

$\mathcal{A}_g$ and its tautological ring

Fix a number $g \geq 1$. We are interested in the intersection theory of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties (X, θ) of dimension g . It is a smooth Deligne-Mumford stack of dimension $\frac{g(g+1)}{2}$.

The vector bundle associating to an abelian variety X the space $H^0(X, \Omega_X)$ is the *Hodge bundle*, denoted by $\mathbb{E}$. It is the source of the first collection of interesting classes, the *lambda classes*:

$$\lambda_i = c_i(\mathbb{E}) \in \mathrm{CH}^i(\mathcal{A}_g).$$

The *tautological ring* is the subring $R^*(\mathcal{A}_g)$ generated by the lambda classes. It was introduced and first studied by Gerard van der Geer (4), who also found the ideal of relations between lambda classes.

Goals of intersection theory

Let $Z \subseteq \mathcal{A}_g$ be the subset of abelian varieties satisfying a certain geometric property. If Z is closed, we can associate to it a cycle $[Z] \in \mathrm{CH}^*(\mathcal{A}_g)$. The two main questions that we can ask from the point of view of intersection theory are:

- **Question 1:** Does $[Z]$ belong to the tautological ring?
- **Question 2:** Given closed subsets $Z_1, \dots, Z_k \subseteq \mathcal{A}_g$, which algebraic relations hold between $[Z_1], \dots, [Z_k]$?

Which cycles will we consider?

Let $d > 0$, and define the *Noether-Lefschetz locus*

$$\mathrm{NL}_{g,d} = \left\{ (X, \theta) \in \mathcal{A}_g \left| \begin{array}{l} \text{there is an homomorphism } f : E \rightarrow X \\ \text{with } E \text{ an elliptic curve, such that } \deg(f^*\theta) = d \end{array} \right. \right\},$$

which has codimension $g - 1$. For example, $\mathrm{NL}_{g,1} = \mathcal{A}_1 \times \mathcal{A}_{g-1}$. We will also define $[\mathrm{NL}_{g,0}] = \frac{(-1)^g}{24} \lambda_{g-1}$.

Another interesting locus is given by Jacobians:

$$\mathcal{J}_g = \left\{ (X, \theta) \in \mathcal{A}_g \left| \begin{array}{l} \text{there are smooth curves } C_1, \dots, C_k \\ \text{such that } (X, \theta) \cong \mathrm{Jac}(C_1) \times \dots \times \mathrm{Jac}(C_k) \end{array} \right. \right\}.$$

Alternatively, $\mathcal{J}_g$ is the image of the *Torelli morphism*:

$$\mathrm{Tor} : \mathcal{M}_g^{\mathrm{ct}} \longrightarrow \mathcal{A}_g,$$

where $\mathcal{M}_g^{\mathrm{ct}} \subseteq \overline{\mathcal{M}}_g$ is the open subset of curves whose dual graph has no loops. One can also consider loci of non-simple abelian varieties, i.e., abelian varieties that have a non-trivial connected subgroup.

Non-tautological classes and the tautological projection

Question 1 is in general very strong; the general answer will be no:

Theorem A

The cycle $[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ is not tautological in the following cases:

- a) (Canning-Oprea-Pandharipande'24 (2)) For $g = 6$, and for $g = 8, 9$ if Pixton's relations on $\overline{\mathcal{M}}_g$ are complete.
- b) (I.L.'24 (6)) For $g = 12$ and any even $g \geq 16$.

We can ask a weaker version of Question 1:

Question 1': For a closed subset $Z \subseteq \mathcal{A}_g$, what is the closest approximation of $[Z]$ by a tautological class?

This is formalized through the following:

Definition (Canning-Molcho-Oprea-Pandharipande, (1)):

Let α be a cycle class on $\mathcal{A}_g$; it's *tautological projection* is the unique class $\mathbf{taut}(\alpha) \in R^*(\mathcal{A}_g)$ such that

$$\int_{\mathcal{A}_g} p(\lambda_i) \lambda_g \cap \bar{\alpha} = \int_{\overline{\mathcal{A}}_g} p(\lambda_i) \lambda_g \cap \mathbf{taut}(\alpha)$$

for any polynomial p in the lambda classes, for any toroidal compactification $\overline{\mathcal{A}}_g$. Implicit in this definition is that the Hodge bundle $\mathbb{E}$ extends to these compactifications.

We can compute these tautological projections:

Theorem B (I.L.'24 (6))

For any $g \geq 2$, we have

$$\mathbf{taut}([\mathrm{NL}_{g,d}]) = \frac{1}{24} \cdot \frac{4g}{|B_{2g}|} \sigma_{2g-1}(d) \lambda_{g-1}.$$

where σ_{2g-1} is the divisor sum function. The tautological projections of the irreducible components of the locus of non-simple abelian varieties have a similar shape.

The homomorphism property

Let α, β in $\mathrm{CH}^*(\mathcal{A}_g)$. We say that (α, β) has the *homomorphism property* if

$$\mathbf{taut}(\alpha) \cdot \mathbf{taut}(\beta) = \mathbf{taut}(\alpha \cdot \beta).$$

If α or β are tautological, then (α, β) has the homomorphism property. Therefore, the homomorphism property detects non-tautological classes. I have not found an example where the homomorphism property fails.

Theorem C

The homomorphism property holds in the following cases:

- a) (I.L.'24 (6)) If $\alpha = [\mathrm{NL}_{g,d}]$ and β is any other cycle.
- b) (Feusi - I.L. (3)) If $\alpha = [Z_1]$, $\beta = [Z_2]$ and the generic elements of Z_1 and Z_2 are non-simple abelian varieties.

Idea of proof: For a), we have that $\lambda_g[\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_{g-1}] = \frac{1}{24} \lambda_{g-1}[pt \times \overline{\mathcal{A}}_{g-1}]$. Then, an application of GRR to the universal compactified family and an input from the Fourier-Mukai transform shows that

$$\lambda_{g-1}[pt \times \overline{\mathcal{A}}_{g-1}] = (-1)^g \frac{4g}{|B_{2g}|} \lambda_g \lambda_{g-1}.$$

This is, $\lambda_g[\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_{g-1}] = \lambda_g \mathbf{taut}([\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_{g-1}])$, which implies a) when $d = 1$. Using the action of Hecke operators, one deduces the result for $d > 1$. b) follows from a calculation of the excess bundle in the fiber product $Z_1 \times_{\mathcal{A}_g} Z_2$.

Relations between NL cycles

We now address Question 2. Let $\mathbf{NL}_g$ be the linear span of the cycles $[\mathrm{NL}_{g,d}]$ for $d \geq 0$, and consider the power series

$$Z_g(q) = \sum_{d \geq 0} [\mathrm{NL}_{g,d}] q^d \in \mathbf{NL}_g[[q]].$$

Based on the results of Theorems A and B, and also in work by François Greer and Carl Lian (5), we formulate the following conjecture

Modularity Conjecture (Greer-I.L.-Lian-Pixton)

Let $g \geq 2$, and let $\mathrm{Mod}_{2g} \subseteq \mathbb{Q}[[q]]$ be the space of Fourier expansions of modular forms of weight $2g$ and level 1. Then:

- a) For any linear functional $\tau : \mathbf{NL}_g \longrightarrow \mathbb{Q}$, $\tau(Z_g) \in \mathrm{Mod}_{2g}$.
- b) $\dim(\mathbf{NL}_g) \geq \dim(\mathrm{Mod}_{2g})$.

a) Imposes a complicated collection of relations between the $[\mathrm{NL}_{g,d}]$, and it implies that $\dim(\mathbf{NL}_g) \leq \dim(\mathrm{Mod}_{2g})$. Then, b) says that there are no more relations.

Part a) of the conjecture in cohomology will follow from variations of the Kudla-Millson program. This is work in progress. I am happy to have discussions about possible proofs in the Chow groups.

Applications to the moduli space of curves

We apply Theorem C to the pair $([\mathrm{NL}_{g,d}], [\mathcal{J}_g])$. On $\overline{\mathcal{M}}_g$ (and any of its open subsets) there is a notion of tautological ring $R^*(\overline{\mathcal{M}}_g)$, which has been extensively studied. There is a λ_g -pairing

$$\mathrm{CH}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}) \times \mathrm{CH}^{g-2}(\mathcal{M}_g^{\mathrm{ct}}) \longrightarrow \mathrm{CH}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \longrightarrow \mathbb{Q},$$

where the last map sends α to $\int_{\overline{\mathcal{M}}_g} \bar{\alpha} \lambda_g$.

Corollary 1: Consider the class

$$\Delta_{g,d} = \mathrm{Tor}^*([\mathrm{NL}_{g,d}] - \mathbf{taut}([\mathrm{NL}_{g,d}])) \in \mathrm{CH}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}).$$

Then $\Delta_{g,d}$ pairs to 0 against any tautological class with respect to the λ_g pairing.

Note that for $d = 1$, $\Delta_{g,d}$ is tautological, and it is non-zero for $g = 6$. We expect it to be non-zero for all $g \geq 8$.

In order to obtain more consequences, one needs to understand the pullback $\mathrm{Tor}^*([\mathrm{NL}_{g,d}])$. Let $\pi : \mathcal{E} \longrightarrow \overline{\mathcal{M}}_{1,1}$ be the universal curve and let $\overline{\mathcal{M}}_g(\pi, d)$ be the moduli space of stable maps of degree d to the fibers of π . It has a virtual class of dimension $2g - 2$. A result of Greer and Lian (5) shows that $\mathrm{Tor}^*([\mathrm{NL}_{g,d}]) = [\overline{\mathcal{M}}_g(\pi, d)]^{\mathrm{vir}}|_{\mathcal{M}_g^{\mathrm{ct}}}$. As a consequence, we obtain:

Corollary 2: We can evaluate any Gromov-Witten invariant of π with a λ_g insertion:

$$\int_{[\overline{\mathcal{M}}_g(\pi, d)]^{\mathrm{vir}}} \lambda_g \Lambda = \frac{1}{24} \frac{4g}{|B_{2g}|} \sigma_{2g-1}(d) \int_{\overline{\mathcal{M}}_g} \lambda_g \lambda_{g-1} \Lambda.$$

If the domain curve is smooth, any stable map comes from a usual cover. Let $\mathcal{H}_{g \rightarrow 1, d} \subseteq \mathcal{M}_g$ be the locus of curves that admit a degree d map to an elliptic curve. Then, $\mathrm{Tor}^*([\mathrm{NL}_{g,d}])|_{\mathcal{M}_g} = [\mathcal{H}_{g \rightarrow 1, d}]$ and so, the Modularity Conjecture yields the following:

Prediction For $g \leq 11$ and $g = 13$, all the cycles $[\mathcal{H}_{g \rightarrow 1, d}] \in \mathrm{CH}^{g-1}(\mathcal{M}_g)$ vanish. For $g = 12, 14, 15, 16, 17, 19$, all the cycles $[\mathcal{H}_{g \rightarrow 1, d}]$ are proportional to each other.

This is known for $g \leq 9$ and all d , because $\mathrm{CH}^{g-1}(\mathcal{M}_g) = 0$, and for $g = 10, 11$ and $d = 2$ by work of Samir Canning and Hannah Larson.

References:

- (1) Samir Canning, Sam Molcho, Dragos Oprea, and Rahul Pandharipande. Tautological projection for cycles on the moduli space of principally polarized abelian varieties, 2024. arxiv preprint 2401.15768.
- (2) Samir Canning, Dragos Oprea, and Rahul Pandharipande. Tautological and non-tautological cycles on the moduli space of abelian varieties, 2024. arxiv preprint 2408.08718.
- (3) Jeremy Feusi and Aitor Iribar López. in preparation, 2025.
- (4) Gerard van der Geer. Cycles on the moduli space of abelian varieties. in *Moduli of Curves and Abelian Varieties*, Aspects of Mathematics, vol 33, p.65–89, 1999.
- (5) François Greer and Carl Lian. d-ellipticity loci and the torelli map, 2024. arXiv preprint 2404.10826.
- (6) Aitor Iribar López. Noether-lefschetz cycles on the moduli space of abelian varieties, 2024. arXiv preprint 2411.09910.