enough,

$$\lambda_j f_j(t) \ge c_1 t^{\frac{n+2}{n-2}+\epsilon} \qquad \forall\, t \ge 0, \qquad \text{for some } c_1 > 0. \tag{3.2}$$

Thanks to this bound, it follows from [9, Proposition B.1] that

$$\|u_j\|_{L^1(\Omega)} \le C_0 = C_0(c_1, \Omega). \tag{3.3}$$

Also, if we define $\Omega_\tau := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \tau\}$, then (3.2) and Proposition 2.6 yield

$$\|\nabla u_j\|_{L^2(\Omega_\tau)} \le C_1 = C_1(c_1, \Omega, \rho) \qquad \forall\, \tau > 0.$$

Since $\tau > 0$ is arbitrary, Proposition 2.3 and a diagonal argument imply that, up to a subsequence,

$$u_j \rightharpoonup u_\infty \text{ in } W^{1,2}_{\mathrm{loc}}(\mathcal{U}), \qquad u_j \to u_\infty \text{ in } C^2_{\mathrm{loc}}(\Omega \setminus \Sigma_\infty), \tag{3.4}$$

for some discrete set $\Sigma_\infty \subset \Omega$ with $\#\Sigma_\infty \le k$, and some function $u_\infty \in C^2(\Omega)$.

Let $\rho_0 \in (0,1)$ and Ω_0 be given by Lemma 2.8, and define

$$\Sigma^0_\infty := \Omega_0 \cap \Sigma_\infty = \{\hat{x}_1, \ldots, \hat{x}_\ell\} \quad (\ell \le k), \qquad r_0 := \frac{1}{2} \min\left\{\rho_0,\, \min_{1 \le i,l \le \ell} |x_i - x_l|\right\}.$$

Then it follows from (3.4) that, for any $\rho \in (0, r_0)$,

$$\max_{1 \le i \le \ell} \|u_j - u_0\|_{C^2(B_{r_0}(\hat{x}_i) \setminus B_\rho(\hat{x}_i))} \to 0 \qquad \text{as } j \to \infty.$$

In particular, since $u_\infty \in C^2(\Omega)$, there exists a constant $\bar{C} > 0$ such that that following holds: for any $\rho \in (0, r_0)$ there exists $j_\rho \in \mathbb{N}$ such that

$$\max_{1 \le i \le \ell} \|\nabla u_j\|_{L^\infty(B_{r_0}(\hat{x}_i) \setminus B_\rho(\hat{x}_i))} \le \bar{C} \qquad \forall\, j \ge j_\rho. \tag{3.5}$$

We now make the following:

Claim: *There exist $\hat{C}, \hat{r} > 0$ such that $\max_{1 \le i \le \ell} \|u_j\|_{L^\infty(B_{\hat{r}}(\hat{x}_i))} \le \hat{C}$ for all j sufficiently large.*

Assuming for a moment that the claim is proved, since $u_j \to u_\infty$ in $C^2_{\mathrm{loc}}(\Omega \setminus \Sigma_\infty)$ and $u_\infty \in C^2(\Omega)$, it follows from the claim that

$$\sup_j \|u_j\|_{L^\infty(\Omega_0)} < \infty,$$

where Ω_0 is given by Lemma 2.8. But then Lemma 2.8 implies that $\sup_j \|u_j\|_{L^\infty(\Omega)} < \infty$, a contradiction to our initial assumption. Hence, in the case $f_\infty(0) > 0$, the theorem is proved provided we can show the claim.

To prove the claim, it suffices to control $\|u_j\|_{L^\infty(B_{r_0}(\hat{x}_i))}$ for each i. With no loss of generality, we can fix $i=1$ and assume that $\hat{x}_1=0$. Then, thanks to (3.2) we can apply Proposition 2.6 to get the following estimate: for any $r\in(0,r_0/2)$ and any $z\in B_r$, given $\rho\in(0,2r)$ it follows from (3.5) that

$$\int_{B_r(z)}|\nabla u_j|^2\,dx\le\int_{B_{2r}}|\nabla u_j|^2\,dx=\int_{B_\rho}|\nabla u_j|^2\,dx+\int_{B_r\setminus B_\rho}|\nabla u_j|^2\,dx$$
$$\le C\rho^\delta+\bar{C}|B_r|\le C'\Big(\rho^\delta+r^n\Big)\qquad\forall\, j\ge j_\rho. \quad (3.6)$$

Also, it follows from (3.3) that

$$⨍_{B_{r_0}}u_j\le|B_{r_0}|^{-1}C_0=:M,\qquad M=M(n,r_0,c_1,\Omega). \quad (3.7)$$

Fix $\gamma:=1/2$, and let $m_0\in\mathbb{N}$ be the constant provided by Proposition 2.7 with $g=\hat{\lambda}\hat{f}$ (see (3.1)). Then, with C' as in (3.6), we choose first $\bar{r}\in(0,r_0)$ such that $C'\bar{r}\le\frac{1}{2}$, and then we fix $\rho\in(0,2\bar{r})$ such that $C'\rho^\delta\le 2^{-m_0(n-1)-1}\bar{r}^{n-1}$. With these choices it follows from (3.6) that, for any $r\in[2^{-m_0}\bar{r},\bar{r}]$ and any $z\in B_{2^{-m_0}\bar{r}}$,

$$\int_{B_r(z)}|\nabla u_j|^2\,dx\le C'\Big(\rho^\delta+r^n\Big)\le 2^{-m_0(n-1)-1}\bar{r}^{n-1}+C'\bar{r}r^{n-1}\le\frac{1}{2}r^{n-1}+\frac{1}{2}r^{n-1}=r^{n-1}, \quad (3.8)$$

provided j is sufficiently large. Hence, applying Proposition 2.7 to the functions $u_{j,z}(x):=u_j(z+\bar{r}x)$ with $f=\bar{r}^2\lambda_jf_j$ (note that $0\le\bar{r}^2\lambda_jf_j\le\lambda_jf_j\le\hat{\lambda}\hat{f}$), thanks to (3.7) we conclude that $|u_j(z)|=|u_{j,z}(0)|\le M+C(n)$ for all j sufficiently large, for all $z\in B_{2^{-m_0}\bar{r}}$. Choosing $\hat{r}:=2^{-m_0}\bar{r}$, this proves the claim and concludes the proof of this case.

3.2. **The case $\lambda_\infty=0$.** Let $M_j=\|\nabla u_j\|_{L^2(\Omega)}$. We prove the result by contradiction, distinguishing between two cases.

Case 1: $M_j\to 0$ as $j\to\infty$. In this case, Proposition 2.3 implies that $u_j\to 0$ in C^2_{loc} outside a set Σ_∞ consisting of at most k points. Since $\|\nabla u_j\|_{L^2(\Omega)}\to 0$, with the same notation as in the case $\lambda_\infty>0$, we deduce that (3.8) holds around each point $\hat{x}_i\in\Sigma_\infty\cap\Omega_0$. Also, by Poincaré and Hölder inequalities,

$$\|u_j\|_{L^1(\Omega)}\le C(n,\Omega)\|u_j\|_{L^2(\Omega)}\le C(n,\Omega)\|\nabla u_j\|_{L^2(\Omega)}\to 0.$$

Hence, arguing exactly as the previous case, thanks to Proposition 2.7 we deduce that $|u_j(z)|\le o(1)+C(n)$ for all j sufficiently large, for all $z\in B_{2^{-m_0}\bar{r}}(\hat{x}_i)$. This implies that $\sup_j\|u_j\|_{L^\infty(\Omega)}<\infty$, a contradiction.

Case 2: M_j are uniformly bounded away from 0. Consider $v_j:=\frac{u_j}{M_j}$, so that

$$-\Delta v_j=\lambda_jh_j(v_j),\qquad h_j(t):=M_j^{-1}f_j(M_jt),\qquad\|\nabla v_j\|_{L^2(\Omega)}=1. \quad (3.9)$$

As in the proof of Proposition 2.7, we multiply the equation satisfied by v_j both by v_j and by $x\cdot\nabla v_j$. Since $v_j\ge 0$, $v_j|_{\partial\Omega}=0$, and Ω convex, as in the classical Derrick-Pohozaev argument (see, e.g., [17, Proof of Theorem 1, Page 515]) the boundary terms "have the right sign", and we get

$$\frac{\lambda_j}{M_j^2}\int_\Omega f_j(M_jv_j)\,M_jv_j\,dx=\int_\Omega|\nabla v_j|^2dx+o(1)=1+o(1),$$

and

$$\frac{\lambda_j}{M_j^2}\int_\Omega F_j(M_jv_j)\,dx\ge\frac{n-2}{2n}\int_\Omega|\nabla v_j|^2dx+o(1)=\frac{n-2}{2n}+o(1).$$

Now, set $S_0 := \{x \in \Omega : M_j v_j \le t_0\}$ and note that, since f_j satisfies (1.4),

$$f_j(M_j v_j)\, M_j v_j \ge \left(\frac{2n}{n-2} + \epsilon\right) F_j(M_j v_j) \quad \text{in } \Omega \setminus S_0.$$

Also, since $\lambda_j \to 0$ and M_j is bounded away from 0,

$$\frac{\lambda_j}{M_j^2} \int_{S_0} f_j(M_j v_j)\, M_j v_j \, dx \le \frac{\lambda_j}{M_j^2} \int_{S_0} f_j(t_0)\, t_0 \, dx \to 0,$$

$$\frac{\lambda_j}{M_j^2} \int_{S_0} F_j(M_j v_j) \, dx \le \frac{\lambda_j}{M_j^2} \int_{S_0} F_j(t_0) \, dx \to 0.$$

Therefore, combining all together,

$$\begin{aligned} 1 + o(1) &= \frac{\lambda_j}{M_j^2} \int_{\Omega \setminus S_0} f_j(M_j v_j)\, M_j v_j \, dx \\ &\ge \left(\frac{2n}{n-2} + \epsilon\right) \frac{\lambda_j}{M_j^2} \int_{\Omega \setminus S_0} F_j(M_j v_j) \, dx \ge \left(\frac{2n}{n-2} + \epsilon\right) \frac{n-2}{2n} + o(1), \end{aligned}$$

a contradiction for j large enough, which concludes the proof of the theorem.

Appendix A. Boundedness of stable solutions in $B_2 \setminus \{0\}$ for $3 \le n \le 9$

It was shown in [7] that, if $3 \le n \le 9$ and $u \in W^{1,2}(B_2) \cap C^2(B_2 \setminus \{0\})$ is a radially symmetric stable solution to (1.1) in $\Omega = B_2 \setminus \{0\}$, then

$$\|u\|_{L^\infty(B_1)} \le C \|u\|_{L^1(B_2)}.$$

Namely removing a point does not influence the interior estimate of the radially symmetric stable solutions.

The aim of this appendix is to show that, combining the approximation argument in [7] with some modifications of the arguments in [9], we can prove the following generalization of [9, Theorem 1.2] which is used in the proof of Proposition 2.3:

Proposition A.1. *Let $3 \le n \le 9$, and let $u \in W^{1,2}(B_2) \cap C^2(B_2 \setminus \{0\})$ be a stable solution to*

$$-\Delta u = f(u) \qquad \text{in } B_2 \setminus \{0\},$$

with $f : \mathbb{R} \to \mathbb{R}$ nonnegative, increasing, and of class C^1. Then

$$\|u\|_{L^\infty(B_1)} \le C \|u\|_{L^1(B_2)}$$

for some universal constant $C > 0$.

We begin by proving the following generalization of [9, Lemma 2.1]:

Lemma A.2. *Let $3 \le n \le 9$, and let u and f be as in Proposition A.1. Then, for any $y \in B_1$ and $0 < \rho < 2 - |y|$ we have*

$$\int_{B_{2\rho/3}(y)} |(x-y) \cdot \nabla u|^2 |x-y|^{-n} \, dx \le C \rho^{2-n} \int_{B_\rho(y) \setminus B_{2\rho/3}(y)} |\nabla u|^2 \, dx, \tag{A.1}$$

and

$$\int_{B_1} |\nabla u|^2 \, dx \le C \int_{B_{3/2} \setminus B_1} |\nabla u|^2 \, dx. \tag{A.2}$$

Proof. For simplicity of notation, given $0 < r < s < 1$, we define $A(s,r) := B_r \setminus B_s$.

We shall first prove the following improved version of (A.1): for any $y \in B_1$ and $0 < \rho < 2 - |y|$ we have

$$\int_{B_{7\rho/8}(y)} |(x-y)\cdot\nabla u|^2 |x-y|^{-n}\,dx \le C\rho^{2-n}\int_{B_\rho(y)\setminus B_{7\rho/8}(y)} |\nabla u|^2\,dx. \tag{A.3}$$

We only prove (A.3) in the case $y = 0$, the general case being analogous[9].

Fix $0 < \theta \ll \epsilon \ll \rho$, $\eta \in C^1_c(A(\theta,2))$, and consider $\xi = (x\cdot\nabla u)\eta$ as test function in the stability inequality for u. Then, by the very same computation as the one in [9, Proof of Lemma 2.1, Step 1], we have

$$0 \le \int_{A(\theta,\,2)} \Big((x\cdot\nabla u)^2|\nabla\eta|^2 + 2(x\cdot\nabla u)\nabla u\cdot\nabla(\eta^2) - |\nabla u|^2 x\cdot\nabla(\eta^2) - (n-2)|\nabla u|^2\eta^2\Big)\,dx.$$

If we now choose $\eta = \min\big\{|x|^{1-\frac{n}{2}}, \epsilon^{1-\frac{n}{2}}\big\}\zeta$ with $\zeta \in C^1_c(A(\theta,2))$, then inside $A(\epsilon,2)$ the formulas are identical to the ones in [9, Proof of Lemma 2.1, Step 2]. Therefore, in the integrals over $A(\epsilon,2)$ we have exactly all the terms appearing in [9, Equation (2.2)], and the only difference concerns the integrals over $A(\theta,\epsilon)$. Note that, inside $A(\theta,\epsilon)$, it holds

$$|\nabla\eta|^2 = \epsilon^{2-n}|\nabla\zeta|^2, \qquad \nabla(\eta^2) = 2\epsilon^{2-n}\zeta\nabla\zeta.$$

Thus, we obtain

$$\begin{aligned} 0 \le \int_{A(\epsilon,\,2)} \Big(&-\frac{(n-2)(10-n)}{4}|x|^{-n}(x\cdot\nabla u)^2\zeta^2 - 2|\nabla u|^2|x|^{2-n}\zeta\; x\cdot\nabla\zeta + 4|x|^{2-n}(x\cdot\nabla u)\zeta\;\nabla u\cdot\nabla\zeta \\ &- (n-2)|x|^{-n}\zeta(x\cdot\nabla\zeta)(x\cdot\nabla u)^2 + |x|^{2-n}(x\cdot\nabla u)^2|\nabla\zeta|^2\Big)\,dx \\ &+ \epsilon^{2-n}\int_{A(\theta,\,\epsilon)} \Big((x\cdot\nabla u)^2|\nabla\zeta|^2 + 4(x\cdot\nabla u)\zeta\,\nabla u\cdot\nabla\zeta - 2|\nabla u|^2\zeta(x\cdot\nabla\zeta) - (n-2)|\nabla u|^2\zeta^2\Big)\,dx. \end{aligned}$$

We now choose $\zeta \in C^1_c(A(\theta,\rho))$ such that $0 \le \zeta \le 1$, $\zeta = 1$ inside $A(2\theta,\rho/2)$, $|\nabla\zeta| \le C/\theta$ in $A(\theta,2\theta)$, and $|\nabla\zeta| \le C/\rho$ in $A(7\rho/8,\rho)$. With this choice, the formula above implies

$$\frac{(n-2)(10-n)}{4}\int_{A(\epsilon,\,7\rho/8)} |x|^{-n}(x\cdot\nabla u)^2\,dx \le C\rho^{2-n}\int_{A(7\rho/8,\,\rho)}|\nabla u|^2 + C\epsilon^{2-n}\int_{A(\theta,\,2\theta)}|\nabla u|^2\,dx,$$

so (A.3) follows by letting first $\theta \to 0$ and then $\epsilon \to 0$ (recall that $3 \le n \le 9$).

Note now that (A.3) readily implies (A.1). Also, as a consequence of (A.3) applied with $y \in B_{1/8}$ and $\rho = \frac{11}{8}$, we have

$$\int_{B_1}\Big|\frac{x-y}{|x-y|}\cdot\nabla u\Big|^2 dx \le \int_{B_{\frac{77}{64}}(y)}\Big|\frac{x-y}{|x-y|}\cdot\nabla u\Big|^2 dx \le C\int_{B_{\frac{11}{8}}(y)\setminus B_{\frac{77}{64}}(y)}|\nabla u|^2\,dx \le C\int_{B_{3/2}\setminus B_1}|\nabla u|^2\,dx.$$

Hence (A.2) follows by averaging the inequality above with respect to $y \in B_{1/8}$. □

We can now prove the following analogue of [9, Lemma 3.1]:[10]

[9] Actually the case $y \neq 0$ is simpler, since for $y = 0$ the function $x \mapsto |x-y|^{-n/2}(x-y)\cdot\nabla u(x)$ (that is used as a test function is the stability inequality) is more singular at the origin.

[10] In Lemma A.3 we require $3 \le n \le 9$ since we proved (A.2) as a consequence of (A.3), and the latter bound requires this dimensional restriction. However, for $n \ge 10$ one could combine our approximation argument with [9, Proof of Theorem 7.1] to show that

$$\int_{B_{7\rho/8}(y)} |(x-y)\cdot\nabla u|^2|x-y|^{-a}\,dx \le C\rho^{2-a}\int_{B_\rho(y)\setminus B_{7\rho/8}(y)}|\nabla u|^2\,dx.$$

Lemma A.3. *Let $3 \le n \le 9$, and let $u \in W^{1,2}(B_2) \cap C^2(B_2 \setminus B_{1/2})$ be a stable solution to*

$$-\Delta u = f(u) \qquad \text{in } B_2 \setminus B_{1/2},$$

with $f : \mathbb{R} \to \mathbb{R}$ nonnegative, increasing, and of class C^1. Assume that

$$\int_{B_1} |\nabla u|^2 \, dx \ge \delta \int_{B_2} |\nabla u|^2 \, dx$$

for some $\delta > 0$. Then there exists a constant C_δ such that

$$\int_{B_{3/2} \setminus B_1} |\nabla u|^2 \, dx \le C_\delta \int_{B_{3/2} \setminus B_1} |x \cdot \nabla u|^2 \, dx.$$

Proof. Assume the result to be false. Then, there exists a sequence of stable solutions to $-\Delta u_k = f_k(u_k)$ in $B_2 \setminus \{0\}$, with $f_k : \mathbb{R} \to \mathbb{R}$ nonnegative, increasing, and of class C^1, such that

$$\int_{B_1} |\nabla u_k|^2 \, dx \ge \delta \int_{B_2} |\nabla u_k|^2 \, dx, \qquad \int_{B_{3/2} \setminus B_1} |\nabla u_k|^2 \, dx = 1, \quad \text{and} \quad \int_{B_{3/2} \setminus B_1} |x \cdot \nabla u_k|^2 \, dx \to 0. \tag{A.4}$$

Now, thanks to (A.4) and (A.2),

$$\int_{B_2} |\nabla u_k|^2 \, dx \le \frac{1}{\delta} \int_{B_1} |\nabla u_k|^2 \, dx \le \frac{C}{\delta} \int_{B_{3/2} \setminus B_1} |\nabla u_k|^2 \, dx = \frac{C}{\delta}. \tag{A.5}$$

Therefore, since u_k is stable in $B_2 \setminus B_{1/2}$, it follows from [9, Proposition 2.4] and a standard scaling and covering argument that

$$\|\nabla u_k\|_{L^{2+\gamma}(B_{3/2} \setminus B_1)} \le C \|\nabla u_k\|_{L^2(B_2 \setminus B_{1/2})} \le \frac{C}{\delta}.$$

This implies that the sequence of superharmonic functions

$$v_k := u_k - \fint_{B_{3/2} \setminus B_1} u_k$$

satisfies

$$\|v_k\|_{L^1(B_{3/2} \setminus B_1)} \le C \|v_k\|_{L^2(B_{3/2} \setminus B_1)} \le C$$

(thanks to (A.5) and Hölder and Poincaré inequalities), as well as

$$\|\nabla v_k\|_{L^2(B_{3/2} \setminus B_1)} = 1, \quad \|v_k\|_{W^{1,2+\gamma}(B_{3/2} \setminus B_1)} \le C, \qquad \int_{B_{3/2} \setminus B_1} |x \cdot \nabla v_k|^2 \, dx \to 0.$$

Thus, as in the proof of [9, Proposition 2.4], up to a subsequence we have that $v_k \to v$ strongly in $W^{1,2}(B_{3/2} \setminus B_1)$, where v is a superharmonic function in $B_{3/2} \setminus B_1$ satisfying

$$\|\nabla v\|_{L^2(B_{3/2} \setminus B_1)} = 1 \qquad \text{and} \qquad x \cdot \nabla v \equiv 0 \quad \text{a.e. in } B_{3/2} \setminus B_1.$$

Again as in the proof of [9, Proposition 2.4], this implies that v is constant in $B_{3/2} \setminus B_1$, a contradiction that proves the result. □

We can now prove the main result of this appendix.

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for any $a < 2(1 + \sqrt{n-1})$. In particular, choosing $a = 2$ and arguing as in the proof of Lemma A.2, one proves the validity of (A.2) in every dimension. As a consequence, one can show that Lemma A.3 holds in every dimension.

Proof of Proposition A.1. The argument is similar to the one in [9, Proof of Theorem 1.2], with some minor modifications.

Given $y \in B_1$ and $\rho \in (0,1)$ we define the quantities

$$\mathcal{D}(\rho, y) := \rho^{2-n} \int_{B_\rho(y)} |\nabla u|^2 \, dx \qquad \text{and} \qquad \mathcal{R}(\rho, y) := \int_{B_\rho(y)} |x-y|^{-n} |(x-y)\cdot \nabla u|^2 \, dx.$$

We claim that there exists a dimensional exponent $\alpha > 0$ such that

$$\mathcal{R}(\rho, y) \le C\rho^{2\alpha} \|\nabla u\|^2_{L^2(B_{3/2})} \qquad \forall\, \rho \in (0, 1/4),\ y \in B_1. \tag{A.6}$$

To prove this claim, note that (A.3) implies that

$$\mathcal{R}(\rho, y) \le C\rho^{2-n} \int_{B_{3\rho/2(y)} \setminus B_\rho(y)} |\nabla u|^2 \, dx \qquad \forall\, \rho \in (0, 1/4),\ y \in B_1. \tag{A.7}$$

Hence, if $\mathcal{D}(\rho, y) \ge \frac{1}{2}\mathcal{D}(2\rho, y)$ and $0 \notin B_{2\rho(y)} \setminus B_{\rho/2(y)}$, then we can apply Lemma A.3 with $\delta = 1/2$ to the function $u(y + \rho\, \cdot)$ to we deduce that

$$\rho^{2-n} \int_{B_{3\rho/2}(y)\setminus B_\rho(y)} |\nabla u|^2 \, dx \le C\rho^{-n} \int_{B_{3\rho/2}(y)\setminus B_\rho(y)} |(x-y)\cdot\nabla u|^2 \, dx \le C\big(\mathcal{R}(3\rho/2, y) - \mathcal{R}(\rho, y)\big)$$

for some universal constant C. Combining this bound with (A.7) and using that $\mathcal{R}$ is nondecreasing, we deduce that

$$\mathcal{R}(\rho, y) \le C\big(\mathcal{R}(2\rho, y) - \mathcal{R}(\rho, y)\big) \qquad \text{provided } \mathcal{D}(\rho, y) \ge \tfrac{1}{2}\mathcal{D}(2\rho, y) \text{ and } 0 \notin B_{2\rho(y)} \setminus B_{\rho/2(y)}. \tag{A.8}$$

Note that $0 \notin B_{2\rho(y)} \setminus B_{\rho/2(y)}$ is equivalent to saying that either $\rho \ge 2|y|$ or $\rho \le |y|$.

Thus, fixed $y \in B_1$, if we define $a_j := \mathcal{D}(2^{-j-2}, y)$, $b_j := \mathcal{R}(2^{-j-2}, y)$, and $N := \lfloor -\log_2 |y| \rfloor$ (so $N = \infty$ if $y = 0$), then there exists a universal constant $L > 1$ such that:

(i) $b_j \le b_{j-1}$ for all $j \ge 1$ (since $\mathcal{R}$ is nondecreasing);
(ii) $a_j + b_j \le La_{j-1}$ for all $j \ge 1$ (by (A.7));
(iii) if $a_j \ge \frac{1}{2}a_{j-1}$ then $b_j \le L(b_{j-1} - b_j)$, for all $j \in \mathbb{N} \setminus \{N-2, N-1\}$[11] (by (A.8)).

Therefore, if we choose $\epsilon > 0$ such that $2^{-\epsilon} = \frac{L^{1+\epsilon}}{1+L}$, and we define $c_j := a_j^\epsilon b_j$ and $\theta := (2^{-\epsilon})^{\frac{1}{1+\epsilon}} \in (0,1)$, then the proof of [9, Lemma 3.2] shows that

$$c_{j+1} \le \theta c_j \qquad \text{for all } j \in \mathbb{N} \setminus \{N-2, N-1\},$$

which implies that

$$c_j \le \theta^j c_0 \quad \text{for } 1 \le j \le N-2, \qquad c_j \le \theta^{j-N} c_N \quad \text{for } j \ge N. \tag{A.9}$$

Also, as a consequence of (i) and (ii) above, we have

$$c_{N-1} = a_{N-1}^\epsilon b_{N-1} \le (La_{N-2})^\epsilon b_{N-2} \le L^\epsilon c_{N-2}, \qquad c_N = a_N^\epsilon b_N \le (L^2 a_{N-2})^\epsilon b_{N-2} \le L^{2\epsilon} c_{N-2}. \tag{A.10}$$

Hence, combining (A.9) and (A.10) we easily deduce that

$$c_j \le L^{2\epsilon} \theta^{j-2} c_0 \qquad \forall\, j \ge 1.$$

As in the proof of [9, Lemma 3.2], this implies that

$$b_j \le C(a_0 + b_0)\theta^j \le C\|\nabla u\|^2_{L^2(B_{3/2})} \theta^j \qquad \forall\, j \ge 1,$$

so (A.6) follows by choosing $\alpha > 0$ so that $2^{-2\alpha} = \theta$.

[11] The condition on j guarantees that either $2^{-j-2} \le |y|$ or $2^{-j-2} \ge 2|y|$.

We now observe that, thanks [9, Proposition 2.5] and a standard scaling and covering argument, we have $\|\nabla u\|_{L^2(B_{3/2}\setminus B_1)} \le C\|u\|_{L^1(B_2\setminus B_{1/2})}$. Hence, combining this bound with (A.6) and (A.2), we obtain

$$\mathcal{R}(\rho, y) \le C\rho^{2\alpha}\|u\|^2_{L^1(B_2)} \qquad \forall\, \rho \in (0, 1/4),\, y \in B_1.$$

Thanks to this estimate, the argument in [9, Proof of Theorem 1.2, Step 2] implies that

$$[u]_{C^\alpha(B_1)} := \sup_{x\neq y\in B_1} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C\|u\|_{L^1(B_2)},$$

and the conclusion follows by the interpolation estimate

$$\|u\|_{L^\infty(B_1)} \le C\Big([u]_{C^\alpha(B_1)} + \|u\|_{L^1(B_1)}\Big).$$

□

Appendix B. Uniform boundedness of solutions with spectrum bounded below

Although not relevant for this paper, it is interesting to observe that, by simply adapting the arguments in [9], one can deduce an a priori bound in L^∞ for solutions of $-\Delta u = f(u)$ whenever the spectrum of the linearized operator $-\Delta - f'(u)$ is contained inside $[-\Lambda, +\infty)$ for some finite constant $\Lambda \ge 0$. Also, for finite Morse index solutions, the constant Λ depends only on n and on a maximal finite dimensional subspace X_k on which Q_u is negative definite. Unfortunately one cannot hope in general to control Λ in terms only on the Morse index, as can be seen by considering the family of solutions (1.2) (which has index 1).

To present this result, consider $u \in C^2(B_2)$ a solution to $-\Delta u = f(u)$ in B_2 with $\mathrm{ind}(u, B_2) \le k$, and define

$$\widehat{Q}_u[\xi,\zeta] := \int_{B_2} \Big(\nabla\xi\cdot\nabla\zeta - f'(u)\,\xi\,\zeta\Big)\,dx.$$

Since $\mathrm{ind}(u, B_1) \le k$, there exists a k-dimensional set $X_k \subset C^1_c(B_1)$ such that, for any $\xi \in C^1_c(B_1)$, we can write $\xi = \xi_k + \xi'$ with $\xi_k \in X_k$, $\widehat{Q}_u[\xi', \xi'] \ge 0$, and $\widehat{Q}_u[\xi', \xi_k] = 0$.

Now, since X_k is finite dimensional,

$$\sup_{\xi\in X_k,\, \|\xi\|_{L^2(B_1)}} \|\xi\|_{L^\infty(B_1)} =: A_k < \infty,$$

so it follows from Lemma 2.2(ii) (and a covering argument) that

$$\inf_{\xi\in X_k,\, \|\xi\|_{L^2(B_1)}=1} \int_{B_1} \Big(|\nabla\xi|^2 - f'(u)\xi^2\Big)\,dx \ge -\sup_{\xi\in X_k,\, \|\xi\|_{L^2(B_1)}} \|\xi\|^2_{L^\infty(B_{3/4})} \int_{B_1} f'(u)\,dx \ge -C_n A_k^2,$$

which implies that

$$\int_{B_1} |\nabla\xi|^2\,dx \ge \int_{B_1} \big(f'(u) - \Lambda\big)\xi^2\,dx \qquad \forall\, \xi \in C^1_c(B_1), \tag{B.1}$$

where $\Lambda := C_n A_k^2$. In other words, the spectrum of the operator $-\Delta - f'(u)$ on $L^2(B_1)$ is bounded from below by $-\Lambda$.

In [9, Theorem 1.1], whenever $3 \le n \le 9$, the authors proved an a priori L^∞ estimate[12] for solutions of $-\Delta u = f(u)$ satisfying (B.1) with $\Lambda = 0$. The goal of this appendix is to show how to extend such a result to the general case $\Lambda \ge 0$.

[12] Actually, [9, Theorem 1.1] provides a universal C^α bound for some $\alpha > 0$. Analogously, also in the general case $\Lambda \ge 0$ one can prove an interior bound on $\|u\|_{C^\alpha}$.

Proposition B.1. *Let $3 \le n \le 9$, let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative, increasing, and of class C^1, and let $u \in C^2(B_2)$ solve $-\Delta u = f(u)$ and satisfy* (B.1) *for some $\Lambda \ge 0$. Then*

$$\|u\|_{L^\infty(B_{1/2})} \le C(\Lambda)\|u\|_{L^1(B_1)}.$$

The proof of Proposition B.1 is very similar to that in [9, Sections 2 & 3], the main differences being in two interior estimates that we present here. Once the two lemmas below are available, the proof follows by the same argument as in [9], and we leave the details to the interested reader.

Lemma B.2. *Let $u \in C^2(B_1)$ be as in Proposition B.1. Then, for any $\eta \in C^1_c(B_1)$, we have*

$$\int_{B_1} ((n-2)\eta + 2x\cdot\nabla\eta)\eta|\nabla u|^2 - 2(x\cdot\nabla\eta)\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2(|\nabla\eta|^2 + \Lambda\eta^2)\,dx \le 0. \tag{B.2}$$

Thus, for any $\varphi \in C^1_c(B_1)$, we have

$$\begin{aligned}\frac{(n-2)(10-n)}{4}&\int_{B_1}|x|^{-n}|x\cdot\nabla u|^2(1-\Lambda|x|^2)\varphi^2\,dx\\ \le \int_{B_1}\Big(&-2|x|^{2-n}|\nabla u|^2\varphi(x\cdot\nabla\varphi) + 4|x|^{2-n}(x\cdot\nabla u)\varphi\nabla u\cdot\nabla\varphi\,dx\\ &+(2-n)|x|^{-n}|x\cdot\nabla u|^2\varphi(x\cdot\nabla\varphi) + |x|^{2-n}|x\cdot\nabla u|^2|\nabla\varphi|^2\Big)\,dx\end{aligned} \tag{B.3}$$

In particular, for $0 < r < \frac{1}{2}\min\{1, \Lambda^{-1/2}\}$,

$$\int_{B_r}|x|^{-n}|x\cdot\nabla u|^2\,dx \le C(n)r^{2-n}\int_{B_{3r/2}\setminus B_r}|\nabla u|^2\,dx. \tag{B.4}$$

Proof. We proceed as in [9, Lemma 2.1] and sketch the proof here. First we choose $\xi = (x\cdot\nabla u)\eta$ in (B.1), with $\eta \in C^1_c(B_1)$, to get

$$\int_{B_1}\big(\Delta(x\cdot\nabla u) + f'(u)(x\cdot\nabla u)\big)(x\cdot\nabla u)\eta^2\,dx \le \int_{B_1}(x\cdot\nabla u)^2(|\nabla\eta|^2+\Lambda\eta^2)\,dx. \tag{B.5}$$

Then by noticing that

$$\Delta(x\cdot\nabla u) = x\cdot\nabla\Delta u + 2\Delta u = -f'(u)(x\cdot\nabla u) + 2\Delta u, \tag{B.6}$$

we conclude (B.2). Then (B.3) follows by taking $\eta = |x|^{-\frac{n-2}{2}}\varphi$, and (B.4) follows by further choosing φ as a cut-off function supported in $B_{3r/2}$ with $\varphi = 1$ on B_r. □

Lemma B.3. *Let $u \in C^2(B_1)$ be as in Proposition B.1. Write*

$$\mathcal{A} = \left(|D^2u|^2 - \frac{|D^2u\cdot\nabla u|^2}{|\nabla u|^2}\right)^{\frac{1}{2}} \quad \text{when } |\nabla u| \ne 0, \qquad \text{and } \mathcal{A} = 0 \quad \text{when } |\nabla u| = 0.$$

Then, for any $\eta \in C^1_c(B_1)$, we have

$$\int_{B_1}\mathcal{A}^2\eta^2\,dx \le (1+\Lambda)\int_{B_1}|\nabla u|^2|\nabla\eta|^2\,dx$$

Proof. We follow the argument of [9, Lemma 2.3] and again sketch the proof. Set $u_i := \partial_i u$. Multiplying both side of the equation $-\Delta u_i = f'(u)u_i$ by $u_i\eta^2$, and summing over $i = 1, \dots, n$, we get

$$\int_{B_1}\left(\sum_i|\nabla(u_i\eta)|^2 - |\nabla u|^2|\eta|^2\right)dx = \int_{B_1}f'(u)|\nabla u|^2\eta^2\,dx.$$

On the other hand, choosing $\xi = |\nabla u|\eta$ in (B.1), we have

$$\int_{B_1} |\nabla(|\nabla u|\eta)|^2 + \Lambda |\nabla u|^2 \eta^2 \, dx \geq \int_{B_1} f'(u)|\nabla u|^2 \eta^2 \, dx.$$

Thus we obtain

$$\int_{B_1} |\nabla u|^2 |\eta|^2 + \Lambda |\nabla u|^2 \eta^2 \, dx \geq \int_{B_1} \left(\sum_i |\nabla(u_i \eta)|^2 - |\nabla(|\nabla u|\eta)|^2 \right) dx,$$

and we aconclude the lemma by noticing that

$$\sum_i |\nabla(u_i \eta)|^2 - |\nabla(|\nabla u|\eta)|^2 = \mathcal{A}^2 \eta^2.$$

□

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