TRABAJO DE FIN DE GRADO

Holomorphic curves

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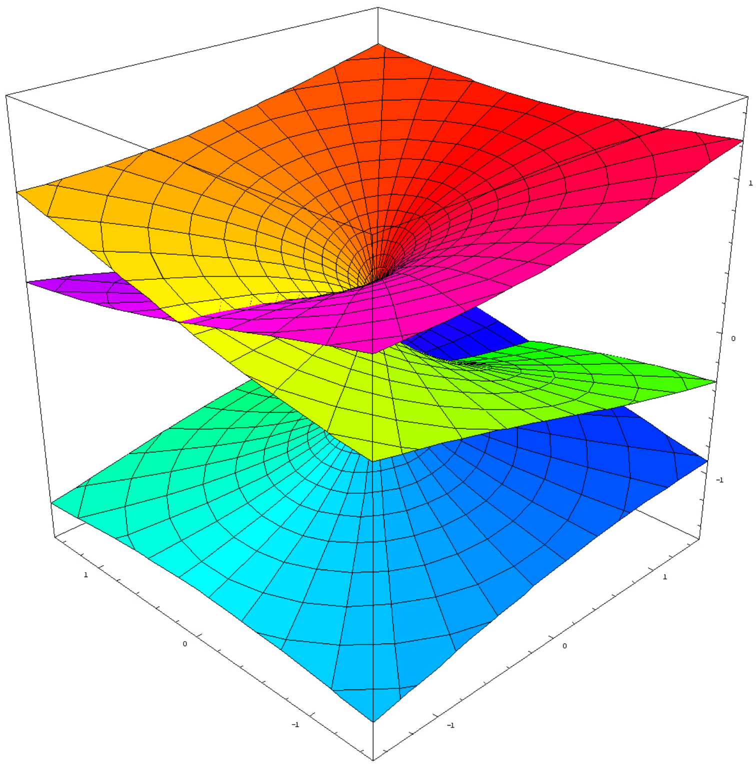

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Abstract

The main goal of this work is to prove two recently published theorems by Antonio Alarcón, Franc Forstneric and Francisco López regarding the existence of *holomorphic legendrian curves*. To that end, we define and develop the concepts of *Riemann surfaces* and *holomorphic contact manifolds* and explain the techniques of *holomorphic approximation* necessary for the proofs.

Keywords: *Contact geometry, holomorphic approximation, legendrian curves, Mergelyan theorem, Riemann surfaces, symplectic geometry*

Resumen

El objetivo principal de este trabajo es demostrar dos teoremas recientemente publicados por Antonio Alarcón, Franc Forstneric y Francisco López sobre la existencia de *curvas legendrianas holomorfas*. Para ello, definimos y desarrollamos los conceptos de *superficie de Riemann* y *variedad de contacto holomorfa* y explicamos las técnicas de *aproximación holomorfa* necesarias para las demostraciones.

Palabras clave: *Aproximación holomorfa, curvas legendrianas, geometría de contacto, geometría simpléctica, superficies de Riemann, teorema de Mergelyan*

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General index

Introduction and main results

The original Bachelor's Project "Curvas holomorfas" was presented in the 14 of July of 2021 in Complutense University of Madrid, being awarded a full mark and a honorific mention. The project was presented in Spanish. This is a translation, carried out by the author, of the original work.

The study of symplectic and contact smooth manifolds appears naturally to formalise diverse problems from classical mechanics. Its study during the past century has been a fruitful are of research, with important consequences in both Physics and Geometry.

A symplectic manifolds is a smooth manifolds M of even dimension $2n$ together with a closed, nondegenerate 2-form ω. A symplectic manifolds (M, ω) together with a function $H : M \to \mathbb{R}$ form a hamiltonian contact system and represents the phase space of a conservative mechanical system.
The most paradigmatic example of these manifolds is the cotangent space of a smooth manifold T^*M. In canonical coordinates, (x^j, y_j) the symplectic form is given by $\sum_j dx^j \wedge dy_j$.
Immersed submanifolds $f : N \to M$ such that $f^*\omega = 0$ are of special interest, and are known as isotropic submanifolds. Among them, the ones with maximal dimension, which is n since ω is non degenerate are named lagrangian submanifolds and play an important role in diverse theorems of symplectic geometry such as the Arnold-Liouville Theorem or Weinstein's Tubular Neighbourhood Theorem. For example, a section $s : M \to T^*M$ is a closed differential form if and only if it is a legendrian immersion.

Contact manifolds are the odd-dimensional analog of symplectic manifolds. A contact structure on a $2n+1$-dimensional manifold M is a subbundle Ψ of the tangent bundle of codimension 1, which is given locally as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. Given a smooth manifolds M, the space of 1-jets, which is isomorphic to $T^*M \times \mathbb{R}$, te projectivised cotantent bundle $\mathbb{P}(T^*M)$ or the unit cotangent bundle in case M is given a metric are examples of contact manifolds, among others.

Now, if M is as above, an immersion $f : N \to M$ is said to be isotropic if $f_*(TN) \subset \Psi$. The condition $\alpha \wedge (d\alpha)^n \neq 0$ and Frobenius Integrability Theorem imply that n is the maximum dimension that N can have, and in that case the immersion is said to be legendrian. Legendrian submanifolds are of special interest:

- Legendrian embeddings of $\mathbb{S}^1$ into a contact 3-dimensional manifold are called legendrian knots. Two legendrian knots $K, K' \subset \mathbb{R}^3$ with the standard contact form given by the kernel of $\alpha = dz - xdy$ are contact-isotopic (this is, there exists a continuous family of legendrian knots ϕ_t such that $\phi_0 = K$, $\phi_1 = K'$) if and only if their projections onto the xy-plane can be related by a series of elementary moves, which correspond to the movements one can do to untie the knot. A proof of this can be foun in [Swi92].
- In his famous book [Gro86], Gromov proves using convex integration that any continuous path $\gamma : [0,1] \to (M, \Psi)$, where Ψ is a contact structure, can be approximated by isotropic embeddings.

Contact and symplectic manifolds enjoy similar or analog properties. For instance, both satisfy the Darboux Theorems, which basically show that locally, any sympletcic or contact manifold is equivalent to

$$\left(\mathbb{R}^{2n}, \sum_{j=1}^{n} dx^j \wedge dy^j\right) \qquad \left(\mathbb{R}^{2n+1}, \ker(dz - \sum_j x^j dy^j)\right),$$

respectively. This is why these two strutures are known as the standard symplectic and contact structures in affine space.

The definitions that we have in smooth manifolds can be generalised to complex manifolds without problem, and the resulting structures are known as holomorphic symplectic and contact structures. Isotropic, lagrangian and legendrian submanifolds can also be define in an analogous way and therefore one may wonder which theorems about real geometry can be extended to complex manifolds.
In [Bry82], R. Bryant proved that every compact Riemann Surface can be embedded in $\mathbb{CP}^3$ as a legendrian submanifold, with the standard contact structure in projective space.
The problem was unsolved for open Riemann surfaces until A. Alarcón, F. Fonstneric y F. López proved in 2017, using techniques of holomorphic approximation, an analogue of Bryant's result. More precisely, in [AFL17] they prove the following stronger statements:

Theorem 1. *Let X be a Riemann surface and let $f : \overline{Y} \to \mathbb{C}^{2n+1}$ be an isotropic holomorphic map*[1]*, where $Y \subset\subset X$ is a Runge open set. Then f can be approximated uniformly over Y by isotropic, proper embeddings $f : X \to \mathbb{C}^{2n+1}$ with the standard contact structure of affine space.*

Theorem 2. *Let X be a compact, bordered Riemann surface, and let (M, Ψ) be a holomorphic, contact manifold of dimension $2n + 1$. Then there exists an isotropic embedding of X into M.*

The objective of this work is to give a complete, rigorous proof of these two Theorems in the case $n = 1$, illustrating the main ideas that the authors use for generic n.

In the first part, we will define complex manifolds and in particular, Riemann surfaces. Contrary to the usual treatment given in some books like [For81] or [GR65], we will avoid the introduction of sheaves and sheaf cohomology, since we will be focusing on open Riemann surfaces and a sistematic study of their sheaves woould exceed the limits of this work. Following the ideas in [Var11], we will instead use Green functions to prove analogues of the well known Cauchy formulas in arbitrary Riemann surfaces, which will be useful to prove the classical theorems of Runge and Weierstrass.
The main feature of open Riemann surfaces for holomorphic approximation is Mergelyan-Bishop's Theorem. To prove it, we will explain the ideas in [Sak72].
Finally, we will explain what holomorphic symplectic and contact structures are, giving simple proofs of the holomorphic Darboux Theorems, which were not found in the references, adapting the proof of the real Darboux Theorems in [Lee12].

The second part is devoted to the proof of the two results and a systematic approach to some of the techniques in holomorphic approximation. This techniques have been exploited by the authors in other similar articles like [AFL17] or [AFL16].
To do so, we start with an exposition of the recurring ideas that appear in the proofs of the two theorems, and we will also formalise some results that the authors use in an implicit way. In particular, propositions 2.1.1, 2.1.2 and 2.1.3 have not been found in any of the references or other articles by the authors, and therefore their proof is original. We will exemplify the use of these ideas to strengthen Mergelyan Theorem and prove a classical result by R. Gunning and M. Narasimhan about the existence of immersions of Riemann surfaces in $\mathbb{C}$, originally proved in [GN67]. The remaining part is rather technical and it's dedicated to the proof of theorems 1 and 2.

Although this project is mostly self contained (based on the Bachelor's knowledge), we will use some ideas from functional analysis to prove Runge and Mergelyan Theorems, and an argument using transversality appears in the proof of 2.3.2. This two topics are treated in a succinct way in the Appendices.

[1] We will see that it is necessary that f is isotropic

Part 1

Riemann surfaces and holomorphic contact structures

1.1 Functions of several complex variables. Complex manifolds

First we will develop the local theory of holomorphic functions.

Definition 1.1.1. *Let Ω an open subset of $\mathbb{C}^m$. A continuous function $f : \Omega \to \mathbb{C}$ is holomorphic if it is holomorphic in each variable (as functions between open subsets of $\mathbb{C}$). A function $f : \Omega \to \mathbb{C}^m$ is holomorphic if all of its components are holomorphic*

A smooth function $f = u + iv$ is holomorphic precisely when, for each j, the Cauchy-Riemann equations hold:

$$\frac{\partial u}{\partial x^j} = -\frac{\partial v}{\partial y^j} \qquad \frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}.$$

It is convenient to use the Wirtinger derivatives

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \qquad \frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$$

and then, the Cauchy-Riemann equations are equivalent to $\frac{\partial f}{\partial \bar{z}^j} = 0$ for all j. To do calculus on complex manifolds, we will follow the standard notation for indices up and down, as well as multi-index notation. Einstein summation is not necessary since the formulas that will appear are small. An immediate consequence of the usual Cauchy Formula is the following:

Lemma 1.1.2. *Let $f : \Omega \to \mathbb{C}$ holomorphic. Then for any $z \in \Omega$,*

$$f(z) = \frac{1}{(2\pi i)^n}\int_{|\eta_1|=r_1} \dots \int_{|\eta_n|=r_n} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1)\dots(\eta_n - z_n)} d\eta_n \dots d\eta_1$$

Proof. Using Cauchy formula for the first variable, one gets

$$f(z) = \frac{1}{2\pi i}\int_{|\eta_1|=r_1} \frac{f(\eta_1, z_2, \dots, z_n)}{(\eta_1 - z_1)} d\eta_1.$$

If one applies the theorem again, inside the integral,

$$f(z) = \frac{1}{(2\pi i)^2}\int_{|\eta_1|=r_1}\int_{|\eta_2|=r_2} \frac{f(\eta_1, \eta_2, \dots, z_n)}{(\eta_1 - z_1)(\eta_2 - z_2)} d\eta_1 d\eta_2.$$

And one can continue like this to get the desired formula. □

Corollary 1.1.3. *Every holomorphic function is smooth, and the following formula holds*

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha} f(z) = \frac{\alpha!}{(2\pi i)^n}\int_{|\eta_1|=r_1} \dots \int_{|\eta_n|=r_n} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1)^{\alpha_1+1}\dots(\eta_n - z_n)^{\alpha_n+1}} d\eta_n \dots d\eta_1$$

Corollary 1.1.4. *If a sequence of holomorphic functions f_n converges uniformly on compact sets to f, then f is holomorphic and for any multi-index α, $\frac{\partial^{|\alpha|}}{\partial z^\alpha} f_n$ converges uniformly over compact subsets to $\frac{\partial^{|\alpha|}}{\partial z^\alpha} f$.*

The classical theorems of calculus can be easily generalised; for that purpose, the following lemma will be necessary. We don't prove it but not that it is a direct consequence of the chain rule and the definitions of $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$.

Lemma 1.1.5. *If f and g are functions between open subsets of complex affine space whose composition makes sense, then*

$$\frac{\partial}{\partial z^k}(f \circ g) = \sum_j \frac{\partial f}{\partial w^j}\frac{\partial g^j}{\partial z^k} + \frac{\partial f}{\partial \bar{w}^j}\frac{\partial \bar{g}^j}{\partial z^k}, \quad \frac{\partial}{\partial \bar{z}^j}(f \circ g) = \sum_j \frac{\partial f}{\partial w^j}\frac{\partial g^j}{\partial \bar{z}^k} + \frac{\partial f}{\partial \bar{w}^j}\frac{\partial \bar{g}^j}{\partial \bar{z}^k}.$$

Corollary 1.1.6. *The composition of two holomorphic functions is again holomorphic, and*

$$\frac{\partial}{\partial z^k}(f \circ g) = \sum_j \frac{\partial f}{\partial w^j}\frac{\partial g^j}{\partial z^k}.$$

If a holomorphic functions between open sets of equal dimension has regular derivative

$$\frac{\partial f}{\partial z} = \left(\frac{\partial f^k}{\partial z^j}\right)_{k,j}$$

in a point a, then on a neighbourhood of a, the function is invertible and its inverse is holomorphic. If a holomorphic function $F = F(z, w)$ between open subsets of $\mathbb{C}^{q+n}$ and $\mathbb{C}^n$ verifies that $\frac{\partial F}{\partial w}(a, b)$ is invertible and $F(a, b) = 0$, then there exists a holomorphic function g such that $g(a) = b$ and $F(t, g(t)) = 0$.

Proof. These theorems follow from the usual chain rule and inverse function theorem and the formulas in the previous lemma. □

With all these tools, one can replicate the same process one would follow to define a smooth manifold to define a complex manifold. It can be seen in full details in [GR65], so we will do it quickly.

Definition 1.1.7. *A complex manifold of dimension n es a topological space X locally homeomorphic to $\mathbb{C}^n$, Hausdorff and second countable, together with a (maximal) atlas of charts $\{(U_j, z_j)_{j\in I}\}$ such that $z_j : U_j \to \mathbb{C}^n$ is a homeomorphism onto its image and $z_j \circ (z_k)^{-1}$ is holomorphic, with holomorphic inverse whenever $U_j \cap U_k$ is nonempty.*

As it was already noted, under the natural identification $\mathbb{C}^n = \mathbb{R}^{2n}$, $\det_{\mathbb{R}}\left(\frac{\partial f}{\partial(x,y)}\right) = \left|\det_C\left(\frac{\partial f}{\partial z}\right)\right|^2$ and thus, in particular, holomorphic change of coordinates have positive jacobian so a complex manifold is orientable

Example 1.1.8.

a) *Every open subset $\Omega \subset \mathbb{C}^n$ is a complex manifold.*

b) *Projective spaces $\mathbb{CP}^n$ are n-dimensional complex manifolds of with charts (U_j, u_j), where U_j is given by $z_j \neq 0$ and $u_j([z_0 : \ldots : z_n]) = \left(\frac{z_0}{z_j}, \ldots, \frac{z_n}{z_j}\right)$, omitting the j-th coordinate.*

c) *If z, w are two complex numbers that are not collinear with 0, the set $L = z\mathbb{Z} + w\mathbb{Z}$ forms a lattice in $\mathbb{C}$ and the quotient $\mathbb{C}/L$, homeomorphic to a torus inherits a complex structure in a natural way.*

d) *Although this is not a trivial fact, every orientable real surface X has a structure of 1-complex manifold, also known as Riemann surface. The idea is to endow the surface with a metric and finding local coordinates around each point such that $g_{12} = 0$ y $g_{11} = g_{12}$. Using the orientability of X, these coordinates can be assumed to verify $g_{11} > 0$ and they constitute a holomorphic atlas.*

e) *Not every even dimensional smooth manifold has a complex structure. Orientability is one of the requisites and, contrary to what the previous example shows, in higher dimensions there are more requisites. For example, as it can be seen in [BS53], $\mathbb{S}^n$ is not a complex manifold if $n \neq 2, 6$, and the case $n = 6$ is an open problem.*

A continuous function $f : X \to \mathbb{C}$ is said to be holomorphic if $f \circ z^{-1}$ is holomorphic for each chart (U, z), and a function $f : X \to Y$ between two complex manifolds is holomorphic if for any pair or charts (U, z) in Y and (V, w) in X, $z \circ f \circ w^{-1}$ is holomorphic.
As it is usual to see in the literature, if $Y \subset X$ is an open set, $\mathcal{O}(Y)$ and $\mathcal{C}^\infty(Y)$ represent the spaces of holomorphic and smooth functions, respectively, in Y.

Definition 1.1.9. *The tangent space to a complex manifold is the complexification of its tangent space as a smooth manifold, that is to say, $T_{\mathbb{C}}X$ is a vector bundle with fibers $(T_{\mathbb{C}}X)_p = TX_p \otimes_{\mathbb{R}} \mathbb{C}$, and inherits a natural structure of complex manifold in such a way that $\pi : T_{\mathbb{C}}X \to X$ is holomorphic.*

Since $TX_p \otimes \mathbb{C}$ is a complex vector space in a natural way, given holomorphic coordinates $z = x + iy$, it makes sense to define in $T_{\mathbb{C}}X$ the vectors $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$ using the same formulas as in $\mathbb{C}^n$. The interesting phenomena is that these vectors depend of the coordinates one takes, but the subspaces generated by the $\frac{\partial}{\partial z^j}$ and the $\frac{\partial}{\partial \bar{z}^j}$ are invariant. We prove it in the following way::
For any p in X, given holomorphic coordinates $z = x+iy$ around p, we define a linear map $J_p : TX_p \to TX_p$ in a basis as

$$J_p\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j} \quad \text{y} \quad J_p\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}.$$

J_p can be extended to $TX_p \otimes \mathbb{C}$ by the formula $J_p(v \otimes \alpha) = J_p(v) \otimes \alpha$. Since $J_p^2 = -1$, and J_p is $\mathbb{C}$-linear, J_p can be diagonalised. The vectors $\frac{\partial}{\partial z^j}$ have i as eigenvalue, while the vectors $\frac{\partial}{\partial \bar{z}_j}$ have $-i$. Therefore, it is sufficient to prove that J_p is independent of the holomorphic coordinates taken.
If $w = u+iv$ are another holomorphic coordinates, and $\widehat{J}_p$ is the map defined if we started with w, the chain rule 1.1.5 shows that

$$\frac{\partial}{\partial z^j} = \frac{\partial w^l}{\partial z^j}\frac{\partial}{\partial w^l} \qquad \frac{\partial}{\partial \bar{z}^j} = \frac{\partial \bar{w}^l}{\partial \bar{z}^j}\frac{\partial}{\partial \bar{w}^l},$$

and therefore, $\widehat{J}_p\left(\frac{\partial}{\partial z^j}\right) = i\frac{\partial}{\partial z^j}$ and $\widehat{J}_p\left(\frac{\partial}{\partial \bar{z}^j}\right) = -i\frac{\partial}{\partial \bar{z}^j}$ for all j. Since these form a basis of $TX_p \otimes \mathbb{C}$, $J_p = \widehat{J}_p$.[1]
It makes sense to make the following definition:

Definition 1.1.10. *$T^{1,0}X$ is the subbundle generated by $\frac{\partial}{\partial z^j}$ and $T^{0,1}X$ is the subbundle generated by $\frac{\partial}{\partial \bar{z}^j}$. They are also complex manifolds in a natural way. In particular, holomorphic section of $T^{1,0}X$ are what we will refer to as holomorphic vector fields, since in local coordinates they can be expressed as $X = X^j \frac{\partial}{\partial z^j}$ where $X^j : X \to \mathbb{C}$ is holomorphic.*

The space of alternating tensors are defined in the same way as in the smooth case, that is to say, $\mathcal{E}^n X = \wedge^n(T_{\mathbb{C}}X^*)$, where * denotes the dual. To the basis formed by $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}$ correspond the dual basis $dz^j, d\bar{z}^j$ of $T_{\mathbb{C}}X^*$, where

$$dz^j = dx^j + idy^j \quad \text{and} \quad d\bar{z}^j = dx^j + idy^j.$$

The decomposition $TX = T^{1,0}X \oplus T^{0,1}X$ gives raise to a decomposition

$$\mathcal{E}^k X = \bigoplus_{p+q=r} \mathcal{E}^{p,q}X := \bigoplus_{p+q=r} (\wedge_p T^{1,0}X^*) \wedge (\wedge_q T^{0,1}X^*).$$

The space $\mathcal{E}^{p,q}$ is called the space of $(p,q)-$forms, and are the ones that can be expressed in local coordinates as

$$\omega = \sum_{|I|=p, |J|=q} f_{IJ} dz^I \wedge d\bar{z}^J.$$

The exterior derivative d also decomposes. If $\pi_{p,q} : \mathcal{E}^{p+q} \to \mathcal{E}^{p,q}$ is the projection, we have $d = \sum_{r+s=p+q+1} \pi_{r,s} \circ d$, seeing $d : \mathcal{E}^{p,q} \to \mathcal{E}^{p+q+1}$. However, most of these terms are 0:

Proposition 1.1.11. *With the previous notation, in a complex manifold X all the projections $\pi_{r,s} \circ d$ vanish except when $(r,s) = (p+1, q)$ and $(r,s) = (p, q+1)$.*

Proof. Let $\omega \in \mathcal{E}^{p,q}$, and write it in local coordinates as

$$\omega = \sum_{|I|=p, |J|=q} f_{I,J} dz^I \wedge d\bar{z}^J.$$

[1] The vector bundle isomorphism J is usually called an almost complex structure in X.

Then, using the elementary properties of d,

$$d\omega = \left(\sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\bar{z}^J \right) + \left(\sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\partial f_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J \right) \in \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}.$$

□

Definition 1.1.12. *We define the operators*

$$\partial = \pi_{p+1,q} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}, \qquad \bar{\partial} = \pi_{p,q+1} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}.$$

From the equalities $d^2 = 0$ and $d = \partial + \bar{\partial}$ one deduces that $\partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$, but this equality occurs in different spaces of forms, so actually

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

From the formula in last proposition one gets the coordinate formulas of these operators. In particular, a function f is holomorphic precisely when $\bar{\partial} f = 0$. A naive definition of a holomorphic form would be something that can be expressed as $\sum_I f_I dz^I$ where f_I is holomorphic. With these operators we can give an invariant definition:

Definition 1.1.13. *A form ω is holomorphic if it is a $(k,0)$-form and $\bar{\partial}\omega = 0$. The set of holomorphic k-forms is denoted by $\mathcal{E}^k_{hol}(X)$.*

We finish this section with a result about smooth manifolds that we are not going to prove, but it will be necessary in the future. It is a classical result in riemannian geometry and differential topology, and can be easily proved taking a riemannian metric and using a covering of the manifold with geodesic balls.

Theorem 1.1.14 (Existence of good coverings)**.** *Let X be a smooth manifold, and let $\mathcal{U}$ be an open covering of X. Then there exists a refinement $\mathcal{V} = \{V_j\}_{j\in I}$ of $\mathcal{U}$ such that all the sets V_j and their finite intersections are contractible.*

1.2 Riemann surfaces

A Riemann surface is simply a 1-dimensional complex manifold. Unless the contrary is said, Riemann surfaces are assumed to be connected. For the purpose of this work, we are interested in non-compact Riemann surfaces, and we refer to them as open Riemann surfaces. The differences between the compact case and the open case are abysmal; for instance, a compact Riemann surface does not admit non constant holomorphic functions, but an open Riemann surface admits plenty of them. A complete study of compact Riemann surfaces can be found in [For81].

We will use the term bordered Riemann surface to refer to an open, relatively compact subset of a Riemann surface with smooth boundary or to refer to the closure of such an open set, but it will be clear by the context which one are we talking about. The notation $Y \subset\subset X$ says that Y is a relatively compact subset of X.
for instance, the unit disk is a bordered Riemann surface but the complex plane is not. BY a holomorphic curve we mean a holomorphic map $f : X \to \mathbb{C}^N$ when X is an open Riemann surface. The second most important Riemann surface after open subsets of $\mathbb{C}$ is the Riemann sphere, $\mathbb{C}_\infty = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Indeed, we can define meromorphic functions using it:

Definition 1.2.1. *A meromorphic function on X is a holomorphic map $f : X \to \mathbb{C}_\infty$. If f is meromorphic and $p \in X$, the order of f at p is the order of $f \circ z^{-1} : \Omega \subset \mathbb{C} \to \mathbb{C}_\infty$ at $z(p)$, where (U, z) are holomorphic coordinates around p. We denote it by $ord_p(f)$ and it is of course well defined.*

In the same way we can define meromorphic forms:

Definition 1.2.2. *A meromorphic form on a Riemann surface X is a holomorphic form $\omega \in \mathcal{E}^1_{hol}(X \smallsetminus A)$, where A is a discrete subset of X, such that for every $p \in X$ there are holomorphic coordinates (U, z) and a meromorphic form f such that $\omega = f\,dz$. In that situation, we define the order of ω in p to be the order f at p.*

Again, one can prove that these are well defined ideas. Most theorems of complex analysis have their analogues in Riemann surfaces:

Theorem 1.2.3 (Identity principle)**.** *Let f, g be meromorphic functions in a Riemann surface X. If there is a subset A with an accumulation point where $f = g$, then $f = g$ in the whole of X.*
Let α, β be two meromorphic forms in X. If there is a subset A having an accumulation point where $\alpha = \beta$, then $\alpha = \beta$ in the whole X.

Theorem 1.2.4 (Maximum principle)**.** *If $f : X \to \mathbb{C}$ is holomorphic and $|f|$ attains an extreme value then f is constant*

In particular, the only holomorphic functions on a compact Riemann surface are the constants, as we claimed before.

The notions of exact and closed differential forms can be defined in a Riemann surface, and one can also integrate complex valued differential forms: if $\omega = \alpha + i\beta$ is a k-form , where α and β are real-valued, then the integral of ω along a k-submanifold Y where ω has compact support is just

$$\int_Y \omega = \int_Y \alpha_{|Y} + i \int_Y \beta_{|Y}$$

and one can easily check that Stokes theorem can be applied. In particular, Stokes theorem can be used to define the De Rham map $\mathrm{dR} : H^1(X) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(X, \mathbb{Z}), \mathbb{C})$

$$dR(\omega)(n_1\gamma_1 + \ldots + n_k\gamma_k) = n_1 \int_{\gamma_1} \omega + \ldots + n_k \int_{\gamma_k} \omega,$$

where we are thinking of $H_1(X, \mathbb{Z})$ as the set of linear combinations of smooth paths in X modulo the paths that are the boundary of some open. The fact that dR is well defined and in fact is an isomorphism can be seen in [For81].

Example 1.2.5. *Every holomorphic 1-form is closed in a Riemann surface, since if can be written locally as $f dz$, with f holomorphic, and then*

$$d(f dz) = df \wedge dz = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = 0.$$

However, they are not exact in general. If $X = \mathbb{C} \setminus \{0\}$ and $\omega = \frac{dz}{z}$, ω but its integral along a circle centred at 0 is $\pm 2\pi i$, by Cauchy formula, so it is not exact.

IN the complex plane, one has

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

So a smooth function f is harmonic if and only if

$$\partial \bar{\partial} f = \frac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = 0.$$

This motivated the following definition:

Definition 1.2.6. *The Laplacian operator in a Riemann surface is $\Delta = \partial \bar{\partial}$. A smooth function f is said to be harmonic if $\Delta f = 0$.*

We would like to define subharmonic functions also, but we will do it in other way for two reasons: first, because it is difficult to make sense to the formula $\Delta f \geq 0$ in an abstract surface and second because we will need a weaker notion of subharmonic function, allowing singularities. To motivate the definition, let U be an open subset of $\mathbb{C}$ and $u : \overline{U} \to \mathbb{R}$ a continuous function such that $\Delta u \geq 0$ in U. If

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt$$

then by the divergence theorem,

$$f'(r) = \int_0^{2\pi} e^{it} \nabla u(z_0 + re^{it}) dt = \frac{1}{2\pi r} \int_{bD(z_0,r)} \nabla u(z) \cdot N(z) = \frac{1}{2\pi r} \int_{D(z_0,r)} \Delta u(z) \geq 0$$

so f is increasing. Also, $\lim_{r \to 0} f(r) = u(z_0)$, so

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt.$$

If u attained its maximum in a point $z_0 \in U$ and we take r such that $bD(z_0, r) \cap bU \neq \emptyset$, then

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} \left[u(z_0 + re^{it}) - u(z_0) \right] dt,$$

but $u(z_0 + re^{it}) - u(z_0) \leq 0$. The only possible option is that $u(z_0) = u(z_0 + re^{it})$ for all t and in particular, $u(z) = u(z_0)$ for some z in bU. Therefore we obtain the weak maximum principle: If u is subharmonic, v is harmonic and both are continuous in $\overline{U}$ and U is bounded, then if $v \geq u$ in bU then $v \geq u$ in U.

Definition 1.2.7. *Let X be a Riemann surface and $u : X \to \mathbb{R} \cup \{-\infty\}$ an upper-semicontinuous function.*[2] *We say u is subharmonic if whenever K is compact and $v : K \to \mathbb{R}$ is continuous in K and subharmonic in its interior, if $v \geq u$ en bK, then $v \geq u$ in K.*

Despite this strange definition, subharmonic functions enjoy some nice properties: if $u, v : X \to \mathbb{R} \cup \{-\infty\}$, then

a) If u and v are subharmonic, $u + v$ and $\max(u, v)$ also are.

b) If u is locally subharmonic, then it is subharmonic.

c) If u is subharmonic, v is continuous in a compact K, harmonic in its interior and $u = v$ en bK, then the function defined as

$$u \# v(z) = \begin{cases} v(z) & \text{if } z \in K \\ u(z) & \text{si } z \in X \setminus K \end{cases}$$

is subharmonic.

[2] ie, $u^{-1}([-\infty, y))$ is open for all $y \in \mathbb{R}$

d) If $f : Y \to X$ is holomorphic and u is subharmonic, so is $u \circ f$.

e) If $f : X \to \mathbb{C}$ holomorphic, $\log |f|^2$ is subharmonic.

f) (The maximum principle) If K is compact, u is continuous in K and subharmonic in its interior,

$$\sup_{bK} u = \sup_{K} u.$$

We will use Perron method to obtain Green functions.

Definition 1.2.8. *A family of subharmonic functions $\mathcal{F}$ in a Riemann surface X is a Perron family when:*

1. *If $u, v \in \mathcal{F}$, $\max(u, v) \in \mathcal{F}$.*
2. *If $u \in \mathcal{F}$, $v : K \to \mathbb{R}$ is continuous in a compact K, harmonic in its interior that agrees with u on its boundary, then $u \# v \in \mathcal{F}$.*

And the main result we will need is the following:

Theorem 1.2.9 (Perron)**.** *Let $\mathcal{F}$ be a Perron family in a Riemann surface X. If*

$$v(z) = \sup_{u \in \mathcal{F}} u(z),$$

then either $v(z) = +\infty$ for all z or v is harmonic in X.

Proof. We refer to the proof in p. 180 of [For81]. □

Our first application will be to solve the Dirichlet problem, which consists in the following: Given an open $Y \subset X$ and a continuous function $f : bY \to \mathbb{R}$, we seek a continuous function $\overline{Y}$, harmonic in its interior that agrees with f in the border. We have to impose some conditions to f and to the boundary of Y, as the following example shows:

Example 1.2.10. *There is no continuous function u in $\overline{D(0,1)}$, harmonic in $D(0,1) \setminus \{0\}$ such that $u(0) = 0$ and $u(z) = 1$ whenever $|z| = 1$. To see this, define*

$$\phi(r, s) = \frac{1}{2\pi} \int_{bA(s,1)} u(rz) dz,$$

where $A(s, 1)$ is the annulus $\{w \in \mathbb{C} : s \leq |w| \leq 1\}$. Then, by the divergence theorem, $\frac{\partial}{\partial r}\phi(r, s) = 0$ if $s, r > 0$, but ϕ is continuous in r, so for all $s > 0$,

$$0 = \phi(0, s) = \phi(1, s) = \frac{1}{2\pi} \left[\int_{|w|=1} u(w) dw - \int_{|w|=s} u(w) dw \right] = 1 - \frac{1}{2\pi} \int_{|w|=s} u(w) dw$$

But ϕ is also continuous in s, so

$$1 = \lim_{s \to 0} \frac{1}{2\pi} \int_{|w|=s} u(w) dw = \lim_{s \to 0} \frac{1}{2\pi} \int_0^{2\pi} u(se^{it}) se^{it} dt = 0,$$

a contradiction.

Definition 1.2.11. *Let $Y \subset X$ be open. We say $x \in bY$ is regular for the Dirichlet problem if there is a neighbourhood U of x and a continuous function $\beta : \overline{Y} \cap U \to \mathbb{R}$ such that*

1. *β is subharmonic in $Y \cap U$.*
2. *$\beta \leq 0$, with equality just in x.*

Such a function is a barrier in x. Most open subsets of $\mathbb{C}$ have regular boundary, as the following example shows:

Example 1.2.12. *If $Y \subset \mathbb{C}$, $x \in bY$ and there is a disk $D = D(a,r)$ such that $x \in bD$ and $Y \cap \overline{D} = \emptyset$. Then*

$$\beta(z) = \log \frac{r}{|2z - x - a|}$$

is a barrier in x.

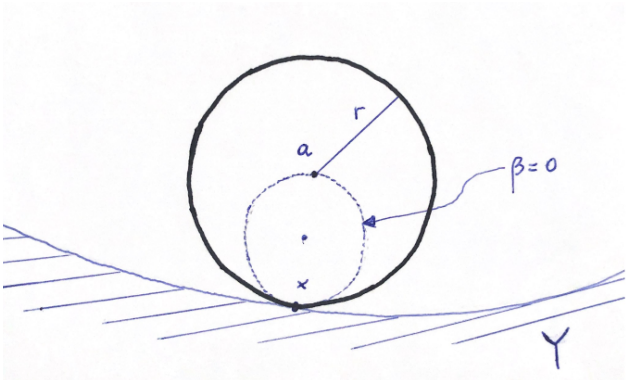

Figure 1: Situation of example 1.2.12

Using this, one can easily prove that

Lemma 1.2.13. *If $Y \subset\subset Z \subseteq X$ are open sets, where Y. Then there is an open set W such that $Y \subset\subset W \subset\subset Z$ and the boundary of W is smooth and regular for Dirichlet problem.*

If $Y \subset\subset X$ y $f : bY \to \mathbb{R}$ is continuous and $\mathcal{P}_f$ is the set of continuous functions $u : \overline{Y} \to \mathbb{R}$ that are subharmonic in Y and less than f in bY. $\mathcal{P}_f$ is nonempty because it contains constant functions and is clearly a Perron family. By Perron method,

$$B_f = \sup_{u \in \mathcal{P}_f} u$$

is harmonic in Y, but we have to check if it extends continuously to the boundary.

Proposition 1.2.14. *If x is a regular point in bY for Dirichlet problem, where $Y \subset\subset X$, and $f : bY \to \mathbb{R}$ is continuous,*

$$\lim_{y \to x} B_f(y) = f(x).$$

Proof. It can be found in p. 183 of [For81]. □

So we see that if Y has regular boundary, one can always find a solution for Dirichlet problem. We now turn to our main objective, which is to find Green functions in Riemann surfaces.

Definition 1.2.15. *Let X be a Riemann surface and let $Y \subseteq X$ be an open subset. A Green function in Y with singularity in x is a function $G_x : \overline{Y} \to [-\infty, 0]$ that is continuous in $\overline{Y} \setminus \{x\}$ and verifies:*

(G1) G_x is subharmonic in Y and harmonic in $Y \setminus \{x\}$.

(G2) If (U, z) are coordinates around x such that $z(x) = 0$, $G_x - \log|z|^2$ es is harmonic in U.

(G3) If H is another continuous function satisfying G1 and G2, then $G_x \geq H$.

Last property guarantees that G_x is unique, so we can refer to it as **the** Green function in Y with singularity in x. Not all Riemann surfaces have Green functions, but the ones that we will need to do approximation do:

Theorem 1.2.16. *If X is a Riemann surface and $Y \subset\subset X$ is an open set, with regular boundary, then Y has a Green function with singularity in all of its points.*

Proof. We use Perron method. Let $\mathcal{G}$ be the set of restrictions to $Y \setminus \{x\}$ of functions $u : \overline{Y} \setminus \{x\} \to [0, \infty)$ with compact support and contained in Y, that are subharmonic in $Y \setminus \{x\}$ and such that $u(z) + \log|z|^2$ can be extended in a subharmonic way to x whenever z are coordinates around p with $z(x) = 0$.
It is clear that $\mathcal{G}$ is nonempty because

$$g_0(z) = \begin{cases} -\log|z|^2 & \text{if } |z| < 1 \\ 0 & \text{in other case} \end{cases}$$

for a chart (U, z) around x where $z(U) = D(0, 2)$, which will remain fixed for the rest of the proof, belongs to $\mathcal{G}$.
The conditions of a Perron family are automatically satisfied. It is enough to take into account that if $u + \log|z|^2$ and $v + \log|z|^2$ extend to 0 in a subharmonic way, its maximum, which is

$$\max\{u + \log|z|^2, v + \log|z|^2\} = \max\{u, v\} + \log|z|^2,$$

can also be extended and that, if K is a compact contained in $Y \setminus \{x\}$, x and bY are closed and disjoint from K, so $u\sharp v$ is still in $\mathcal{G}$.
Let $0 < r < 1$ be fixed, and consider the harmonic function ω_r in $\overline{Y} \setminus z^{-1}(D(0, r))$ which is 0 in bY and 1 if $|z| = r$, whose existence is granted by the solution to Dirichlet theorem. For any $u \in \mathcal{G}$, define also

$$a_r = \sup_{|z|=1} \omega_r(z), \qquad b_r = \sup_{|z|=r} u(z).$$

Then, $u - b_r\omega_r$ is subharmonic, 0 in bY and is less than $b_r - b_r = 0$ in $|z| = r$. By the maximum principle, $u - b_r\omega_r \leq 0$ if $|z| = 1$. Together with the fact that $u + \log z^2$ is subharmonic in $D(0, 2)$, we get that

$$b_r + \log r^2 = \max_{|z|=r}(u + \log|z|^2) \leq \max_{|z|=1}(u + \log|z|^2) = \max_{|z|=1} u \leq \max_{|z|=1} b_r\omega_r = b_r a_r,$$

so $b_r \leq \frac{-\log r^2}{1-a_r}$. From here it follows, by the maximum principle again, that since u has compact support,

$$\sup_{\overline{Y} \setminus z^{-1}(D(0,r))} u = \sup_{|z|=r} u = b_r \leq \frac{-\log r^2}{1 - a_r} < +\infty, \tag{1}$$

$$\sup_{|z| \leq r} u + \log|z|^2 \leq b_r + \log r^2 \leq \frac{-\log r^2}{1 - a_r} + \log r^2 = \frac{-a_r \log r^2}{1 - a_r} < +\infty. \tag{2}$$

But these bounds do not depend on u, so from (1) one deduces that

$$U = \sup_{u \in \mathcal{G}} u$$

is harmonic in $Y \setminus \{x\}$ by Perron method. Let $G_x = -U$. By (2), $U(z) + \log|z|^2 = \sup_{u \in \mathcal{G}}(u + \log|z|^2)$ is harmonic since it is the supremum of a bounded Perron family, so $G_x - \log|z|^2$ is harmonic. If H satisfies (G1) and (G2), and $u \in \mathcal{G}$, then $u + H$ is subharmonic in Y (it is subharmonic in x because $u + H = u + \log|z|^2 + H - \log|z|^2$). Since H is negative and u has compact support, $u + H \leq 0$ in Y. Taking the supremum among all u, $U + H \leq 0$. In other words, $G_x \geq H$. □

We will denote the Green function in Y with pole in x by $G_X(x, \cdot)$.

Example 1.2.17. *The Green function for the unit disk is* $G(x, y) = \log\left|\frac{1-\bar{y}x}{x-y}\right|^2$*, since it clearly verifies (G1) y (G2), and since it is* 0 *in the boundary, (G3) follows from the maximum principle.*
However, the complex plane does not have a Green function in 0*, since it would have the form* $G(0, y) = H(y) + \log|y|^2$ *for some harmonic harmonic* H*. If* f *is holomorphic with* $\Re f = H$*, since* $G \leq 0$ *we would have*

$$\left|e^{f(y)}\right| = e^{\Re f(y)} \leq e^{\log\frac{1}{|y|^2}} \leq \frac{1}{|y|^2},$$

so e^f *is bounded, But by Liouville Theorem, it is constant and therefore so is* f *and as a consequence,* H*, but* $\log|y|^2 \to \infty$ *as* $y \to \infty$ *so* G *cannot be non-negative.*

Theorem 1.2.18. *[Properties of Green functions] Let X be a Riemann surface and $Y \subset\subset X$ an open subset with smooth and regular boundary. Then:*

a) $G_Y(x,y) = 0$ *for all* $y \in bY$.

b) $G_Y(x,y) = G_Y(y,x)$.

c) For all p there is a neighbourhood U of (p,p) in $X \times X$ and holomorphic coordinates (x,y) in U such that $x(p) = y(p) = 0$ and

$$G_Y(x,y) = H(x,y) + \log|x-y|^2$$

for some continuous function H, harmonic in each variable.

Proof. Going through the proof of 1.2.16, one can see that G_Y is the supremum of some functions that are 0 in bY, and this proves a).
b) Let Z be a bordered, compact Riemann surface u, v smooth functions in Z. By Stokes theorem, we have the first Green identities:

$$\int_Z v\partial\bar{\partial}u + \int_Z \partial v \wedge \bar{\partial}u = \int_{bZ} v\bar{\partial}u \qquad \int_Z u\bar{\partial}\partial v + \int_Z \bar{\partial}u \wedge \partial v = \int_{bZ} u\partial v$$

and adding them we get the second Green identity:

$$\int_Z v\partial\bar{\partial}u - u\partial\bar{\partial}v = \int_{bZ} v\bar{\partial}u + u\partial v$$

Now let ξ_1, ξ_2 different points in Y. Let z_1, z_2 two charts around them, with $z_j(\xi_j) = 0$. Let D_j be the disk defined by $|z_j| = \varepsilon$ and assume they don't intersect. Define $g_j(z) = G_Y(\xi_j, z)$. Applying the second Green identity $Z = Y \setminus (D_1 \cup D_2)$ to g_1, g_2. They are harmonic and by a) they vanish in the boundary of Y, so

$$0 = \int_{bD_1 \cup bD_2} g_1\bar{\partial}g_2 + g_2\partial g_1$$

Now we write $g_1 = G_1 + \log|z_1|^2$ where G_1 is harmonic, and so

$$\int_{bD_1} g_1\bar{\partial}g_2 + g_2\partial g_1 = \int_{bD_1} G_1\bar{\partial}g_2 + g_2\partial G_1 + \log|z_1|^2\partial g_2 + \int_{bD_1} g_2\partial\log|z_1|^2.$$

Since $|z_1| = \varepsilon$ is constant along the path, by Stokes Theorem the first integral is 0, since g_2 and G_1 are harmonic,

$$d(G_1\bar{\partial}g_2 + g_2\partial G_1 + \log\varepsilon^2\partial g_2) = \partial G_1 \wedge \bar{\partial}g_2 + \bar{\partial}g_2 \wedge g_1 + G_1\partial\bar{\partial}g_2 + g_2\bar{\partial}\partial G_1 + \log\varepsilon^2\bar{\partial}\partial g_2 = 0$$

while the second integral is

$$\int_{bD_1} g_2\partial\log|z_1|^2 = \int_{|z_1|=\varepsilon} \frac{g_2(z_1)}{z_1}dz_1 = i\int_0^{2\pi} g(\xi_1 + \varepsilon e^{it}) = 2i\pi g_2(\xi_1)$$

Similarly,

$$\int_{bD_2} g_1\bar{\partial}g_2 + g_2\partial g_1 = -2i\pi g_1(\xi_2)$$

and therefore, $g_1(\xi_2) - g_2(\xi_1) = 0$, proving c)
c) Take a chart (U, z), and for each $q \in U$, let z_q be defined as $z_q(r) = z(r) - z(q)$. Then by property (G2), for each q there is a harmonic function H^q such that

$$G_Y(q,r) = H^q(z_q(r)) + \log|z_q(r)|^2 = H^q(z_q(r)) + \log|z(r) - z(q)|^2$$

so $H^q(z_q(r)) = G_Y(q,r) - \log|z(r) - z(q)|^2 = H^r(z_r(q))$ by the symmetry of Green function, so H^q is harmonic in q and in particular continuous. □

By an abuse of notation, we will denote

$$\int_X \omega = \lim_{t\to 0}\int_{X\smallsetminus B(t)} \omega$$

when ω is a compactly supported form with a singularity in x, where $B(t)$ family of neighbourhoods of x that decreases uniformly to x. Generally we will take $B(t) = z^{-1}(D(z(x),t))$ where z is some x. We will often say that the integral is in an improper sense.
We can now prove the Cauchy-Green formulas:

Proposition 1.2.19 (Cauchy-Green formulas)**.** *Let $Y \subset\subset Z \subset\subset X$ be open subsets of a Riemann surface where Y has smooth, regular border X, and Z has regular border. If $f : \overline{Y} \to \mathbb{C}$ is smooth, then*

$$f(x) = \frac{1}{2\pi i}\int_{y\in bY} f(y)\partial_y G_Z(x,y) + \frac{1}{2\pi i}\int_{y\in Y} \partial_y G_Z(x,y)\wedge \bar\partial_y f(y),$$

where the second integral is improper. In particular, if f is holomorphic in a neighbourhood of $\overline{Y}$, then

$$f(x) = \frac{1}{2\pi i}\int_{y\in bY} f(y)\partial_y G_Z(x,y),$$

whereas if f vanishes in the boundary of Y,

$$f(x) = \frac{1}{2\pi i}\int_{y\in Y} \partial_y G_Z(x,y)\wedge \bar\partial_y f(y).$$

Proof. For a fixed x, if (z,U) s a chart around x with $z(x)=0$ and if $B(\varepsilon) = z^{-1}(B(0,\varepsilon))$, the first Green identity tells us that:

$$\int_{bM} u\partial v = \int_M \bar\partial u\wedge\partial v + u\bar\partial\partial v,$$

so if we apply it to f and $G_Z(x,\cdot)$ we obtain that

$$\int_{y\in b(Y\smallsetminus B(\varepsilon))} f(y)\partial_y G_Z(x,y) = \int_{y\in Y\smallsetminus B(\varepsilon)} \left[\bar\partial_y f(y)\wedge\partial_y G_Z(x,y) + f(y)\bar\partial_y\partial_y G_Z(x,y)\right].$$

Since G is harmonic, $\bar\partial_y\partial_y G_Z(x,y) = 0$, and after reordering,

$$\int_{y\in bY} f(y)\partial_y G_Z(x,y) + \int_{y\in Y\smallsetminus B(\varepsilon)} \partial_y G_Z(x,y)\wedge\bar\partial_y f(y) = \int_{|z|=\varepsilon} f(z)\frac{\partial G_Z(x,z)}{\partial z}dz,$$

but by property (G2), $G_Z(x,z) = \log|z|^2 + H(z)$ with H continuous. Therefore,

$$\int_{|z|=\varepsilon} f(z)\frac{\partial G_Z(x,z)}{\partial z}dz = \int_{|z|=\varepsilon}\left[\frac{f(z)}{z} + f(z)\frac{\partial H(z)}{\partial z}\right]dz = i\int_0^{2\pi} f(\varepsilon e^{it})dt + O(\varepsilon).$$

So after letting $\varepsilon\to 0$ we obtain the first formula, and the other two follow from it. □

This formula should be reminiscent of the well-known Cauchy-Pompeiu formula, which is used to solve the equation $\frac{\partial}{\partial\bar z}f = g$; following the same ideas, we can prove an analogous result, that will be improved in 1.3.10, concerning the solution of $\bar\partial f = \alpha$.

Proposition 1.2.20. *Let X be an open Riemann surface, $Y \subset\subset X$ an open subset with smooth, regular boundary, and let $\alpha\in\mathcal{E}^{0,1}(X)$. Then there exists $f\in\mathcal{C}^\infty(Y)$ such that $\bar\partial f = \alpha$.*

Proof. We can assume that α has compact support, contained in Y, because if Z is an open set with smooth boundary such that $Y\subset\subset Z\subset\subset X$, and ξ is a bump function with support in Z which is 1 in $\overline{Y}$, then a solution to $\bar\partial f = (\xi\alpha)$ in Z is a solution for $\bar\partial f = \alpha$ en Y.
The last formula in 1.2.19 suggests defining

$$f(x) = \frac{1}{2\pi i}\int_{y\in Y}\partial_y G_Y(x,y)\wedge\alpha(y),$$

but f does not necessarily have compact support even if α does[3]. S slight modification is enough. We define

$$f(x) = \frac{1}{2\pi i} \int_{y \in Y} \partial_y (G_Y(x,y)\alpha(y)),$$

in the improper sense. Since α is a $(0,1)$-form, $\bar{\partial}_y(G_Y(x,y)\alpha(y)) = 0$ so by Stokes Theorem, for fixed x_0,

$$f(x_0) = \int_{y \in Y} d_y(G_Y(x_0,y)\alpha(y)) = \int_{y \in bY} G_Y(x_0,y)\alpha(y) - \lim_{t \to 0} \int_{bB(t)} G_Y(x_0,y)\alpha(y) = -\lim_{t \to 0} \int_{bB(t)} G_Y(x_0,y)\alpha(y).$$

We take holomorphic coordinates (η, ξ) as in b) from 1.2.18. We also write $\alpha = h d\bar{\xi}$, and let $B(\varepsilon) = \eta^{-1}(D(0,\varepsilon)$ for $0 < \varepsilon < M$, and we have

$$f(x) = \lim_{\varepsilon \to 0} \frac{-1}{2\pi i} \int_{|\xi|=\varepsilon} G_Y(x_0,\xi) h(\xi) d\bar{\xi},$$

But this works for fixed x_0. However, if $|\eta(x) - \xi(x)| \leq M/2$, (η, ξ) is also a holomorphic chart for x, and so in a neighbourhood of x_0,

$$f(\eta) = \lim_{\varepsilon \to 0} \frac{-1}{2\pi i} \int_{|\eta - \xi|=\varepsilon} G_Y(\eta,\xi) h(\xi) d\bar{\xi} = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|\zeta|=\varepsilon} G_Y(\eta, \eta - \zeta) h(\eta - \zeta) d\bar{\zeta}.$$

We derive under the integral sign and undo the change of variables to get

$$\frac{\partial f}{\partial \bar{\eta}}(\eta) = \frac{-1}{2\pi i} \lim_{\varepsilon \to 0} \int_{|\xi|=\varepsilon} \left[\left(\frac{\partial G_Y}{\partial \bar{\eta}} + \frac{\partial G_Y}{\partial \bar{\xi}} \right) (\eta,\xi) h(\xi) + G_Y(\eta,\xi) \frac{\partial h}{\partial \bar{\xi}}(\xi) \right] d\bar{\xi}.$$

After reordering the terms,

$$\frac{\partial f}{\partial \bar{\eta}}(0) = \frac{-1}{2\pi i} \lim_{\varepsilon \to 0} \int_{|\xi|=\varepsilon} \left[\frac{\partial G_Y}{\partial \bar{\eta}}(0,\xi) h(\xi) + \frac{\partial}{\partial \bar{\xi}} (G_Y(0,\xi) h(\xi)) \right] d\bar{\xi},$$

and finally, using that $G_Y(x,y) = H(x,y) - \log|x-y|^2$ where H is continuous,

$$\frac{\partial G_Y}{\partial \bar{\eta}}(0,\xi) = \frac{\partial H}{\partial \bar{\eta}}(0,\xi) + \frac{1}{\bar{\eta}} \quad \text{y} \quad G_Y(0,\xi) = H(0,\xi) - \log \varepsilon^2.$$

Substitution of this in the previous equation yields

$$\int_{|\xi|=\varepsilon} \left[\frac{\partial G_Y}{\partial \bar{\eta}}(0,\xi) h(\xi) + \frac{\partial}{\partial \bar{\xi}} (G_Y(0,\xi) h(\xi)) \right] d\bar{\xi} = \int_{|\xi|=\varepsilon} \frac{h(\xi) d\bar{\xi}}{\bar{\xi}} + O(\varepsilon) = -i \int_0^{2\pi} h(\varepsilon e^{it}) dt + O(\varepsilon),$$

so $\frac{\partial f}{\partial \bar{\eta}}(0) = h(0)$, proving $\bar{\partial}_x f(x_0) = \alpha(x_0)$, and since x_0 was arbitrary, that $\bar{\partial} f = \alpha$. □

[3] for instance, if $g : \mathbb{C} \to \mathbb{C}$ is formed by gluing $\frac{1}{z}$ and $\bar{z}$ with bump functions, $\bar{\partial} g$ has compact support, and all functions h such that $\bar{\partial} h = \bar{\partial} g$ are of the form $g + f$ with f holomorphic in $\mathbb{C}$, but there is no f giving a compactly supported function

1.3 The theorems of Runge and Weierstrass

The classical Runge theorem in the complex plane says the following.

Theorem 1.3.1 (Runge theorem in $\mathbb{C}$)**.** *If $K \subset \mathbb{C}$ is a compact whose complement is connected, Ω is an open subset containing K and $f : \Omega \to \mathbb{C}$ is a holomorphic function, then there is a sequence of polynomials that converge to f uniformly in K.*

Its proof can be found in page 342 from [Gam01] or in page 270 from [Rud86]. In order to extend this result to arbitrary Riemann surfaces, we need to change the assumption on K, and have a notion of what it means "not to have holes". If $Y \subseteq X$, $h(Y)$ a la is the union of Y with all the connected components of $X \setminus Y$ whose closure (in X) is compact. We say that Y is Runge (en X) if $h(Y) = Y$.

Example 1.3.2. *The definition depends strongly in the ambient space X, so if it is not clear by the context we will use $h_X(Y)$. For instance, if $Y = \{z \in \mathbb{C} : |z| = 1\}$, $h_{\mathbb{C}}(Y)$ is the closed unit disk, although Y is Runge en $\mathbb{C} \setminus \{0\}$.*

Before heading to Runge theorem, we start by looking at some properties of the operator h.

Proposition 1.3.3. *Let X be a Riemann surface, Y, Z subsets of X.*

a) $h(h(Y)) = h(Y)$ and $h(Y) \subset h(Z)$ if $Y \subset Z$.

b) If $Y \subset Z \subset X$, where Z is open in X and $h_X(Y) = Y$ then $h_Z(Y) = Y$.

c) If Y is compact, $h(Y)$ is compact.

d) If $Y \subset\subset X$ is an open set with smooth boundary, Y is Runge if and only if $\overline{Y}$ is.

e) If $Y \subset X$ is an open, Runge subset, all of its connected components are also Runge.

Proof. It can be found along pages 187-189 of [For81] □

We can also characterise Runge subsets using algebraic topology. This will be more useful in the second part, but it clarifies why Runge means "no holes"

Proposition 1.3.4. *An open $Y \subset X$ is Runge if and only if the natural map $H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is injective.*

Proof. By Poincaré duality it is enough to check when the natural map $H^1_c(Y, \mathbb{Z}) \to H^1_c(X, \mathbb{Z})$ is injective. However, there is a long exact sequence, whose proof can be found in [Bre97]

$$H^0_c(X, \mathbb{Z}) \to H^0_c(X \setminus Y, \mathbb{Z}) \to H^1_c(Y, \mathbb{Z}) \to H^1_c(X, \mathbb{Z})$$

But since X is non compact, $H^0_c(X, \mathbb{Z}) = 0$ so the map is injective if and only if $H^0_c(X \setminus Y, \mathbb{Z}) = 0$, which is again equivalent to the definition of Y being Runge in X. □

We want to use Runge compacts to do induction in a Riemann surface. IN order to do so, we will need to use exhaustions by compact, Runge sets:

Theorem 1.3.5. *[Existence of good exhaustions from outside] If $Y \subset\subset X$ is open, there is a sequence of Runge open sets such that Y_j*

a) $Y \subset Y_k \subset\subset Y_{k+1}$

b) Y_k has smooth, regular boundary for all $k \geq 0$.

c) Any compact in X is contained in some Y_k.

d) If Y is Runge and has smooth, regular boundary, we can assume $Y_0 = Y$

Proof. This is in p. 189 of [For81]. □

Proposition 1.3.6 (Existence of good exhaustions from inside)**.** *Let $Y \subset\subset X$ be a Runge open set. Then there are open sets with smooth, regular boundary W_k such that* *a)* $W_k \subset\subset W_{k+1}$.

b) $\overline{W}_k$ *is Runge in* X.

c) *Any compact set in* Y *is contained in some* W_k.

Proof. This can be easily proven using Lemma 23.7 in p. 188 of [For81]. □

Examples of such exhaustions can be seen in Figure 2

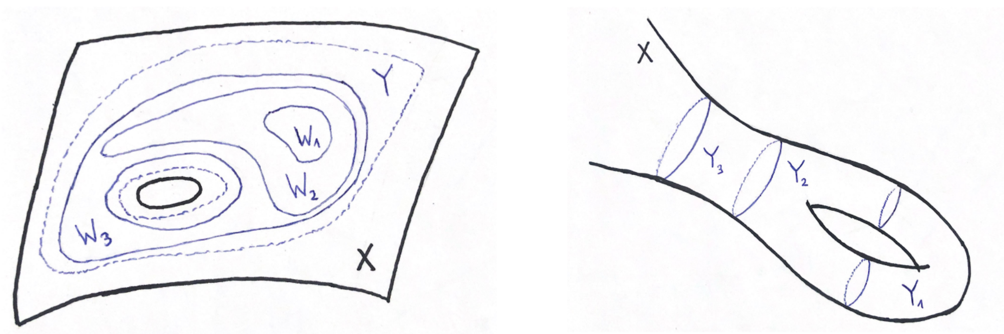

Figure 2: Exhaustions by Runge, compact sets

The proof of the theorems of Runge and Mergelyan use tools from functional analysis that can be found in the appendix. If K is a compact in a Riemann surface, the space of all continuous functions $f : K \to \mathbb{C}$, that we will denote by $\mathcal{C}(K)$ is a Banach space with the sup norm:

$$\|f\|_K = \sup_{z \in K} |f(z)|.$$

If $\int_K g d\mu = 0$ for all $g \in \mathcal{F}$, we will say that μ is orthogonal to $\mathcal{F}$. Combining Riesz representation theorem A.2 and Hahn-Banach theorem A.1, we see that if $\mathcal{F} \subset \mathcal{C}(K)$ is a family of functions, $f \in \mathcal{C}(K)$ belongs to the closure of $\mathcal{F}$ (in other works, there exists a sequence of functions f_n of $\mathcal{F}$ converging uniformly to f) if and only if given a Borel measure μ in K orthogonal to $\mathcal{F}$, $\int_K f d\mu = 0$.
It is convenient to define the following subspaces of $\mathcal{C}(K)$: If Y is an open subset containing K, $\mathcal{O}(Y)_{|K}$ are the restrictions to K of the holomorphic functions in Y, and $\mathcal{O}(K)$ denotes the set of all functions that are holomorphic in some neighbourhood of K; in other words,

$$\mathcal{O}(K) = \bigcup_{K \subset Y} \mathcal{O}(Y)_{|K}.$$

We start proving a relative version or Runge theorem, from which the Runge theorems immediately follow.

Proposition 1.3.7. *Let* X *be a Riemann surface and let* $K \subset Y \subset\subset Z \subset\subset X$*, where* K *is a Runge compact in* X *and* Y*,* Z *are open sets. Let* $f : Y \to \mathbb{C}$ *be a holomorphic function. Then* f *can be approximated uniformly in* K *by holomorphic functions defined in the whole* Z*.*

Proof. In light of the previous discussion, we take a complex Borel measure μ with support K such that $\int_K g d\mu = 0$ for all $g \in \mathcal{O}(Z)_{|K}$. Using 1.2.20, we define a linear functional S assigning to each $(0,1)$-form $\alpha \in \mathcal{E}^{0,1}(Z)$ with compact support the number

$$S(\alpha) = \int_Y f d\mu, \text{ where } \bar{\partial} f = \alpha \text{ en } Z$$

S is well defined precisely because μ is orthogonal to the holomorphic functions in Z. It is adequate to use

$$\sigma(y) = \int_Y G_Z(x,y) d\mu(x)$$

(which is well defined $\log |z|^2$ is locally integrable). By the properties of Green functions 1.2.18 (in particular, its symmetry and behaviour in the boundary), σ is harmonic in $Z \setminus K$, and 0 in bZ. Therefore, σ is also 0

in all the non-compact connected components of $Z \setminus K$; this is, σ is 0 en $Z \setminus h_Z(K) = Z \setminus K$, since K is Runge.
If f is holomorphic in Y, and h is a smooth extension of f to Y that vanishes outside Z, $\bar{\partial} h = 0$ in K. Therefore, by Fubini theorem and the explicit solution to $\bar{\partial} f = \alpha$ that we obtained 1.2.20, $S(\theta) = \int_Z \partial(\sigma\theta)$, and therefore

$$\int_K f d\mu = \int_Y h d\mu = S(\bar{\partial} h) = \int_Z \partial(\sigma \bar{\partial} h) = \int_Z \partial(0) = 0.$$

In other words, f can be approximated uniformly in K by holomorphic functions in Z. □

Theorem 1.3.8 (Runge theorem for compact sets)**.** *If X is a Riemann surface and K is a compact, Runge subset, then every holomorphic function in a neighbourhood of K can be approximated uniformly in K by holomorphic functions defined in X.*

Proof. Let $f : Y \to \mathbb{C}$ holomorphic, with $K \subset Y$. Since K is Runge, we can assume, after restricting f, that Y is also Runge. We take an exhaustion of Y by compact Runge subsets Y_k. We apply the relative version of Runge theorem 1.3.7 successive times. Let $\varepsilon > 0$, and we start with a function f_1 holomorphic in Y_1 such that

$$\|f - f_1\|_K \leq \frac{\varepsilon}{2}.$$

We get, inductively, holomorphic functions $f_n : Y_n \to \mathbb{C}$ such that

$$\|f_n - f_{n-1}\|_{\overline{Y}_{n-2}} \leq \frac{\varepsilon}{2^n}.$$

The sequence f_n is then uniformly convergent to some F, which will be holomorphic in X and $\|f - F\|_K \leq \varepsilon$. □

Theorem 1.3.9 (Runge theorem for open sets)**.** *Let $Y \subset X$ be a Runge open subset. Then every holomorphic function in Y can be approximated by holomorphic functions in X, uniformly over the compact subsets of Y.*

Proof. Let W_n be an exhaustion of Y by compact Runge subsets of X as in 1.3.6, and let f be holomorphic in Y. For each n, we take $f_n : X \to \mathbb{C}$ holomorphic and such that

$$\|f_n - f\|_{W_n} \leq \frac{1}{n}.$$

Then the sequence f_n converges to f uniformly over the compact subsets of Y, since every compact of Y is contained in some W_n. □

Now we can prove the non-relative version of 1.2.20:

Proposition 1.3.10. *Let X be an open Riemann surface, $\alpha \in \mathcal{E}^{0,1}(X)$. Then there exists a smooth function f such that $\bar{\partial} f = \alpha$.*

Proof. Take an exhaustion of X by open Runge subsets $Y_0 \subset Y_1 \subset \dots$. Let $f_1 \in \mathcal{C}^\infty(Y_1)$ be a solution in Y_0 of $\bar{\partial} f_1 = \alpha$, which exists due to 1.2.20. We construct in a recursive way functions $f_k \in \mathcal{C}^\infty(Y_k)$ such that

$$\bar{\partial} f_k = \alpha \text{ in } Y_{k-1} \quad \text{y} \quad \|f_k - f_{k-1}\|_{Y_{k-1}} \leq \frac{1}{2^k}.$$

To construct f_{k+1}, we start with a solution u_{k+1} of $\bar{\partial} u_{k+1} = \alpha$ in Y_k. Since $u_{k+1} - f_k$ is holomorphic in Y_{k-1}, we can find a function g_k holomorphic in Y_k as close to is as we wish, and define $f_{k+1} = u_{k+1} - g_k$. It is clear that f_{k+1} meets the requirements. Now, since $f_{k+1} - f_k$ is holomorphic in Y_k and is bounded by $\frac{1}{2^k}$, the series

$$\sum_{k=n}^{\infty} (f_{k+1} - f_k)$$

converges to a holomorphic h_n, and

$$h_n + f_n = \lim_{k \to \infty} f_k$$

uniformly on compact subsets of Y_k. Therefore $\{f_n\}$ converges in X to a smooth function f, and $\bar{\partial} f = \alpha$ in Y_k for all k. □

This is the point where we will use the existence of good coverings. The following theorem can be seen as a generalisation of the classical factorisation theorem.

Theorem 1.3.11 (Weierstrass theorem)**.** *Let X be an open Riemann surface and $m : X \to \mathbb{Z}$ a function that is supported in a discrete set. Then there is a meromorphic function s defined in X such that $ord_p(s) = m(p)$ for all p.*

Proof. We cover X by charts (U_j, z_j). Using the existence of good coverings as in 1.1.14, after refining the covering we can assume that all U_j and $U_j \cap U_k$ are simply connected and relatively compact. Let A be the support of m. Since U_j is relatively compact, $A \cap U_j$ is finite, and using quotients of polynomials we obtain functions, $g_j : U_j \to \mathbb{C}$ such that $\mathrm{ord}_p(g_j) = m(p)$ for all $p \in U_j$. The functions $\frac{g_j}{g_k}$ do not vanish in $U_j \cap U_k$, and since $U_j \cap U_k$ is simply connected, there exist holomorphic functions p_{jk} such that

$$e^{p_{jk}} = \frac{g_j}{g_k} \text{ en } U_j \cap U_k.$$

Let ξ_k be a partition of unity subordinate to the covering $\{U_k\}$, and we consider the smooth functions $\widehat{f}_j = \sum_k \xi_k p_{jk}$. Since $p_{jk} - p_{kl} + p_{lj} = p_{kj} - p_{jk} = 0$ for any indices,

$$\widehat{f}_j - \widehat{f}_k = \sum_l \xi_l (p_{jl} - p_{kl}) = \sum_l \xi_l p_{jk} = p_{jk} \sum_l \xi_l = p_{jk}$$

in $U_j \cap U_k$ and in particular, $\bar{\partial}\widehat{f}_j = \bar{\partial}\widehat{f}_k$ and therefore $\bar{\partial}\widehat{f}_j$ ranging over all j defines a $(0,1)$-form α in X. If f is such that $\bar{\partial} f = \alpha$, and we define

$$f_j = \widehat{f}_j - f,$$

then it clear that f_j is holomorphic, but still satisfies $f_j - f_k = p_{jk}$. Then, the functions

$$h_j = \frac{g_j}{e^{f_j}}$$

are holomorphic in U_j and agree in the overlaps, so they define a meromorphic function h in X such that $\mathrm{ord}_p(h) = m(p)$. □

Corollary 1.3.12. *If X is an open Riemann surface, there is a holomorphic 1-form that does not vanish in any point.*

Proof. Let f be a non-constant holomorphic function X (to show that such an f exists, it suffices to apply Runge theorem to a chart defined in a Runge open set),and let $\alpha = df$. Since α is not 0, by the identity principle 1.2.3, its zeros are isolated. BY Weierstrass theorem there is $s \in \mathcal{M}(X)$ such that $\mathrm{ord}_p(s) = -\mathrm{ord}_p(\alpha)$ for all p and so $s\alpha$ is the desired form. □

The theorems of Runge and Weierstrass prove that in an open Riemann surface X there are plenty of holomorphic functions. In particular, that:

- If p, q are distinct points then there is some $f \in \mathcal{O}(X)$ such that $f(p) = f(q)$.
- If $K \subset X$ is compact, its holomorphic envelope

  $$\widehat{K} = \{p \in X : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{O}(X)\}$$

  is compact.

The first one is immediate and the second one follows after proving that $\widehat{K} = h(K)$ and recalling the properties of h.
A complex manifold of any dimension with these two properties is a Stein manifold. It can be seen in [GR65] that any Stein manifold of dimension n can be embedded in $\mathbb{C}^{2n+1}$ and in particular, Riemann surfaces can be embedded in $\mathbb{C}^3$.

1.4 Mergelyan Theorem

In Runge theorem for compact sets the hypotheses are slightly stronger than necessary, as the following example shows:

Example 1.4.1. *The series*

$$\sum_{n=0}^{\infty} \frac{z^{n!}}{n!}$$

converges uniformly if $|z| \leq 1$*, because* $\sum_n \left|\frac{z^{n!}}{n!}\right| \leq \sum_n \frac{1}{n!} < \infty$ *and therefore it defines a function in* $\overline{D(0,1)}$ *as the limit of entire functions. However, the radius of convergence of this series is 1, so it does not have any analytic extension to any open subset containing the closed disk.*

In general, if f_n is a sequence of holomorphic functions in X that converge uniformly on a compact K to f, it follows that f is continuous in K and holomorphic in its interior. Mergelyan theorem proves that these conditions are also sufficient for f to be approximated by holomorphic functions in X. There are very few proofs of Mergelyan theorem for Riemann surfaces. We show here how its proof can be reduced to the proof of a lemma, known as Bishop localisation lemma. We start with Mergelyan theorem in the complex plane:

Theorem 1.4.2. *If* $f : K \to \mathbb{C}$ *is continuous and holomorphic in the interior of* K*,* K *is compact and* $\mathbb{C} \setminus K$ *is connected, then* f *can be approximated uniformly on* K *by polynomials.*

Proof. It can be found in chapter 20 of [Rud86]. □

After seeing Runge theorem, it should not be surprising that its generalisation to Riemann surfaces is the following:

Theorem 1.4.3 (Mergelyan-Bishop theorem)**.** *Let* X *be a Riemann surface,* $K \subset X$ *a compact set such that* $h(K) = K$ *and let* $f : K \to X$ *be a continuous function in* K*, holomorphic in its interior. Then there is a sequence of holomorphic functions in* X *that approximate* f *uniformly in* K*.*

From Mergelyan theorem in the complex plane one can derive a local Mergelyan theorem in a Riemann surface: If (V, z) is a chart such that $z(V) = D(0, 2)$, then the complement in $\mathbb{C}$ of $z(K) \cap \overline{D(0,1)}$ is connected, so we can apply Mergelyan theorem: if $U = z^{-1}(D(0, 1)$, f can be approximated in $K \cap \overline{U}$ by holomorphic functions in a neighbourhood of such set. In view of Runge theorem 1.3.8, it is sufficient to prove the following

Proposition 1.4.4 (Bishop localisation lemma)**.** *If* K *is a compact subset of* X*, covered by a finite number of charts* $U_1, \ldots, U_k$ *such that for all* j*,* f *can be approximated in* $\overline{U}_j \cap K$ *by holomorphic functions in a neighbourhood of* $\overline{U}_j \cap K$*, then* f *can be approximated in* K *by holomorphic functions in a neighbourhood of* K*.*

We will explain the proof in [Sak72] of this lemma. Using the same ideas from functional analysis that we used to prove Runge theorem (A.1 and A.2), Bishop localisation lemma follows from the following:

Proposition 1.4.5. *If* K *is a compact in* X*, covered by a finite number of charts* $U_1, \ldots, U_k$ *and* μ *is a Borel measure in* K *orthogonal to* $\mathcal{O}(K)$*, then there are Borel measures* μ_j *in* $\overline{U}_j$ *for all* j *that are orthogonal to* $\mathcal{O}(K \cap \overline{U}_j)$ *and such that* $\mu = \mu_1 + \ldots + \mu_k$*.*

To carry out the proof of this statement, we will use elementary differentials, whose existence was proven in [BS47] by Behnke and Stein. They are very related to Green functions: If $K \subset Y$ where Y is a relatively compact open set that we will keep fixed, having Green function G, and $\omega(x, y) = \partial_y G(x, y)$, then the Cauchy equations 1.2.19 read

$$f(x) = \frac{1}{2\pi i} \int_{y \in bZ} f(y)\omega(x, y) + \frac{1}{2\pi i} \int_{y \in Z} \omega(x, y) \wedge \bar{\partial} f(y) \tag{1}$$

In particular, with $f = 1$, $\int_{bW} \omega(x, y) = 2\pi i$, so $\omega(x, y)$ has residue 1. However, ω is not holomorphic in x, it is only harmonic.

An elementary differential is a holomorphic form $\omega \in \Omega^1(Y \times Y \setminus D_Y)$, where D_Y is the diagonal, such that for fixed x, is a meromorphic form in y, with a unique pole in $y = x$ and residue 1, and for fixed y, is meromorphic in x with a unique pole in $x = y$. If (U, z) are some coordinates, we can write

$$\omega(x, z) = k(x, z)dz$$

where k is meromorphic and has residue 1 and therefore, for fixed x, if $z(x) = 0$, $k(x, z) = \frac{1}{z} + g(z)$, and so

$$\int_U |k(x, y)||dz \wedge d\bar{z}| \leq \int_U |g(z)||dz \wedge d\bar{z}| + \int_U \frac{|dz \wedge d\bar{z}|}{|z|} < \infty.$$

Elementary differentials still verify the Cauchy formula in (1)We will also use the following formula: If (U, z) are some coordinates and p, q are different,

$$\begin{gathered}
\frac{1}{2\pi i} \int_{x \in Y} k(x, p)\bar{\partial} h(x) \wedge \omega(q, x) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{x \in Y \setminus D(p,\varepsilon) \cup D(q,\varepsilon)} k(x, p)\bar{\partial} h(x) \wedge \omega(q, x) = \\
= -\lim_{\varepsilon \to 0} \left(\frac{1}{2\pi i} \int_{|x-p|=\varepsilon} k(x, p)h(x) \wedge \omega(q, x) + \frac{1}{2\pi i} \int_{|x-q|=\varepsilon} k(x, p)h(x) \wedge \omega(q, x) \right) = \\
= [h(p) - h(q)]k(q, p).
\end{gathered} \tag{2}$$

If μ is a compactly supported measure in Y, fixed, we define the $(1, 0)$-form

$$T\mu(y) = \int_{x \in Y} \omega(x, y)d\mu(x).$$

In local coordinates,

$$T\mu(z) = \left(\int_{x \in Y} k(x, z)d\mu(x) \right) dz.$$

Since $x \mapsto \int_{z \in U} |k(x, z)||dz \wedge d\bar{z}|$ is continuous (because it is bounded as was seen before), $\omega(x, \cdot)$, and μ have compact support, so by Fubini theorem,

$$\int_{(z,x) \in U \times Y} |k(x, y)|d|\mu|(x)|dz \wedge d\bar{z}| = \int_{x \in Y} \left(\int_{z \in U} |k(x, z)||dz \wedge d\bar{z}| \right) d|\mu|(x) < \infty,$$

and in particular by Fubini theorem again, except on a measure 0 set A (with respect to $|dz \wedge d\bar{z}|$),

$$\int_{x \in Y} |k(x, z)|d|\mu|(x) < \infty, \tag{3}$$

so $T\mu(z)$ is defined.

Lemma 1.4.6. *If $T\mu(y) = 0$ for almost all y in Y, then $\mu = 0$.*

Proof. By Cauchy formula (1), if g is smooth and has has compact support contained in some $Z \subset\subset Y$,

$$\int_{x \in Y} g(x)d\mu(x) = \int_{x \in Y} \int_{y \in Z} \omega(x, y) \wedge \bar{\partial}_y g(y)d\mu(x) = \int_{y \in Z} T\mu(y) \wedge \bar{\partial}_y g(y) = 0.$$

Since these functions are dense in the space of compactly supported continuous maps, $\mu = 0$. ☐

Lemma 1.4.7. *If μ has compact support contained in K, then μ is orthogonal to $\mathcal{O}(K)$ if and only if $T\mu(y) = 0$ for all $y \in Y \setminus K$.*

Proof. If μ is orthogonal to $\mathcal{O}(K)$ and $y \notin K$, $\omega(x, y)$ is holomorphic in x, on a neighbourhood of K not containing y, so $T\mu(y) = 0$.
If $T\mu(y) = 0$ for all $y \in X \setminus K$. Then by Cauchy formula (1), if f is holomorphic in some $Z \subset\subset Y$ gives

$$f(x) = \frac{1}{2\pi i} \int_{y \in bZ} \omega(x, y)f(y).$$

So, if $f \in \mathcal{O}(K)$, we can take $bZ \subset Y \setminus K$ and so by Fubini theorem,

$$\int_{x \in Y} f(x)d\mu(x) = \int_{x \in X} \int_{y \in bY} f(y)\partial_y G(x, y)d\mu(y) = \int_{y \in bY} f(y)T\mu(y) = 0.$$

☐

If h is continuous, for any $y \in Y \setminus A$, where (3) happens, if $z = z(y)$

$$\int_{x\in Y} |k(x,z)| d|h\mu|(x) = \int_{x\in Y} |k(x,z)||h| d|\mu|(x) \leq \|h\|_K \int_{x\in Y} |k(x,z)| d|\mu|(x) < \infty,$$

so $T(h\mu)(y)$ is also defined.

Lemma 1.4.8. *If U is a coordinate, relatively compact domain, h is smooth with compact support contained in U and μ is a compactly supported measure in U, there is a compactly supported measure μ_1 in U such that*

$$hT\mu(y) = T\mu_1(y)$$

for almost all y

Proof. Let ν be defined by $d\nu = \bar{\partial} h \wedge T\mu$, which has compact support in U. By equation (2),

$$T\nu(y) = \int_{x\in Y}\int_{z\in Y\setminus A} \omega(x,y)\bar{\partial}h(x) \wedge \omega(z,x) d\mu(z) = \int_{z\in Z\setminus A}\left[\int_{x\in Y} k(x,y)\bar{\partial}h(x)\wedge\omega(z,x)\right] d\mu(z) dy =$$

$$= 2\pi i \int_{z\in Y\setminus A} [h(y) - h(z)] k(y,z) d\mu(z) dy = 2\pi i h(y) T\mu(y) - 2\pi i T(h\mu)(y).$$

So $\mu_1 = h\mu + \frac{\nu}{2\pi i}$ is the desired measure. □

Finally we can prove 1.4.5:

Proof of 1.4.5. Let h_j be a partition of unity subordinate to the U_j. By 1.4.8, there are measures μ_j, supported in U_j, such that $T\mu_j = h_j T\mu$. Since h_j is 0 outside U_j, and $T\mu$ also vanishes outside K, we have $T\mu_j(y) = 0$ if $y \notin K \cap \overline{U}_j$, so, by 1.4.7 μ_j is orthogonal to $\mathcal{O}(\overline{U}_j \cap K)$.
Since the h_j sum up to 1, $T(\mu - \sum \mu_j) = 0$ and finally, using 1.4.6 we obtain the result. □

We will use Mergelyan theorem repeatedly in the second part of this work, as well as the following version:

Theorem 1.4.9 (Mergelyan Theorem with fixed points)**.** *Let X be a Riemann surface, $K \subset X$ a compact such that $h(K) = K$ and $f : K \to X$ a continuous function in K, holomorphic in its interior. Let A be a finite set disjoint from K and $m : A \to \mathbb{N}$ a function. Then f can be approximated uniformly in K by holomorphic functions in X having zeros of order at least $m(a)$ in each $a \in A$.*

It is clear that it is sufficient to prove the following:

Lemma 1.4.10. *If K is a compact, Runge subset of X and $x_1, \ldots, x_r \in X \setminus K$, then for each ε there is a holomorphic function h in X such that $h(x_j) = 0$ y $|h(x) - 1| < \varepsilon$.*

Proof. For any $\delta > 0$, by Mergelyan theorem 1.4.3, applied to $K \cup \{x_j\}$, and the function that is 1 in K and 0 in the x_j, there are holomorphic functions $h_j : X \to \mathbb{C}$ such that $|h_j(x) - 1| < \delta$ and $|h_j(x_j) < \delta$. If we define

$$h(x) = \prod_{j=1}^{r} (h_j(x) - h_j(x_j))$$

Then for any $\varepsilon > 0$, we can choose δ such that h verifies $|h(x)| < \varepsilon$by the continuity of a product. □

1.5 Holomorphic contact structures

Let X be a complex manifold of odd dimension $2n+1$. A contact structure in X is a holomorphic subbundle Ψ of the holomorphic tangent bundle $T^{1,0}X$ of complex codimension 1 such that for all p there is a neighbourhood U of p and a holomorphic 1-form α in U such that:

a) $\Psi_{|U} = \ker \alpha$

b) $\alpha \wedge (d\alpha)^n \neq 0$

Such a form is a said to be a contact form for $\Psi_{|U}$. The pair (X, Ψ) is a contact holomorphic manifold. If α works for all X, we will say also that (X, α) is a contact manifold and α is a contact form.
Real contact structures are the odd dimensional analogue of symplectic structures. There is a notion of holomorphic symplectic structures:
If X is a complex manifold of even dimension $2n$, a holomorphic, symplectic form is a holomorphic, closed 2-form such that $\omega^n \neq 0$. The pair (X, ω) is a holomorphic symplectic manifold.

The following are the usual examples of holomorphic symplectic and contact structures:

Example 1.5.1.

a) In $\mathbb{C}^{2n}$, with its usual (complex) coordinates $x^1, y^1, \ldots, x^n, y^n$, the standard symplectic form is

$$\omega = \sum_j dx^j \wedge dy^j.$$

Since in the expansion of ω^n all products vanish excepting the ones that involve all the coordinates, $\omega^n = n! dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n$.

b) In $\mathbb{C}^{2n+1}$, with usual coordinates $z, x^1, y^1, \ldots, x^n, y^n$, the standard contact structure is defined globally by the contact form

$$\eta = dz - \sum_j x^j dy^j.$$

Since $d\eta = -\sum_j dx^j \wedge dy^j$, using the same calculation as above $\eta \wedge (d\eta)^n = (-1)^n n! dz \wedge dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n \neq 0$.

c) If X is an n-dimensional complex manifold, its holomorphic cotangent bundle $T^*X = \mathcal{E}^{1,0}X = T^{1,0}X^*$, which is $2n$-dimensional, has a canonical symplectic form: If (U, z) are coordinates for X, they can be extended to T^*U by

$$\widetilde{z}(p, \omega) = \left(z^j(p), \omega\left(\frac{\partial}{\partial z^j}\right)\right) = (x(p,\omega), y(p,\omega))_j$$

and then, $\alpha = \sum_j y_j dx^j$, is independent of the coordinates z, is holomorphic and therefore $d\alpha = \omega$ is closed, holomorphic and $\omega^n \neq 0$. by the same calculation as in a).

d) If X is a complex manifold of dimension $n > 1$, we consider the projectivised cotangent bundle $\mathbb{P}(T^*X)$. These are equivalence classes $(p, [\theta])$ with $\theta \in T_p^{1,0}X^* \setminus \{0\}$ con $(p, [\theta_1]) = (p, [\theta_2])$. The tautological form is not invariant under this equivalence relation, but since it is homogeneous, its kernel gives rise to a subbundle Ψ of codimension 1 of $T^{1,0}\mathbb{P}(T^*X)$. If (x^j, y_j) are coordinates as in c), y k is fixed, we have coordinates for $\mathbb{P}(T^*X)$

$$(p, [y_1 dx^1 + \ldots + y_n dx^n]) \mapsto \left(z(p), \frac{y_1}{y_k}, \ldots, \frac{y_n}{y_k}\right),$$

where y^k is omitted, and so Ψ is given as the kernel of

$$\beta = dx^k + \sum_{j \neq k} y_j dx^j$$

which is holomorphic and $\beta \wedge (d\beta)^n \neq 0$ as in b).

e) *Using the same ideas, considering the form*

$$\sum_j z^j dw^j - w^j dz^j$$

in $\mathbb{C}^{2n+2}$*, one can form projective space* $\mathbb{CP}^{2n+1}$ *as a quotient of* $\mathbb{C}^{2n+2}$*, and the kernel of this form gives a subbundle* Ψ *of codimension 1 in* $T^{1,0}\mathbb{CP}^{2n+1}$ *which is given by the kernel of*

$$\beta = dw^k + \sum_{j\neq k} z^j dw^j - w^j dz^j$$

when $z^k \neq 0$*, so a similar calculation as in b) shows that* $\beta \wedge (d\beta)^n \neq 0$*.*

In the examples d) and e) it can be seen that there is no global contact form, but locally all the contact forms were as in b). This is the case always, as the Darboux theorems prove.

The proofs that we present are adaptations of the ones given in [Lee12] for real contact and symplectic structures. We should point out that all the constructions in real manifolds extend to complex manifolds by $\mathbb{C}$-linearity. For instance, the notions of Lie bracket, Lie derivative, the interior product $\lrcorner$ or the flux associated to a vector field make sense in a complex manifold.

Lemma 1.5.2. *Let* V_t*,* $t \in [0,1]$ *be a time dependent holomorphic vector field in* X *and let* $\varphi_t = \varphi_{t,0}$ *be its associated flux . Then, wherever* φ_t *is defined, it is a holomorphic function.*

Proof. Let $F = \varphi_t$ and take holomorphic coordinates (z, U) and (w, W) around p and $F(p)$. We can write

$$V_t = \sum_j V_t^j \frac{\partial}{\partial z^j},$$

where V_t^j is holomorphic. Then it is clear by the properties of Lie brackets that $[V_t, \frac{\partial}{\partial \bar{z}^j}] = 0$ for all j and therefore the vectors $\frac{\partial}{\partial \bar{z}^j}$ are invariant under F, but this implies $\frac{\partial F^k}{\partial \bar{z}^j} = 0$ for all k, j, so F is holomorphic. ☐

We can now prove the Darboux theorems:

Theorem 1.5.3 (Symplectic Darboux Theorem)**.** *Let* (X, ω) *be a holomorphic symplectic manifold and let* $p \in X$*. There are holomorphic coordinates* $(x^1, y^1, \ldots, x^n, y^n)$ *around* p *such that* $\omega = \sum_j d\,x^j \wedge d\,y^j$

Proof. For any non-degenerate skew-symmetric bilinear form in an even dimensional topological space thee is a basis (X_j, Y_j), with dual basis (φ^j, ξ^j) such that the bilinear form is $\sum_j \varphi^j \wedge \xi^j$. The proof can be done by induction in the dimension of the vector space. We can assume X is the ball $B(0,1)$ in $\mathbb{C}^{2n}$, $p = 0$ and that in the natural coordinates (z^j, w^j), ω agrees with $\omega_0 := \sum_j dz^j \wedge dw^j$ in 0.
Let $\omega_t = t\omega + (1-t)\omega_0$. If we found a time dependent holomorphic vector field V_t such that

$$\phi_t^* \omega_t = \omega_0$$

Where ϕ_t is the associated flux, we would have finished by 1.5.2. Since the equation before works for $t = 0$, it is sufficient to prove that the derivatives of both sides agree. We use the formulas

$$\frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^* \left(\mathcal{L}_{V_t}\omega_t + \frac{d\omega_t}{dt} \right) \qquad \mathcal{L}_{V_t}\omega_t = d(V_t \lrcorner\, \omega_t) + V_t \lrcorner\, d\omega_t,$$

one can prove these following the proofs given in [Lee12] for 14.35 and 22.14. Since ω is closed, by Poincaré lemma, $\omega - \omega_0 = -d\alpha$ for a holomorphic 1-form α, so

$$\frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^* \left(\mathcal{L}_{V_t}\omega_t + \frac{d\omega_t}{dt} \right) = \phi^* \left(d(V_t \lrcorner\, \omega_t) + V_t \lrcorner\, d\omega_t + \omega - \omega_0 \right) = \phi_t^* \circ d(V_t \lrcorner\, \omega_t - \alpha)$$

and then it is sufficient to define V_t in such a way that $V_t \lrcorner\, \omega_t = \alpha$. The can be done because ω_t is non degenerate, and V_t is holomorphic because α and ω_t are. ☐

Theorem 1.5.4 (Contact Darboux theorem)**.** *Let (X, Ψ) be a holomorphic contact manifold and $p \in X$. Then there are holomorphic coordinates $(z, x^1, y^1, \ldots, x^n, y^n)$ around p such that*

$$dz - \sum_j x^j \, dy^j$$

is a contact form for Ψ in a neighbourhood of p.

Proof. As before, we can assume that X is the ball $B(0,1)$ in $\mathbb{C}^{2n+1}$, that $p = 0$ and Ψ is defined by a contact form α. We consider the Reeb vector field $R_\alpha : X \to T^{1,0}X$ which is the only vector field such that

$$\alpha(R_\alpha) = 1 \quad , \quad R_\alpha \lrcorner \, d\alpha = 0.$$

Because α is holomorphic, one can easily see that R_α is holomorphic too. Let (U, u^j) be a chart around p such that $R_\alpha = \frac{\partial}{\partial u^1}$, and let $Y \subset U$ be defined as $u^1 = 0$. Since $\frac{\partial}{\partial u^1}$ is not tangent in any point of Y, $\alpha(\frac{\partial}{\partial u^1}) = 1$ and $\alpha \wedge (d\alpha)^n \neq 0$, $d\alpha_{|Y}$ is non-degenerate, and therefore it is a symplectic form. By Darboux theorem, there are holomorphic coordinates (x^j, y^j) for a neighbourhood of p in Y where $d\alpha = \sum_j dx^j \wedge dy^j$. These can be extended to coordinates in a neighbourhood V of p in U, asking them to be constant along the integral curves of $\frac{\partial}{\partial u^1}$.
Let η be the 1-form $\sum_j y^j dx^j$ en V. Then $d\eta_{|Y} + d\alpha_{|Y} = 0$ in V, but since $R_\alpha \lrcorner \, d\alpha = R_\alpha \lrcorner \, d\eta = 0$, we have $d\eta + d\alpha = 0$ in all points of Y. Also, $\mathcal{L}_{R_\alpha}\eta = \mathcal{L}_{R_\alpha}\alpha = 0$ by Cartan magic formula, so α y η are the same along the integral curves of R_α, and we obtain $d\alpha + d\eta = 0$ in the whole V. By Poincaré lemma there is a holomorphic function z such that $dz = \alpha + \eta$. In other words,

$$\alpha = dz - \sum_j x^j dy^j.$$

Since $0 \neq \alpha \wedge (d\alpha)^n = (-1)^n dz \wedge dx^1 \wedge dy^1 \ldots \wedge dx^n \wedge dy^n$ dz, dx^j, dy^j form a basis in the cotangent space to 0 and therefore are holomorphic coordinates in a neighbourhood of 0. ☐

The result that concerns us is about legendrian subvarieties of complex manifolds.

Definition 1.5.5. *Let (X, Ψ) be a $(2n+1)$-dimensional holomorphic contact manifold and let Y a k-dimensional complex manifold. If $F : Y \to X$ is holomorphic, we will say it is isotropic if*

$$F_*(T^{1,0}Y) \subset \Psi.$$

and if $k = n$ and it is an immersion, we will say it is legendrian.

It is clear that if α is a contact form for Ψ, then F is isotropic if and only if $F^*\alpha = 0$. With some abuse of notation, an isotropic function from a Riemann surface to the complex affine space with its standard contact structure is said to be a *legendrian curve.*
If F is a legendrian immersion, $k = n$ is the maximum one can get since, being an immersion implies it is a local (in the domain) embedding and it is well known (lemma 8.32 in [Lee12]) that the lie bracket of two vector fields that are tangent to a submanifold is again tangent to the submanifold. Therefore, if V, W are tangent to $F(Y)$, $[V, W] \subset \Psi$, so if α is a contact form for Ψ,

$$d\alpha(V, W) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) = 0,$$

but since $\alpha \wedge (d\alpha)^n \neq 0$, $d\alpha_p$ is a non-degenerate 2-form in Ψ_p, which has dimension $2n$ and the maximum dimension of a subspace $E \subset \Psi_p$ such that $d\alpha_{|E} = 0$ is n.

Part 2

Legendrian embeddings of curves

2.1 Main ideas in the proofs

The proof in this second part will make repeated use of several ideas that we state here:

Idea 1. *Refined exhaustions by compact sets*

AS in the proof of Runge theorem, it is easier to obtain legendrian functions on compact subsets. We can refine the exhaustion given in 1.3.5 to see that given a relatively compact, Runge open subset M_0 with smooth boundary, there is a sequence of open subsets

$$M_0 \subset\subset M_1 \subset\subset M_2 \subset\subset \ldots \subset X$$

such that M_k has smooth boundary, is Runge in X, all compact subsets of X are contained in some M_k and for each k one of the following two things happen (there are pictures of this in Figure 3):

(A1.1) $\overline{M}_k$ is a deformation retract of M_{k+1}.

(A1.2) There is a smooth arc $\alpha_k : [0,1] \to M_{k+1}$ such that $\alpha_k(0), \alpha_k(1) \in \overline{M}_k$, $\alpha_k((0,1)) \subset \operatorname{int}(M_{k+1}) \setminus M_k$ and $\overline{M}_k \cup \alpha_k$ is a deformation retract of M_{k+1}.

We will not give a rigorous proof of this, but note that starting from the exhaustion in 1.3.5, in each step $K_m \subset K_{m+1}$, by the classification of compact bordered surfaces, K_{m+1} is diffeomorphic to a sphere with handles and holes. Since K_m is Runge in K_{m+1}, K_m must be a compact subset containing a certain number of these holes and handles, but no more holes. Therefore we can add arcs and compacts as in Figure 3 to obtain the sequence of M_k. A proof can be found in the article [FMM12].

Idea 2. *Metrics in Riemann surface and the Cauchy estimates*

We will assume there is a riemannian metric g in X, and we use it to measure lengths of curves of differential forms: if $\gamma : [a,b] \to X$ is piecewise smooth, its length is defined to be

$$l(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t))^{\frac{1}{2}} dt,$$

and the distance between two points $d(p,q)$ is the infimum of the length of curves joining p and q. It is a standard result (Theorem 2.55 in [Lee18]) that d is a metric and induces the natural topology in X.
If ω, η are smooth, real 1-forms, we define their scalar product pointwise, (or using coordinates) as

$$\langle \omega, \eta \rangle_p = g_p(\omega_p^\sharp, \eta_p^\sharp) = \omega_j(p)\eta_k(p)g^{jk}(p).$$

On each tangent space, one can extend this inner product to complex differential forms, asking the inner product to be hermitic; this is, to be $\mathbb{C}$-lineal in its first entry, and conjugate-symmetric. With this, one can define the norm of a 1-form ω to be $|\omega| = \langle \omega, \omega \rangle^{1/2}$ and it is easy to check that

$$|f\omega| = |f||\omega|, \quad |\omega + \eta| \leq |\omega| + |\eta|, \quad |\omega|_p = 0 \text{ implies } \omega(p) = 0,$$

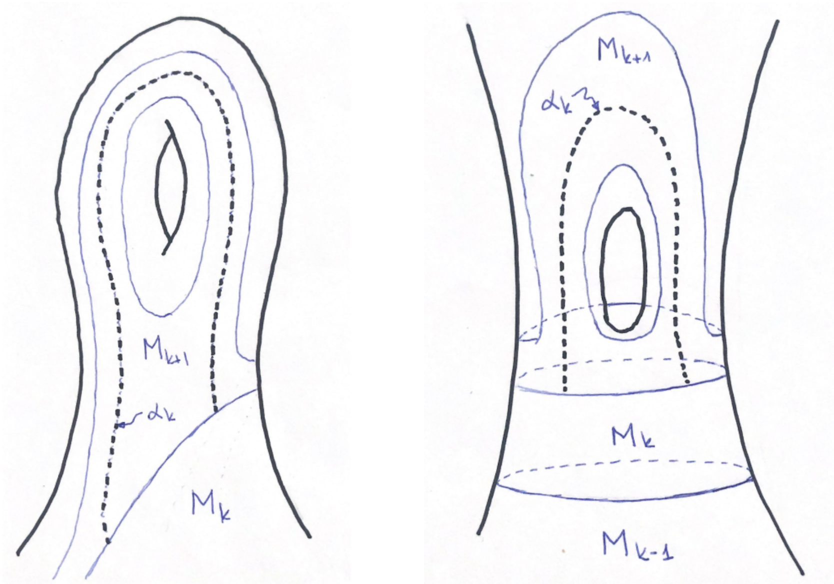

Figure 3: Open sets of the form (A1.1) and (A1.2)

and the inequality that we will use the most, which is that

$$\left|\int_\gamma \omega\right| \leq l(\gamma) \cdot \sup_{p\in\gamma} |\omega(p)|. \tag{A2}$$

As with functions, we use the notation $\|\omega\|_K = \sup_{z\in K} |w|_z$, and we use the supremum norm for maps $F : X \to \mathbb{C}^N$:

$$|dF(p)| = \sup_{j=1,\dots,N} \{|dF_j(p)|\}, \qquad \|dF\|_K = \sup_{j=1,\dots,N} \{\|dF_j\|_K\}.$$

Now we can prove the Cauchy estimates in abstract Riemann surfaces.

Proposition 2.1.1 (Cauchy estimates)**.** *Let X be a Riemann surface, $Y \subset\subset Z \subset\subset X$ open subsets with smooth boundary. Then there is a constant M such that, for any holomorphic function f in X,*

$$\|df\|_{\overline{Y}} \leq M\|f\|_{\overline{Z}}.$$

Proof. Let W be an open subset, properly contained in Y and Z. By the Cauchy-Green formulas in 1.2.19, we have

$$f(x) = \frac{1}{2\pi i}\int_{y\in bW} f(y)\partial_y G_Z(x,y).$$

We cover $\overline{Y}$ by a finite number of open sets V_j which are again properly contained in (U_j, z_j) (by this we mean, $V_j \subset\subset U_j$). The sets $\overline{V}_j \cap \overline{Y}$ and bW are disjoint and therefore the form $\partial_y \frac{\partial}{\partial z_j} G_Z(z_j(p), y)$ does not have singularities if $(p, y) \in \overline{V}_j \cap \overline{Y} \times bW$, which is a compact set, so there is a bound M_j for such form. Therefore, if $x \in V_j \cap \overline{Y}$,

$$|df(x)| = \left|\frac{\partial f}{\partial z_j}(x)dz_j(x)\right| = \left|\frac{1}{2\pi i}\int_{y\in bW} f(y)\partial_y \frac{\partial}{\partial z_j} G_Z(z_j(x), y)dz_j\right| \leq \frac{l(bW)\cdot \|f\|_{bW} M_j \|dz_j\|_{V_j}}{2\pi},$$

And we deduce the proposition. ☐

As a direct corollary, we see that if f_n converges uniformly over compact subsets of X to f, then df_n converges uniformly over compacts of X to df, generalising 1.1.4 to Riemann surfaces. In particular, the uniform limit of legendrian maps is again legendrian, showing that the conditions in Theorem 1 are necessary

Idea 3. *Injective maps and the map of differences.*

Let $f : A \to \mathbb{C}^n$ a smooth function between manifolds. By the inverse function theorem, if f is an immersion, it is locally injective. On the other hand, f is an embedding when it is a diffeomorphism onto its image, but it is a simple result in topology that it is enough for f to be an injective, proper immersion. If A is compact, any continuous f is proper.
It is convenient to use the difference map $\delta f : A \times A \to \mathbb{C}^n$, given by

$$\delta f(a, a') = f(a) - f(a'). \tag{A3}$$

Then f is injective if and only if $f^{-1}(\{0\})$ is the diagonal of $A \times A$. If we start with an immersion,there is an open neighbourhood U of the diagonal in $A \times A$ such that $f^{-1}(\{0\}) \cap U$ is the diagonal.

Idea 4. *Uniformity of holomorphic immersions and embeddings*

We will use the Cauchy estimates 2.1.1 to prove two surprising results, that fail in the smooth case. The proofs are original.

Proposition 2.1.2. *Let $Y \subset\subset Z \subset\subset X$, and let $F : X \to \mathbb{C}^N$ be a smooth map which is a holomorphic immersion in $\overline{Y}$. Then there is an $\varepsilon > 0$ and an $r > 0$ such that, if $G : X \to \mathbb{C}^N$ is holomorphic and $\|F - G\|_{\overline{Z}} \leq \varepsilon$ then G is an immersion and if $0 < d(x, y) < r$ then $G(x) \neq G(y)$.*

Proof. Let M be the constant from the Cauchy inequalities for the pair $Y \subset\subset Z$. Let $c = \|dF\|_{\overline{Y}} > 0$ and take $p \in \overline{Y}$ and a chart (U, z) around it. The function $\frac{\partial F}{\partial z}$ and since z is a chart, $dz \neq 0$ so we can take a disk $V \subset\subset U$ centred in p such that

$$\left| \frac{\partial F}{\partial z}(q) - \frac{\partial F}{\partial z}(p) \right| \leq \frac{c}{6|dz(p)|} \quad \text{if } q \in V.$$

We cover $\overline{Y}$ by a finite number of V_j as above, $j = 1, \ldots, n$. Again, since z_j is a chart, dz_j is nonzero, and V_j is relatively compact, so we take $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{cM}{6} \quad \text{y} \quad \varepsilon \leq \frac{|dz_j(p)|c}{6M \inf_{q \in \overline{V}_j} |dz_j(q)|}, j = 1, \ldots, n.$$

Finally, if $W = \bigcup_{j=1}^n V_j \times V_j$, W is an open neighbourhood of the diagonal

$$\{(x, y) \in X : x = y \in \overline{Y}\}$$

so there is an $r > 0$ such that $d(x, y) \leq r$ whenever $(x, y) \in W$.
If $\|G - F\|_{\overline{Z}} \leq \varepsilon$ and G is holomorphic, $\|dG\|_{\overline{Y}} \geq \|dF\|_{\overline{Y}} - \|d(G - F)\|_{\overline{Y}} \geq \frac{c}{2}$ so G is an immersion. If p is the centre of V_j and $|dF_k(p)| \geq c$ then $|\frac{\partial G_k}{\partial z_j}(p)|dz_j(p) = |dG_k(p)| \geq \frac{c}{2}$. By the triangle inequality and the choice of ε,

$$\left| \frac{\partial G_k}{\partial z_j}(p) - \frac{\partial G_k}{\partial z_j}(q) \right| \leq \left| \frac{\partial G_k}{\partial z_j}(p) - \frac{\partial F_k}{\partial z_j}(p) \right| + \left| \frac{\partial F_k}{\partial z_j}(p) - \frac{\partial F_k}{\partial z_j}(q) \right| + \left| \frac{\partial F_k}{\partial z_j}(q) - \frac{\partial G_k}{\partial z_j}(q) \right| \leq \frac{c}{2|dz_j(p)|} = \left| \frac{\partial G_k}{\partial z_j}(p) \right|$$

for all $q \in V_j$. If we let $h = G_k \circ z_j^{-1}$, we will have that if $q, r \in V_j$ are different points and γ is the segment joining them

$$|h(q) - h(r)| = \left| \int_\gamma [h'(z) - h'(p) + h'(p)] dz \right| \geq \left| \int_\gamma h'(p) dz \right| - \left| \int_\gamma [h'(z) - h'(p)] dz \right| > |h'(a)||q - r| - |h'(a)|q - r| \geq 0.$$

And so G_k, and therefore G, is injective in V_j, but this happens for all j, so we get the desired result. □

We can deduce immediately that

Corollary 2.1.3. *Let $Y \subset\subset Z \subset\subset X$, and $F : X \to \mathbb{C}^N$ a holomorphic map which is an embedding in $\overline{Y}$. Then there is an $\varepsilon > 0$ such that, if $G : X \to \mathbb{C}^N$ is holomorphic and $\|F - G\|_{\overline{Z}} \leq \varepsilon$ then G is an embedding in $\overline{Y}$*

Proof. Let ε and r as before. Since $\overline{Y}$ is compact,

$$c = \inf\{|\delta F(x,y)| : x, y \in \overline{Y}, d(x,y) \geq r\} > 0.$$

If $\varepsilon_1 = \min\{\varepsilon, \frac{c}{3}\}$. Then, if $\|G - F\|_Z \leq \varepsilon_1$ we can use the previous proposition, but also, if $d(x,y) \geq r$,

$$|G(x) - G(y)| \geq |F(x) - F(y)| - |G(x) - F(x)| - |G(y) - F(y)| \geq \frac{c}{3}$$

So G is an injective immersion on a compact space, and therefore an embedding. □

Idea 5. *Holomorphic sprays*

We will distinguish between discrete and continue sprays. Let V, W be complex manifolds and let U be an open set in $\mathbb{C}^N$ containing 0.
A discrete spray of holomorphic functions is a sequence of holomorphic maps $f_n : V \to W$ converging uniformly on compact subsets of U to a holomorphic map f, which we will refer to as the core of the spray. The spray is said to be dominant at $p \in V$ if f is regular in p.
A continuous spray is a family of maps $F_u : V \to W$ parametrised by $v \in V$ such that the induced map $F : V \times U \to W$ is holomorphic. Its core is F_0, and we say that the spray is dominant in $p \in V$ if $u \to F(p,u)$ is a regular map in $u = 0$.
The two following results manifest the similarities between these two concepts:

Proposition 2.1.4. *Let f_n be a discrete spray with core f. If the pray is dominant in p and $f(p) = b$. Then for any neighbourhood of p there is an N as big as wished and a z in this neighbourhood such that $f_N(z) = b$.*

Proof. We can assume that $W = \mathbb{C}^n$, $b = 0$ and $\frac{\partial f}{\partial z}(0)$ is the identity matrix. We will use the following lemma (1.3. in chapter 14 of [Lan93]):

Lemma 2.1.5. *If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ verifies $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial z}(0) = id$ and there are R, s such that $\|\frac{\partial \varphi}{\partial s}(x) - \frac{\partial \varphi}{\partial s}(y)\| \leq s$ whenever $x, y \in \overline{B(0,R)}$then, for if $|z| \leq R(1-s)$, there is a unique $w \in \overline{B(0,R)}$ such that $\varphi(w) = z$.*

Proof. If $h_z(w) = w + \phi(w) - z$, then the bounds in the lemma and the mean value theorem prove that $h_z : \overline{B(0,R)} \to \overline{B(0,R)}$ is contractive and therefore has a (unique) fixed point, corresponding to a solution of $\varphi(w) = z$ □

Since the function appearing are holomorphic, $\frac{\partial f_n}{\partial z}$ converges uniformly over compact sets, so the family of functions $\frac{\partial f_n}{\partial z}$: is equicontinuous. Let $\delta > 0$ be such that if $|x - y| \leq \delta$,

$$\left\| \frac{\partial f_n}{\partial z}(x) - \frac{\partial f_n}{\partial z}(y) \right\| \leq \frac{1}{3}$$

Now define $A_n = \left(\frac{\partial f_n}{\partial z}(0)\right)^{-1}$ and $g_n = f_n \circ A_n - f_n(0)$. Since $A_n \to \text{id}$, from some N_0 onwards $\|A_n\| \leq 2$. If $\varepsilon \leq \delta$ is arbitrary, and $|f_n(0)| \leq \frac{\varepsilon}{12}$ whenever $n > N_1$. If $n \geq N_0, N_1$, we have $g_n(0) = 0$, $\frac{\partial g_n}{\partial s}(0) = \text{id}$, and if $x, y \in B(0, \frac{\varepsilon}{4})$, then $\|A_n x - A_n y\| \leq \varepsilon < \delta$ so

$$\left\| \frac{\partial g_n}{\partial z}(x) - \frac{\partial g_n}{\partial z}(y) \right\| = \left\| \frac{\partial f}{\partial z}(A_n x) A_n - \frac{\partial f}{\partial z}(A_n y) A_n \right\| \leq \frac{\|A_n\|}{3} \leq \frac{2}{3}$$

Since $|-f_n(0)| \leq \frac{\varepsilon}{12} = \frac{\varepsilon}{4}(1 - \frac{2}{3})$, we can apply the lemma so there is some $z \in B(0, \frac{\varepsilon}{4})$ such that $g_n(z) = -f_n(0)$. If $w = A_n z \in B(0, \frac{\varepsilon}{2})$, $f_n(w) = 0$ as desired. □

Proposition 2.1.6. *Let F be a continuous spray which is dominant in p, having f as core and let $b = f(x)$. Then for any y in a neighbourhood of x there is some z such that $F_z(y) = b$ and z depends of y in a holomorphic way, and converges to 0 if y converges to x*

Proof. this is the implicit function theorem for holomorphic functions in 1.1.6. □

Idea 6. *The third component in a legendrian map to $\mathbb{C}^3$*

Let $f:(x,y,z):X\to\mathbb{C}^3$ be a holomorphic map. The standard contact form in $\mathbb{C}^3$ says that f is Legendrian if $dz=xdy$, or, in other words, if xdy is exact and z is a primitive for it. Therefore we can reduce the problem of finding legendrian maps from X to $\mathbb{C}^3$ can be reduced to finding pairs of holomorphic functions (x,y) such that xdy is exact, and then define

$$z(p)=\int_{\gamma_p} xdy+C, \tag{A4}$$

where γ_p is any path joining a fixed p_0 with p, and $C\in\mathbb{C}$ is a constant. By deRham theorem (section 1.2), this integral is independent of the path taken, and by Stokes theorem, if $dw=xdy$, $z(p)=w(p)-w(p_0)+C$ so $dz=xdy$.
If $Y\subset\subset X$, and we start with a legendrian $f=(x,y,z)$ and functions x_n, y_n in a neighbourhood of $\overline{Y}$ such that $x_n\to x$, $dy_n\to dy$ uniformly in $\overline{Y}$ and x_ndy_n is exact for all n, defining z_n as in (A4) with $C=z(p_0)$, since $\overline{Y}$ is compact, the path joining p_0 to p has bounded length L.
Then, by Stokes' theorem:

$$|z(p)-z_n(p)|=\left|\int_{\gamma_p}(x-x_n)dy+x_n(dy-dy_n)\right|\le L\left(\|x_n-x\|_{\overline{Y}}\|dy\|_{\overline{Y}}+\|x_n\|_{\overline{Y}}\|dy-dy_n\|_{\overline{Y}}\right)\to 0$$

uniformly in $p\in\overline{Y}$.

Idea 7. *Symmetry of the first and second coordinates*

The equation $xdy=dz$ does not appear to be symmetric in x and y. However, the map

$$\Phi(x,y,z)=(x',y',z')=(x,-y,z-xy) \tag{A5}$$

is involutive, and $x'dy'-dz'=ydx-dz$ so the roles of x and y can be exchanged.

Idea 8. *The period map*

By deRham Theorem, to check if a differential form is exact in an open set Y it is enough to check that t is closed and its integral along any path in $H_1(Y,\mathbb{Z})$ is 0. If Y is relatively compact in X, this is a free abelian group of finite rank generated by simple paths (this can be found along the pages of [Hat01]). After choosing a basis γ_j for it, we will consider the period map

$$\mathcal{P}(x,y)=\left(\int_{\gamma_j} xdy\right)\in\mathbb{C}^{\dim H_1(Y,\mathbb{Z})} \tag{A4}$$

and make modifications to x and y that don't alter the value of this period map. IN order to combine holomorphic sprays and period maps, we will use this lemma repeated times:

Lemma 2.1.7. *Let $f:[0,1]\to\mathbb{C}$ continuous, with a finite number of zeros and let $c\in\mathbb{C}$. There is a smooth function $g:[0,1]\to\mathbb{C}$ with compact support in $(0,1)$ such that*

$$\int_0^1 f(x)e^{g(x)}dx=c \quad \textit{and} \quad \int_0^1 f(x)g(x)e^{g(x)}dx\neq 0.$$

Proof. We can take a simple function satisfying this (this is, a linear combination of characteristic functions of some intervals) g with these properties. If g_n is a sequence of smooth maps vanishing outside some compact and that converge uniformly to g, and we consider the discrete spray

$$\varphi^n(s)=\int_0^1 f(x)e^{sg_n(x)})$$

with core $\varphi(s)=\int_0^1 f(x)e^{sg(x)}$, we have $\varphi(1)=c$ and $\frac{\partial}{\partial s}\varphi(s)\neq 0$ so, by 2.1.4, we can find $s_n\neq 0$ and define $h=s_ng_n$ so

$$c=\varphi_n(s_n)=\int_0^1 f(x)e^{h(x)} \qquad 0\neq\frac{\partial}{\partial s}\varphi_n(s_n)=\frac{1}{s_n}\int_0^1 f(x)h(x)e^{h(x)}.$$

□

Idea 9. *Admissible sets*

Given a Riemann surface X, an admissible set is a set S of the form $S = K \cup \Gamma$ where K is the closure of a relatively compact open subset of X with smooth boundary and Γ is the union of some pairwise disjoint, embedded, smooth curves that intersect K only in its endpoints, and transversely, as in Figure 4.
If W is an open set with smooth boundary containing S such that S is a deformation retract of it, W is Runge if and only if S is, because there is a commutative diagram

$$\begin{tikzcd} H_1(S,\mathbb{Z}) \arrow[rr] \arrow[dr, "j"] & & H_1(X,\mathbb{Z}) \\ & H_1(\overline{W},\mathbb{Z}) \arrow[ur] & \end{tikzcd}$$

where j is an isomorphism, so 1.3.4 proves the claim.
The classification of compact, bordered surfaces can be used to prove the following

Lemma 2.1.8. *Given an admissible set S there is a set of curves γ_j that intersect each other in finitely many points and can be taken to avoid any finite set, forming a basis of $H_1(S,\mathbb{Z})$.*
If S is Runge in X, the γ_j can be taken in such a way that any union of the γ_j is Runge in X.

If γ is an embedded curve in X,the inclusion $T\gamma \subset TX$ induces a natural inclusion $T_{\mathbb{C}}\gamma = \mathbb{C}\otimes_{\mathbb{R}} T\gamma \to T_{\mathbb{C}}X$. There is not an intrinsic decomposition of $T_{\mathbb{C}}\gamma$ as in 1.1.10.

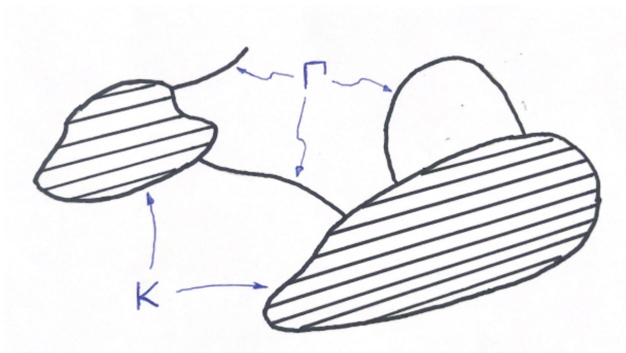

Figure 4: Admissible set

If $S = K \cup \Gamma = \overline{Y} \cup \Gamma$, we say that a function $f : S \to \mathbb{C}$ is of class $\mathcal{C}^r$ if $f_{|K}$ and $f_{|\gamma_j}$ are of class $\mathcal{C}^r$ for any curve γ_j in Γ, and the derivative of f in Γ and K agree up to order r in the endpoints of the paths.
The $\mathcal{C}^1$ norm is $\|f\|_S + \|df\|_S$, with respect to some metric g. Also, a map $f = (x, y, z)$ of class $\mathcal{C}^1$ is a generalised legendrian curve in S if it is legendrian in K and for any parametrisation $\gamma$of the curves in Γ,

$$x \circ \gamma(t) \cdot (y \circ \gamma)'(t) = (z \circ \gamma)'(t),$$

although it is clear that it is enough to check for one such parametrisation.

2.2 First examples of holomorphic approximation

We start by proving a classic result by R. Gunning and M. Narashiman, concerning the existence of exact holomorphic forms with prescribed zeros, presented in [GN67] in 1967, but using the ideas mentioned above, as well as an improved version of Mergelyan theorem in admissible sets.

As usual, we start with a relative version first

Proposition 2.2.1. *Let X be a Riemann surface and $Y \subset Z$ be Runge open sets with smooth boundary of the form (A1.1) or (A1.2) and let ω be a holomorphic 1-form in X such that $\int_\gamma \omega = 0$ for all closed paths $\gamma \subset Y$.*
Then, for any $\varepsilon > 0$ there is a holomorphic function g in X such that $|g| < \varepsilon$ en Y and $\int_\gamma e^g\omega = 0$ for all closed curves $\gamma \subset Z$.

Proof. If ω is 0 or if the compacts are of the form (A1.1), we my take g constant, so assume they are of the form (A1.2) and let $\gamma_1, \ldots, \gamma_p, \gamma_{p+1}$ be paths forming a basis for $H_1(\overline{Y} \cup \alpha, \mathbb{Z})$ as in 2.1.8. Since the zeros of ω are isolated and γ_j is compact, we can use 2.1.7 with $f(t) = \omega(\gamma_j'(t))$ to get continuous functions g_j defined in $\gamma_1 \cup \ldots \cup \gamma_{p+1}$ such that the support of g_j is contained in γ_j and

$$\int_{\gamma_{p+1}} e^{g_{p+1}}\omega = 0, \qquad \int_{\gamma_j} g_{p+1}e^{g_{p+1}}\omega \neq 0 \qquad y \qquad \int_{\gamma_j} g_j\omega \neq 0 \text{para } 1 \leq j \leq p \tag{1}$$

In fact, since γ_{p+1} only intersects bY in a finite number of points, we can also assume that g_{p+1} is defined and equals 0 in $\overline{Y}$. We consider the period map $\mathcal{Q} : \mathcal{O}(X) \to \mathbb{C}^{p+1}$ defined by

$$\mathcal{Q}(g) = \left(\int_{\gamma_j} g\omega\right)_{1\leq j\leq p+1}$$

from which we obtain the following holomorphic map:

$$\begin{array}{rcl} \varphi : \mathbb{C}^{p+1} & \to & \mathbb{C}^{p+1} \\ (s_1, \ldots, s_q) & \mapsto & \mathcal{Q}\left(e^{s_1u_1+\ldots+s_{p+1}u_{p+1}}\right) \end{array}$$

If $a = (0, \ldots, 0, 1)$, the conditions (1) imply that $\varphi(a) = 0$ and $\frac{\partial\varphi}{\partial s}(a)$ is invertible. The sets $\gamma_1 \cup \ldots \cup \gamma_{p+1}$ and $\overline{Y} \cup \gamma_{p+1}$ are Runge compacts by 2.1.8 and the homological characterisation of admissible sets. We use Mergelyan theorem to obtain holomorphic functions $g_j^{(m)}$ in X converging uniformly to g_j in $\gamma_1 \cup \ldots \cup \gamma_{p+1}$ for $1 \leq j \leq p$, and in $\overline{Y} \cup \gamma_{p+1}$ for $j = p+1$. We consider the sequence

$$\varphi^m(s) = \mathcal{Q}\left(e^{s_1v_1^{(m)}+\ldots+s_pv_p^{(m)}+s_{p+1}u_{p+1}}\right).$$

Which converges to φ uniformly on compact sets, and since a has zeros in its first entries, $\varphi^m(a) = 0$ for all m. We can also find $\mu = m$ such that φ^μ is regular in a. Let $h_j = u_j^{(\mu)}$ for $j = 1, \ldots, p$.
The discrete spray

$$\psi^n(s) = \mathcal{Q}\left(e^{s_1h_1+\ldots+s_ph_p+s_{p+1}g_{p+1}^{(n)}}\right)$$

has φ^μ as core, so it is dominant in a. By 2.1.4, for any δ there is some N_0 from which we can find $s^{(n)}$ at distance at most δ from a such that $\psi^{(n)}(s^{(n)}) = 0$.
Let C be a common bound for the functions $w_1, \ldots, w_p$ in $\overline{Y}$. Since $g_{p+1}^{(n)}$ converges uniformly in $\overline{Y}$ to 0, there is some N_1 such that $\|g_{p+1}^{(n)}\|_{\overline{Y}} \leq \frac{\varepsilon}{2q}$ whenever $n > N_1$. Finally, if $\delta \leq \min\left\{1, \frac{\varepsilon}{nC}\right\}$, any s such that $|s - a| \leq \delta$ verifies

$$\|s_1h_1 + \ldots + s_ph_p + s_{p+1}g_{p+1}^{(n)}\|_K \leq \varepsilon,$$

and therefore ,$f = s_1^{(n)}w_1 + \ldots + s_p^{(n)}w_p + s_{p+1}^{(n)}v_{p+1}^{(n)}$ is the desired function if $|s^{(n)} - a| \leq \delta$ and $n > \max\{N_1, N_0\}$. □

Theorem 2.2.2 (Gunning-Narashiman). *Let X be an open Riemann surface and let ω be a holomorphic 1-form. Then there is a function $F \in \mathcal{O}(X)$ such that $e^F\omega$ is exact.*

Proof. Recall that holomorphic forms are closed, as we noted in 1.2.5, and take an exhaustion by compact sets of X as in Idea 1 such that M_0 is a disk. Since in a disk closed forms are exact, $\int_\gamma \omega = 0$ for all paths $\gamma \subset M_0$, we define $f_0 : X \to \mathbb{C}$ to be 0. Using the previous lemma 2.2.1 in an inductive way, we can find holomorphic functions $f_n : X \to \mathbb{C}$ such that:

a) $\int_\gamma e^{f_0+\ldots+f_n}\omega = 0$ for all $\gamma \subset M_n$

b) $|f_n(p)| < \frac{1}{2^n}$ for all $p \in K_{n-1}$

The second condition ensures that $\sum_{n=0}^{\infty} f_n$ converges uniformly over compact subsets of X to some holomorphic F. For each path γ in X, γ is contained in some M_n, so taking limits in the first condition, $\int_\gamma e^F\omega = 0$. Since $e^F\omega$ is closed because it is holomorphic, and all of its periods are 0,it is exact □

In the language of immersions, we obtain the following corollary

Corollary 2.2.3. *If X is an open Riemann surface, there is a holomorphic immersion of X in $\mathbb{C}$*

Proof. By 1.3.12, there is a nonvanishing holomorphic form ω, and by 2.2.2, $e^f\omega = dg$ for some g (g is holomorphic because $d = \partial + \bar{\partial}$ and $e^f\omega \in \Omega^{1,0}$). Since dg is nowhere vanishing, $g : X \to \mathbb{C}$ is an immersion. □

However, these immersions are not dense in $\mathcal{O}(X)$. This is due to a well known theorem of Hurwitz that can be found in p. 231 of [Gam01] which, among other things implies that a sequence of immersions into $\mathbb{C}$ can only converge to a constant function or to an immersion.
However, immersions of Riemann surfaces in higher dimensional affine spaces are dense, and we can prove it easily:

Proposition 2.2.4. *Let X be an open Riemann surface. Any holomorphic map from X to $\mathbb{C}^N$ can be approximated uniformly on compact sets by immersions with non-constant component functions.*

Proof. It is enough to prove it for $N = 2$. Write $f = (x, y)$. If f is constant, take functions $g, h : X \to \mathbb{C}$ that are immersions and consider the sequence

$$f_n = \left(x + \frac{1}{n}g, y + \frac{1}{n}h\right).$$

If y is constant but x is not, and $h : X \to \mathbb{C}$ is any immersion, the sequence

$$f_n = \left(x, \frac{1}{n}h + y\right)$$

is sufficient. In the remaining situation, let $u_1, u_2, \ldots$ be the zeros of dy. Using 1.3.12, there is a holomorphic form ω having zeros precisely in the u_j such that $dx(u_j) \neq 0$. By 2.2.2, we can assume that ω is exact. If $\omega = dh$, h is holomorphic and we can take

$$f_n = \left(x + \frac{1}{n}h, y\right).$$

□

Now we prove a version of Mergelyan theorem for approximations of class $\mathcal{C}^1$. It can be done for functions of class $\mathcal{C}^r$ by induction but we have not defined the $\mathcal{C}^r$ norm and we don't need this generalisation.

Theorem 2.2.5 (Mergelyan Theorem with approximation up to order 1)**.** *Let X be an open Riemann surface and $S = K \cup S \subset X$ an admissible, Runge subset. If $f : S \to \mathbb{C}$ is of class $\mathbb{C}^1$ and holomorphic in the interior of S then it can be approximated uniformly in S by holomorphic functions in X in the $\mathcal{C}^1$ norm.*

Proof. Let $Y \subset\subset X$ be an open set containing S, which is still Runge of whom S is a deformation retract. Let γ_j be a basis for $H_1(Y, \mathbb{Z})$ ($j = 1, \ldots, q$) contained in S and θ a nowhere vanishing 1-form as in 1.3.12. Using 2.1.7 we find continuous functions g_k defined in the paths γ_j such that

$$\int_{\gamma_j} g_k\theta = \delta_{j,k}$$

We consider the period map $\mathcal{Q}(g) = \left(\int_{\gamma_j} g\theta\right)_{j=1,\dots,q}$.
The usual Mergelyan theorem, applied to the functions $g_k$yields holomorphic functions h_k in X, such that the matrix $\left(\int_{\gamma_j} h_k\theta\right)_{j,k}$ is invertible. If $F = \frac{df}{\theta}$, F continuous in S and holomorphic in its interior. By Mergelyan theorem we find a sequence F_n converging to F uniformly on S.
For these reasons, the discrete spray

$$\varphi^{(n)}(s) = \mathcal{Q}\left(F_n + \sum_k s_k h_k\right)$$

has $\varphi(s) = \mathcal{Q}(F + \sum_{k=1}^q s_k h_k)\theta)$ as core, is dominant in 0 and $\varphi(0) = \int_{\gamma_j} F\theta = \int_{\gamma_j} df = 0$. By 2.1.4, for all n there is $s^{(n)}$ such that if

$$\widetilde{F}_n = F_n + \sum_{k=1}^q s_k^{(n)} g_k',$$

these functions converge uniformly to h, $s^{(n)}$ converges to 0 and $\widetilde{F}_n\theta$ is exact in Y, so if

$$f_n(z) = f(p_0) + \int_{p_0}^z \widetilde{F}_n\theta \quad z \in Y,$$

f_n is holomorphic in Y and for all $p \in S$,

$$f_n(z) - f(z) = \int_{p_0}^z (F_n - F)df \qquad df_n(z) - df(z) = [F_n(z) - F(z)]\theta(z).$$

Since all paths can be taken of finite length because $Y \subset\subset X$, f_n approximates f in $\mathcal{C}^1$-norm. Finally, by Runge theorem applied to each F_n we obtain a sequence of holomorphic functions converging to F_n uniformly over compact subsets of Y. By the Cauchy estimates, dG_{nk} also converges to dF_n, so it is enough to approximate f by the sequence G_{nn}. $\square$

2.3 Legendrian embeddings in bordered surfaces

We are now going to prove the main result to get holomorphic embeddings of bordered Riemann surfaces, which in fact strengthens Theorem 2. We will need a theorem about transversality B.2 which can be found in the Appendix. In all the proofs of this sections we will use the period map

$$\mathcal{P}(x, y) = \left(\int_{\gamma_j} x dy \right)_j$$

that we introduced in (A4).
First, we explain how to get immersions:

Proposition 2.3.1. *If X is an open Riemann surface and $f : X \to \mathbb{C}^3$ a Legendrian map, $Y \subset\subset X$ a smoothly bounded domain that is a deformation retract of X. Then f can be approximated uniformly over $\overline{Y}$ legendrian maps in X, which are immersions in $\overline{Y}$, and do not have constant component functions.*

Proof. Write $f = (x, y, z)$. If both x and y are constants, z is also constant. If $h : X \to \mathbb{C}$ is an immersion and $g : X \to \mathbb{C}$ is a non-constant function such that dg vanishes at some point

$$y_n = y + \frac{h}{n} \quad z_n = z + \frac{g}{n^2} + \frac{xh}{n} \quad x_n = \frac{dz_n}{dy_n} = \frac{1}{n}\frac{dg}{dh} + x$$

y $f_n = (x_n, y_n, z_n)$.
If one of x, y is non-constant, using the involution in (A5), we can assume that y is the non-constant function and $u_1, \ldots, u_l$ are the points where dy vanishes in $\overline{Y}$. Let $\gamma_1, \ldots, \gamma_q$ be a basis for $H_1(Y, \mathbb{C})$, and let's assume that no such path contains any if the u_j, and such that $\gamma_1 \cup \ldots \cup \gamma_q$ is Runge.
If (x_n, y) is the sequence of immersion approximating (x, y) given by 2.2.4, and $g_1, \ldots, g_q$ are continuous functions defined in $\gamma_1 \cup \ldots \cup \gamma_q$ such that

$$\int_{\gamma_j} g_k dy = \delta_{j,k}$$

(this can be done thanks to 2.1.7). Using Mergelyan theorem with fixed points 1.4.9, there are holomorphic functions h_k in X such that

$$\det \left(\int_{\gamma_j} h_k dy \right)_{j,k} \neq 0 \quad \text{and} \quad dh_k(u_m) = 0 \text{ for all } m, k.$$

We consider now the discrete spray

$$\varphi^n(s) = \mathcal{P}(x_n + s_1 h_1 + \ldots + s_q h_q, y_n),$$

with core

$$\varphi(s) = \mathcal{P}(x + s_1 h_1 + \ldots + s_q h_q, y)$$

Since f is legendrian, $\varphi(0) = 0$ and by the way we chose the h_j, the spray is dominant in 0. We can apply 2.1.4, and thus from some N_0 and onwards, we can find $s^{(n)}$ such that $\varphi^n(s^{(n)}) = 0$ and the points $s^{(n)}$ converge to 0. Let $\widetilde{x}_n = x_n + s_1^{(n)} h_1 + \ldots + s_q^{(n)} h_q$, $\widetilde{y}_n = y_n = y$ and $\widetilde{z}_n$ defined as in (A3) with $C = z(p_0)$. It can be well-defined in the whole X precisely because Y is a deformation retract of X and so $\gamma_1, \ldots, \gamma_q$ is a basis of $H_1(X, \mathbb{Z})$ also. Now let

$$f_n = (\widetilde{x}_n, \widetilde{y}_n, \widetilde{z}_n).$$

Since $d(\widetilde{x}_n)(u_j) = dx_n(u_j) \neq 0$ for all j, the maps f_n are immersions in $\overline{Y}$; they are clearly legendrian and approximate f because, since $s^{(n)} \to 0$ if $n \to \infty$,$\widetilde{x}_n \to x$ uniformly on $\overline{Y}$ and it is clear that $d\widetilde{y}_n \to dy$, so by the argument in Idea 6, we also have that $\widetilde{z}_n$ converges to z uniformly over $\overline{Y}$. ☐

Recall that on a compact manifold, an injective immersion is an embedding. This is how we will obtain embeddings:

Theorem 2.3.2. *Let $Y \subset\subset X$ be an open, Runge subset with smooth boundary of an open Riemann surface X which, at the same time is a deformation retract of Y. If $f : X \to \mathbb{C}^3$ is a legendrian map, then it can be approximated uniformly in $\overline{Y}$ by legendrian curves defined in X without constant components that are embeddings in $\overline{Y}$.*

Proof. In light of 2.3.1, we may assume that f is an immersion without constant components. We divide the proof into various steps:

Step 1: Given $p \neq q \in \overline{Y}$, and a legendrian map $g : X \to \mathbb{C}^3$ without constant components, we seek a holomorphic family of maps $H : X \times \mathbb{C}^3 \to \mathbb{C}^3$ such that:

a) $H(\cdot, 0) = g$.

b) $H(\cdot, \xi)$ is a legendrian map without constant component functions for all $\xi \in B(0, r)$, for some r.

c) $\delta H(p, q, \cdot) : \mathbb{C}^3 \to \mathbb{C}^3$ is a submersion in 0.

Write $g = (x, y, z)$. If $\gamma_1, \dots, \gamma_q$ is a basis for $H_1(X, \mathbb{Z})$, we look for functions of the form

$$x(\cdot, \xi, s) = x + \xi_1 h_1 + \xi_3 h_2 + \sum_k g_k s_k, \qquad y(\cdot, \xi) = y + \xi_2 h_1,$$

in such a way that the spray

$$(\xi, s) \mapsto \mathcal{P}((x(\cdot, \xi, s)), y(\cdot, \xi))$$

is dominant in $\xi = 0$. Let $\mu > 0$. Since

$$\frac{\partial}{\partial s} \mathcal{P}(x(\cdot, \xi, s), y(\xi)) \bigg|_{\xi = s = 0} = \left(\int_{\gamma_k} g_j dy \right)$$

This can be achieved if

i) $\left| \int_{\gamma_k} g_j dy - \delta_{j,k} \right| < \mu$

and μ is sufficiently small. In this case, using 2.1.6 we can solve $s = \rho(\xi)$ in such a way that $x(\cdot, \xi, \rho(\xi)) dy(\cdot, \xi))$ is exact and therefore we define $z(\cdot, \xi)$ as in (A4) with $C = z(p_0)$ and

$$H(u, \xi) = (x(u, \xi, \rho(\xi)), y(u, \xi), z(u, \xi)) \, .$$

Clearly H satisfies a) y b) because $H(u, \xi)$ converges to g if $\xi \to 0$ and g does not have constant component functions. We are left condition. Note that, if Γ is a segment joining p with q, then

$$\delta z(p, q, \xi) = \int_\Gamma x(z, \xi, \rho(\xi)) dy(z, \xi) = \int_\Gamma \xi_3 h_2 dy(\cdot, \xi) + \left[x + \xi_1 h_1 + \sum_k g_k \rho_k(\xi) \right] d(y + \xi_2 h_1)$$

So using the definitions of $x(\cdot, \xi, \rho(\xi))$ e $y(\cdot, \xi)$, we obtain the formulas

$$\frac{\partial}{\partial \xi_j} \delta x(p, q, \xi, \rho(\xi)) \bigg|_{\xi=0} = \delta_{1,j} \delta h_1(p, q) + \delta_{3,j} \delta h_2(p, q) + \sum_k \frac{\partial \rho_k(\xi)}{\partial \xi_j} \bigg|_{\xi=0} (g_k(p) - g_k(q)).$$

$$\frac{\partial}{\partial \xi_j} \delta y(p, q, \xi) \bigg|_{\xi=0} = \delta_{2,j} \delta h_1(p, q).$$

$$\frac{\partial}{\partial \xi_3} \delta z(p, q, \xi) \bigg|_{\xi=0} = \int_\Gamma h_2 dy + \int_\Gamma \sum_k \frac{\partial \rho_k(\xi)}{\partial \xi_3} \bigg|_{\xi=0} g_k dy.$$

Conditions

ii) $h_1(p) = 1, h_1(q) = 0$.

iii) $|h_2(p)|, |h_2(q)| < \mu$.

iv) $\left| 1 - \int_\Gamma h_2 dy \right| < \mu$.

Ensure that $\delta h_1(p,q) = 1$, $\int_\Gamma h_2 dy = 1 + O(\mu)$ and $\delta h_2(p,q) = 0 + O(\mu)$. On the other hand, to bound $\frac{\partial \rho_k(0)}{\partial \xi_j}$, we derive with respect to ξ the equality $\mathcal{P}(x(\cdot,\xi,\rho(\xi),y(\cdot,\xi)) = 0$ to obtain:

$$\frac{\partial \rho(0)}{\partial \xi} = -\left[\frac{\partial}{\partial s}\mathcal{P}(x(\cdot,\xi,s),y(\xi))\Big|_{\xi=s=0}\right]^{-1} \cdot \frac{\partial}{\partial \xi}\mathcal{P}(x(\cdot,\xi,s),y(\xi))\Big|_{\xi=s=0}.$$

If in condition i) the approximation is strong enough, the first matrix is close to the identity, and the second one is the matrix

$$\left(\int_{\gamma_k} h_1 dy, \int_{\gamma_k} dh_1, \int_{\gamma_k} h_2 dy\right)_k.$$

If we can take h_2 in such a way that

v) $\left|\int_{\gamma_k} h_2 dy\right| < \mu$ along the paths γ_k,

then, if we previously fix h_1 we have

$$\frac{\partial \rho(0)}{\partial \xi_1} = \frac{\partial \rho(0)}{\partial \xi_2} = O(1) \qquad \frac{\partial \rho(0)}{\partial \xi_3} = O(\mu)$$

and thanks to the formulas for $\frac{\partial}{\partial \xi}\delta H(p,q,0)$ that we obtained before,

vi) $\left|\int_\Gamma g_k dy\right| < 1$ en Γ

vii) $|g_k(p) - g_k(q)| < \mu$

then

$$\frac{\partial}{\partial \xi}\delta H(p,q,\xi)\Big|_{\xi=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix} + O(\mu) \tag{2}$$

and so if μ is small enough, H satisfies c). It is therefore enough to check that the functions h_j, g_k can indeed be chosen to satisfy properties i)-iv). To do so, one starts selecting the paths in such a way that Γ is disjoint from the γ_k, and $\gamma_1 \cup \ldots \cup \gamma_q \cup \Gamma$ is Runge.
Then we find a holomorphic function h_1 that satisfies ii) (which can be done, for instance, by Weierstrass theorem 1.3.11) and we let it be fixed. there is some μ_0 such that if $\mu < 0$, condition i) ensures that the matrix $\int_{\gamma_k} g_j dy$ has norm between 1/2 and 2. BY successive uses of 2.1.8, we find continuous functions in $\gamma_1 \cup \ldots \cup \gamma_q \cup \Gamma$ that verify conditions i), iii)-vii). Since these conditions are of open nature, by Mergelyan theorem we may assume that such functions are holomorphic in X, and since the bound $O(\mu)$ that appears only depends on g and its differential, h_1 and of the length of Γ if $\mu < \mu_0$, they don't change after using Mergelyan theorem. Finally, letting $\mu \to 0$, we can ensure condition c). Step 2: We can assume that there is an open set $Z \subset\subset X$ containing $\overline{Y}$ such that f is a legendrian immersion in $\overline{Z}$, without constant component functions.
We are going to improve condition c) in Step 1. Let $H_g^{p,q}$ be the family of function obtained before, given $p, q \in \overline{Y}$ and $g : X \to \mathbb{C}^3$ legendrian and without constant component functions. Since the submersion is an open condition, there is a neighbourhood $V_{p,q} \subset X \times X$ such that $\delta H_f^{p,q}(a,b,\cdot)$ is a submersion in 0 for all $(a,b) \in V_{p,q}$. Thanks to 2.1.2 we find a neighbourhood U (which we can assume has smooth boundary) of the diagonal

$$D_{\overline{Y}} = \{(x,x) : x \in \overline{Y}\} \subset X \times X$$

and an $\varepsilon > 0$ such that

$$\begin{array}{c}\text{if } \|f-g\|_{\overline{Z}} \leq \varepsilon \text{ then } g \text{ is an immersion in } \overline{Y} \text{ and} \\ \delta g \text{ only takes the value 0 in } U \text{ in the points of the diagonal.}\end{array} \tag{$*$}$$

The set $\overline{Y}^2 \setminus U$ is compact, so it can be covered by finitely many $V_{p,q}$, which we name $V_{p_1,q_1}, \ldots, V_{p_N,q_N}$. Now we construct functions $H_k : X \times B(0,r_k) \to \mathbb{C}^3$, where $B(0,r_k) \subset \mathbb{C}^{3k}$, in an inductive way for $1 \leq k \leq N$ as follows:

$$H_1(u,\eta_1) = H_f^{p_1,q_1}(u,\eta_1)$$

$$H_k(u,\eta_1,\ldots,\eta_k) = H^{p_k,q_k}_{H_{k-1}(\cdot,\eta_1,\ldots,\eta_{k-1})}(u,\eta_k),$$

where $\eta_j \in \mathbb{C}^3$. Note that H_k is legendrian in u by construction and that $H_k(\cdot,\eta_1,\ldots,\eta_k)$ approximates f uniformly in $\overline{Y}$ and $|\eta|$, so for small enough η, $H^k(\cdot,\eta)$ has non-constant component function and therefore we can carry on the construction of the previous step. On top of that, $H_k(\cdot,0,\ldots,0) = f$ by construction, so for all j,

$$\begin{aligned} H_N(u,0,\ldots,\eta_j,\ldots,0) &= H^{p_N,q_N}_{H_{N-1}(\cdot,0,\ldots,\eta_j,\ldots,0)}(u,0) = H_{N-1}(u,0,\ldots,\eta_j,\ldots,0) = \\ &= H^{p_{N-1},q_{N-1}}_{H_{N-2}(\cdot,0,\ldots,\eta_j,\ldots,0)}(u,0) = H_{N-2}(u,0,\ldots,\eta_j,\ldots,0) = \\ &= \ldots = \\ &= H_j(u,0,\ldots,\eta_j) = H^{p_j,q_j}_{H_j(\cdot,0,\ldots,0)}(u,\eta_j) = H^{p_j,q_j}_f(u,\eta_j), \end{aligned}$$

and so

$$\frac{\partial}{\partial\eta_j}\delta H_N(a,b,\eta_1,\ldots,\eta_n)\bigg|_{\eta=0} = \frac{\partial}{\partial\eta_j}\delta H_N(a,b,0,\ldots,\eta_j,\ldots,0)\bigg|_{\eta_j=0} = \frac{\partial}{\partial\eta_j}\delta H^{p_j,q_j}_f(a,b,\eta_j)\bigg|_{\eta_j=0}$$

which has range 3 if $(a,b) \in V_{p_j,q_j}$. Since the V_{p_j,q_j} cover $\overline{Y}^2 \smallsetminus U$, and $H = H_N$, we will have found a holomorphic map $H : X \times B(0,r) \to \mathbb{C}^3$, where $B(0,r) \subset \mathbb{C}^{3N}$, such that

a) $H(\cdot,0) = f$.

b) $H(\cdot,\eta)$ is a legendrian immersion in $\overline{Y}$ without constant component functions, for all η

c) $\delta H(a,b,\cdot) : B(0,r) \to \mathbb{C}^3$ is a submersion in 0 for all $a,b \in \overline{Y}\times\overline{Y}\smallsetminus U$

As before, $H(\cdot,\eta)$ converges to f uniformly in $\overline{Z}$ and $|\eta|$ a f, so condition b) is ensured after reducing r, due to $(*)$.

Step 3: Since the submersion condition is of open nature, for all $(p,q,0) \in X \times \mathbb{C}^{3N}$ we can find a neighbourhood V of it such that $\delta H(a,b,\cdot)$ is a submersion in η, if $(a,b,\eta) \in V$. Since $\overline{Y}\times\overline{Y}\smallsetminus U$ is compact, finitely many of them cover it, so we can find an $r' < r$ such that the modified condition

c') $\delta H(a,b,\cdot) : \mathbb{C}^{3N} \to \mathbb{C}^3$ is a submersion in η if $(a,b,\eta) \in (\overline{Y}^2 \smallsetminus U) \times B(0,r')$

holds. If $M = (\overline{Y}^2 \smallsetminus U)$, M is a smooth manifold with boundary, and $\delta H : M \times B(0,r') \to \mathbb{C}^3$ and $b\delta H : bM \times B(0,r') \to \mathbb{C}^3$ are transverse to any submanifold of $\mathbb{C}^3$ (because the differential is surjective onto $\mathbb{C}^3$). In particular, they are transverse to 0, so using B.2, for generic points $\eta \in B(0,r')$, the map

$$G_\eta = \delta H(\cdot,\cdot,\eta) : M \to \mathbb{C}^3$$

is transverse to 0, but in these cases, counting dimensions with B.1,

$$\dim(M) - \dim(G_\eta^{-1}(\{0\})) = \dim(\mathbb{C}^3) - \dim(\{0\}) = 3.$$

However, $\dim(M) = 2$, so $\delta H(\cdot,\cdot,\eta)$ has to omit 0. If $H(\cdot,\eta)$ is the corresponding map and η is small enough so that $\|H(\cdot,\eta) - f\|_{\overline{Z}} \leq \varepsilon$, where ε is the one $(*)$, then $\delta H(\cdot,\eta)$ omits 0 in $\overline{Y}^2 \smallsetminus D_{\overline{Y}}$ also, so $H(\cdot,\eta)$ is injective in $\overline{Y}$. Since η can be taken as small as one wishes, f can be approximated uniformly in $\overline{Y}$ by legendrian embeddings without constant component functions $\square$

2.4 Legendrian approximation in admissible sets

We will prove now two result which will be the crucial for the recursive step in the proof of Theorem 1. We start proving how to approximate generalised legendrian curve by ordinary legendrian curves:

Theorem 2.4.1. *Let X be a Riemann surface, $Y \subset\subset X$ an open, connected open set and $S \subset Y$ an admissible subset which is a deformation retract of Y. Any generalised legendrian curve $f : S \to \mathbb{C}^3$ can be approximated uniformly in S by legendrian curves defined in Y, having no constant component function. Furthermore, if $f = (x, y, z)$ and x (resp. y) is non-constant, and holomorphic in a neighbourhood of $\overline{Y}$, we can assume that the first (resp. second) component of the functions approximating f is in fact x (resp. y).*

Proof. Since Y is connected, $S = K \cup \Gamma$ is also connected. In a similar way to 2.1.8, we can find paths $\gamma_1, \ldots, \gamma_q$ contained in S, whose union is Runge in Y and that have a subarc contained in $\operatorname{int}(K)$.

If f is non-constant, since it s legendrian and $\operatorname{int}(K)$ is nonempty, using the involution (A5) we can assume that y is non-constant in the interior of K, so dy is not identically 0 in some subarc of the γ_j. Therefore we can apply 2.1.7 to get continuous functions g_j in $\gamma_1 \cup \ldots \cup \gamma_q$ with compact support such that

$$\int_{\gamma_j} g_k dy = \delta_{j,k}.$$

We use Mergelyan theorem with approximation up to order 1 to obtain sequences $x^{(n)}$, $y^{(n)}$ of holomorphic functions in Y that converge to x and y in the $\mathbb{C}^1$-norm, and the usual Mergelyan theorem to approximate g_j in $\gamma_1 \cup \ldots \cup \gamma_q$ by holomorphic functions $g^{(n)}$ defined in Y. Let

$$\mathcal{P}(a, b) = \left(\int_{\gamma_j} adb \right)_{j=1,\ldots,q}$$

Since we have approximation of order 1, the spray

$$\varphi^n(s) = \mathcal{P}\left(x^{(n)} + \sum_{k=1}^{q} s_k g_k^{(n)}, y^{(n)} \right)$$

has $\varphi(s) = \mathcal{P}\left(x + \sum_{k=1}^q s_k g_k, y\right)$ as core, is dominant in 0by the choice of the g_k and $\varphi(0) = 0$ because f a generalised legendrian curve. Therefore, by 2.1.4 we can find for all n a $s^{(n)}$ close to 0 sch that $\varphi^n(s^{(n)}) = 0$ and so, since $\gamma_1, \ldots, \gamma_q$ form a basis of $H_1(Y, \mathbb{Z})$, the maps

$$\left(x^{(n)} + \sum_{k=1}^{q} s_k^{(n)} g_k^{(n)}, y^{(n)}, z(p_0) + \int_{p_0}^{\cdot} \left[x^{(n)} + \sum_{k=1}^{q} s_k^{(n)} g_k^{(n)} \right] dy^{(n)} \right)$$

can be defined in the whole Y, are holomorphic and approximate (x, y, z) in S because we are using $\mathcal{C}^1$-approximation and using the argument in Idea 6.
If y is already holomorphic and non-constant we can take $y^{(n)} = y$ for all n. If x is holomorphic, we use the involution in (A5).
If f is constant, we can repeat the argument in the beginning of 2.3.1. □

The following theorem is an analogue of the theorem by Gromov in [Gro86] that continuous real curves in $\mathbb{R}^3$ can be approximated uniformly by legendrian curves. We will adapt the proof of the result by Gromov given in [HMW17] to the complex case to obtain our lemma:

Lemma 2.4.2. *Any smooth curve $\gamma = (x, y, z) : [0, 1] \to \mathbb{C}^3$ can be approximated in $[0, 1]$ by embedded legendrian curves $\lambda : [0, 1] \to \mathbb{C}^3$ such that $\lambda(0) = \gamma(0)$ and $\lambda(1) = \gamma(1)$.*
In fact, if $\gamma'(0), \gamma'(1) \in \Psi$, where $\mathcal{P}if = \ker(xdy - dz)$, we can assume that $\lambda'(0), \lambda'(1) \in \Psi$.

Proof. Let

$$\mathcal{R}_{t,\varepsilon} = \{(u, v) \in \mathbb{C}^2 : |v - x(t)u| \leq \varepsilon \max\{|u|, |u^2|\}\}.$$

If we can find a curve $(b, c) : [0, 1] \to \mathbb{C}^2$ such that $(b'(t), c'(t)) \in \mathcal{R}_{t,\varepsilon}$ then, defining $a(t) = \frac{c'(t)}{b'(t)}$ we would have $\|a - x\|_{[0,1]} \leq \varepsilon$. To show that we can do so, we reduce again to finding a family of functions $\gamma(t, s) : [0, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}^2$ such that $\gamma(t, \cdot) \in \mathcal{R}_{t,\varepsilon}$, because we can then define

$$(b(t), c(t)) = (y(0), z(0)) + \int_0^t \gamma(u, nu)du. \tag{2.1}$$

If we also have

$$\int_{\mathbb{R}/\mathbb{Z}} \gamma(t, s)ds = (y'(t), z'(t)) \tag{2.2}$$

Then as $n \to \infty$, (b, c) approximates (y, z), because, by the periodicity of γ in its second variable, the change $v = nu + (k - 1)$, and the mean value theorem,

$$|(b(t), c(t)) - (y(t), z(t))| = \left|\int_0^t \left[\gamma(u, nu) - \int_0^1 \gamma(u, v)dv\right] du\right| =$$

$$= \left|\sum_{k=1}^{\lfloor tn \rfloor} \int_{(k-1)/n}^{k/n} \gamma(u, nu) - \left[\int_0^1 \gamma(u, v)dv\right] du + \int_{\lfloor tn \rfloor/n}^{t} \int_0^1 \gamma(u, nu) - \gamma(u, v)dvdu\right| \leq$$

$$\leq \left|\sum_{k=1}^{\lfloor tn \rfloor} \frac{1}{n} \int_0^1 \gamma(\frac{v + k - 1}{n}, v)dv - \int_{(k-1)/n}^{k/n} \int_0^1 \gamma(u, v)dvdu\right| + (t - \lfloor tn \rfloor /n)\|\gamma\|_{[0,1]\times\mathbb{R}/\mathbb{Z}} =$$

$$= \left|\sum_{k=1}^{\lfloor tn \rfloor} \int_{(k-1)/n}^{k/n} \int_0^1 \left[\gamma(\frac{v + k - 1}{n}, v) - \gamma(u, v)\right] dvdu\right| + (t - \lfloor tn \rfloor /n)\|\gamma\|_{[0,1]\times\mathbb{R}/\mathbb{Z}} \leq$$

$$\leq \|\frac{\partial}{\partial t}\gamma(\cdot, \cdot)\|_{[0,1]\times\mathbb{R}/\mathbb{Z}} \frac{\lfloor tn \rfloor}{n^2} + (t - \lfloor tn \rfloor /n)\|\gamma\|_{[0,1]\times\mathbb{R}/\mathbb{Z}},$$

which converges to 0 if $n \to \infty$, uniformly in t.
Conditions (1) y (2) can be attained because the convex envelope $\mathcal{R}_{t,\varepsilon}$ is $\mathbb{C}^2$, although in this case we can find an explicit formula for γ:

$$\gamma(t, s) = \left(r \sin 2\pi s + y'(t), (r \sin 2\pi s + y'(t)) \left[x(t) + \frac{2(z'(t) - x(t)y'(t))}{r^2 + 2x'(t)^2}(r \sin 2\pi s + y'(t))\right]\right)$$

For r big enough, also, if (x, y, z) was legendrian inn 0 y 1, we have

$$(b'(0), c'(0)) = \gamma(0, 0) = (y'(0), z'(0))$$

$$(a(0), b(0), c(0)) = \left(\frac{z'(0)}{y'(0)}, y(0), z(0)\right) = (x(0), y(0), z(0))$$

$$(b'(1), c'(1)) = \gamma(1, n) = (y'(1), z'(1))$$

$$(a(1), b(1), c(1)) = \left(\frac{z'(1)}{y'(1)}, (y(0), z(0)) + \int_0^1 \gamma(u, nu)du\right) = (x(1), y(1), z(1))$$

and we can carry out the calculations for $a'(0)$ and $a'(1)$. □

We note that this is an instance of the h-principle, and the fact that $\mathcal{R}_{t,\varepsilon}$ has $\mathbb{C}^2$ as convex envelope would let us use the theorem of convex integration in [EM02], so the h-principle is true in all of its versions. In particular, in its relative version, and from this version follows our lemma. However, we prefer not to prove the result this way, as the machinery necessary to understand the h-principle exceeds by far the objectives and extension of this work.

The following result is the one requiring the most technical difficulty but it ensures that we can control the behaviour of our legendrian curves so they don't fold. In order to visualise the proof it is convenient to remember that if $Y \subset\subset Z$ and Z is a deformation retract of Y, $\overline{Z} \setminus Y$ is a finite union of rings.

Theorem 2.4.3. *Let $Y \subset\subset Z \subset\subset X$ be open subsets with smooth boundary of an open Riemann surface X, such that Y is a deformation retract of Z. If $f = (x, y, z)$ is a legendrian map defined in some neighbourhood of $\overline{Y}$ such that*

$$\max\{|x|, |y|\} > \mu \text{ en } bY$$

for some $\mu > 0$. Then f can be approximated uniformly in $\overline{Y}$ by legendrian maps $\widetilde{f} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ defined in a neighbourhood of $\overline{Z}$ such that

i) $\max\{|\widetilde{x}|, |\widetilde{y}|\} > \mu + 1$ *en bZ.*

ii) $\max\{|\widetilde{x}|, |\widetilde{y}|\} > \mu$ *en $\overline{Z} \setminus Y$.*

Proof. Lets assume, in order to simplify the notation, that $A = \overline{Z} \setminus Y$ consists of only one ring. Only in this proof we will use subindices to denote the components of a function: $F = (F_1, F_2, F_3) : X \to \mathbb{C}^3$. Let Ψ be the standard contact structure in $\mathbb{C}^3$.
The hypotheses of the theorem and compactness of bY allows us to find points $p_1, \ldots, p_n$ such that there are arcs $\alpha_1, \ldots, \alpha_n$ such that p_k is an extreme point for α_k and α_{k-1}, where integers are taken modulo n, and such that

$$\left.\begin{array}{l} |x| > \mu \\ |y| > \mu \end{array}\right\} \text{ en } \alpha_k \text{ if } \left\{\begin{array}{l} k \text{ is odd} \\ k \text{ is even} \end{array}\right. \tag{1}$$

Note that if $|y| > \mu$ in α we can reduce to the case $n = 1$ using the involution in (A5). On the other hand, since p_k is both in α_k and α_{k-1},

$$\max\{|x(p_k)|, |y(p_k)|\} > \mu \text{ if } n > 1 \tag{2}$$

We take points $q_k \in bZ$ and paths $\gamma_k : [0, 1] \to \overline{Z} \setminus Y$ joining p_k with q_k. AS before, the points q_k divide β into subarcs β_k, and we can decompose

$$A = \bigcup_{k=1}^{n} \overline{\Omega}_k = \bigcup_{k=1}^{n} \Omega_k \cup \alpha_k \cup \beta_k \cup \gamma_k$$

Where Ω_k is the open set limited by $\alpha_k, \beta_k, \gamma_k, \gamma_{k-1}$. (See Figure 5).

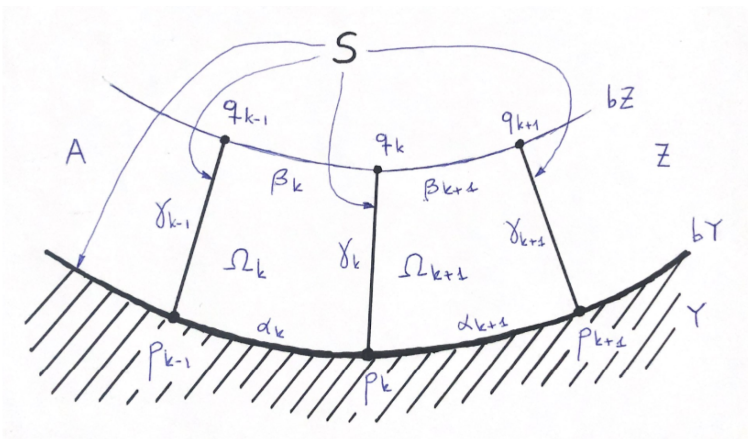

Figure 5: Sets in the proof of 2.4.3

We look for smooth paths $r_k : \gamma_k \to \mathbb{C}^3$ such that $r_k(p_k) = f(p_k)$, $r_k'(p_k) = f_*(\gamma'(p_k))$, $r_k'(q_k) \in \Psi_{r_k(q_k)}$,

$$r_k(\gamma_k) \subset \{(x, y, z) \in \mathbb{C}^3 : \min\{|x|, |y|\} > \mu\} \left(\text{resp } r_1(\gamma_1) \subset \{(x, y, z) \in \mathbb{C}^3 : |x| > \mu\} \text{ if } n = 1\right),$$

$$r_k(q_k) \in \{(x, y, z) \in \mathbb{C}^3 : \min\{|x|, |y|\} > \mu + 1\} \left(\text{resp } r_1(\gamma_1) \in \{(x, y, z) \in \mathbb{C}^3 : |x| > \mu + 1\} \text{ if } n = 1\right).$$

This can be done because the sets appearing are open subsets of $\mathbb{C}^3$ and by condition (1). Now, after using 2.4.2 we can assume that the paths r_k are indeed legendrian so we have a generalised legendrian map defined in the admissible set

$$S = \overline{Y} \cup \gamma_1 \cup \ldots \cup \gamma_n$$

that can be seen in Figure 5. Using the theorem of approximation over admissible sets 2.4.1, we get a legendrian map $g = (g_1, g_2, g_3)$ without constant components, defined in a neighbourhood of $\overline{Z}$ such that

(B1) g approximates f uniformly in $\overline{Y}$

(B2') $\min\{|g_1(q_k)|, |g_2(q_k)|\} > \mu + 1$ for all k (resp. $|g_1(q_1)| > \mu + 1$ if $n = 1$)

(B3') $\min\{|g_1(u)|, |g_2(u)|\} > \mu$ for all $u \in \gamma_k$ and for all k (resp. $|g_1(u)| > \mu + 1$ for all $u \in \gamma_1$ if $n = 1$)

(B4') $|g_j(u)| > \mu$ if $u \in \alpha_k$, $j \in \{1, 2\}$ and j, k share parity

The function g is continuous and conditions B2', B3', B4' are of open nature, so they occur in open neighbourhoods of the sets appearing. Therefore, there are sets $T_k, R_k,\ \lambda_k$ (see Figure 6) such that $\overline{\Omega}_k = \overline{R}_k \cup \overline{T}_k$ and $\lambda_k = \overline{R}_k \cap \beta_k$, and we have the improved conditions

(B2) $\min\{|g_1(u)|, |g_2(u)|\} > \mu + 1$ if $u \in \lambda_k$, (resp. $|g_1(u)| > \mu + 1 \mathrm{if} u \in \lambda_1$ when $n = 1$)

(B3) $|g_j(u)| > \mu$ if $u \in \overline{R}_k$, $j \in \{1, 2\}$ and j, k share parity.

Let δ_k be paths joining T_k con α_k

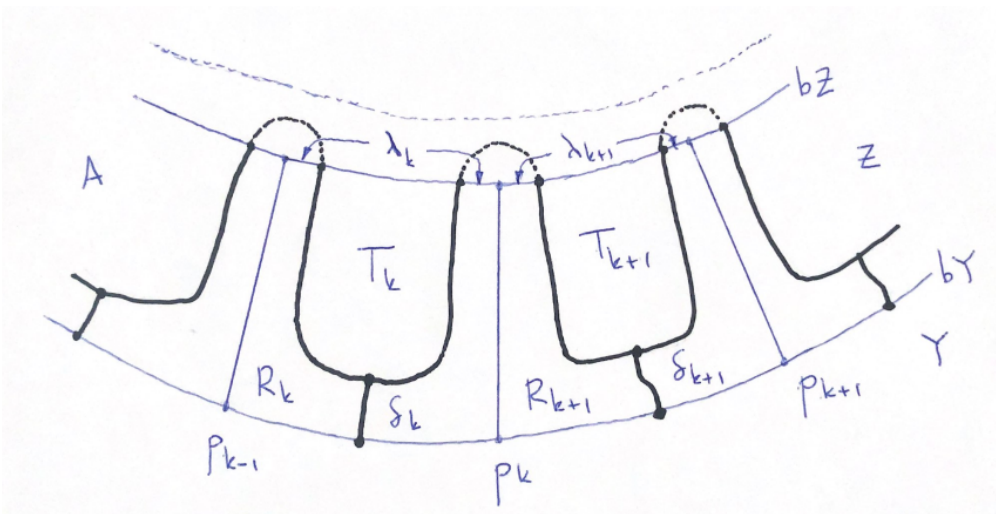

Figure 6: Sets in the proof of 2.4.3

We distinguish two cases:
If $n = 1$, we consider the set

$$S' = \overline{Y} \cup \delta_1 \cup \overline{T}_1,$$

which is admissible, and the map $\widehat{g} = (\widehat{g}_1, \widehat{g}_2, \widehat{g}_3) : S' \to \mathbb{C}^3$ given by

$$\widehat{g}_1 = g_1 \tag{3.1}$$

$$\widehat{g}_2 = \begin{cases} g_2 & \text{in } \overline{Y} \\ \text{Any holomorphic function with } |\widehat{g}_2(u)| > \mu + 1 & \text{in } T_1 \\ \text{Any smooth function making } \widehat{g}_2 \text{ of class } \mathcal{C}^1 & \text{in } \delta_1 \end{cases} \tag{3.2}$$

$$\widehat{g}_3 = g_3(p_0) + \int_{p_0}^{p} \widehat{g}_1 d\widehat{g}_2 \tag{3.3}$$

Where $p_0 \in Y$ is any point. The integral is independent of the path because in $\overline{Y}$, $\widehat{g}_1 d\widehat{g}_2 = g_1 dg_2$ so $\widehat{g}_3 = g_3$, and any curve joining p_0 with $\delta_1 \cup T_1$ has to go through $p_1 = \delta_1 \cap \alpha_1$. Since $\delta_1 \cup T_1$ is simply connected, we conclude that the integral is independent of the path. Therefore, $\widehat{g}$ is a generalised legendrian map and g_1 is holomorphic and non-constant in a neighbourhood of $\overline{Z}$.
Since S' is a deformation retract of $\overline{Z}$ and so of some neighbourhood of it, by 2.4.1 there is a map $h = (h_1, h_2, h_3)$ in a neighbourhood of $\overline{Z}$ such that

(C1) $h_1 = \widehat{g}_1$.

(C2) h approximates $\widehat{g}$ in S'.

In this case, (B3) and (C1) prove that $|h_1| > \mu$ in $\overline{R}_1$; (B2) and (C1) prove that $|h_1| > \mu + 1$ in λ_1;if the approximation in (C2) is string enough, the way we chose $\widehat{g}_2$ in (3.2) proves that $|h_2| > \mu + 1$ in $\overline{T}_2$; and finally, (C2), (B1) and the fact that $\widehat{g} = g$ in $\overline{Y}$ prove that h approximates f, so it is enough to take $\widetilde{f} = h$.

If $n > 1$, let's suppose that k walks through odd indices and j through even ones. We consider the admissible set

$$S_1 = \overline{Y} \cup \bigcup_k (\overline{T}_k \cup \delta_k) \cup \bigcup_j \overline{\Omega}_j$$

and construct the function $\widehat{g} : S_1 \to \mathbb{C}^3$ given by

$$\widehat{g}_1 = g_1. \tag{4.1}$$

$$\widehat{g}_2 = \begin{cases} g_2 & \text{in } \overline{Y} \cup \bigcup_j \overline{\Omega}_j \\ \text{Any holomorphic function } |\widehat{g}_2(u)| > \mu + 1 & \text{in } \bigcup_k T_k \\ \text{Any smooth function making } \widehat{g}_2 \text{ of class } \mathcal{C}^1 & \text{in } \bigcup_k \delta_k \end{cases}. \tag{4.2}$$

$$\widehat{g}_3 = g_3(p_0) + \int_{p_0}^{p} \widehat{g}_1 d\widehat{g}_2. \tag{4.3}$$

In an analogous way to the constructions in (3.1), (3.2) and (3.3), conditions (4.1), (4.2) y (4.3) make $\widehat{g}$ a generalised legendrian map with non-constant first component and so, using 2.4.1, we find a legendrian map with non-constant components h in a neighbourhood of $\overline{Z}$ such that

(D1) $h_1 = \widehat{g}_1$.

(D2) h approximates $\widehat{g}$ in S_1.

We repeat this construction exchanging S_1 with

$$S_2 = \overline{Y} \cup \bigcup_j (\overline{T}_j \cup \delta_j) \cup \bigcup_k \overline{\Omega}_k,$$

and the definitions (4.1), (4.2), (4.3) with

$$\widehat{h}_1 = \begin{cases} h_1 & \text{in } \overline{Y} \cup \bigcup_k \overline{\Omega}_k \\ \text{Any holomorphic function such that } |\widehat{h}_1(u)| > \mu + 1 & \text{en } \bigcup_j T_j \\ \text{Any smooth function making } \widehat{h}_1 \text{ of class } \mathcal{C}^1 & \text{in } \bigcup_j \delta_j \end{cases}. \tag{5.1}$$

$$\widehat{h}_2 = h_2. \tag{5.2}$$

$$\widehat{h}_3 = h_3(p_0) + \int_{p_0}^{p} \widehat{h}_1 d\widehat{h}_2. \tag{5.3}$$

We use again 2.4.1 to obtain a legendrian map b defined in a neighbourhood of $\overline{Z}$ such that

(E1) $b_2 = \widehat{h}_2$.

(E2) b approximates $\widehat{h}$ in S_2.

Recapitulating everything,

$$|b_2| \overset{E1}{=} |\widehat{h}_2| \overset{5.2}{=} |h_2| \overset{D2}{\approx} |\widehat{g}_2| \overset{4.2}{=} |g_2| \overset{B2}{>} \mu + 1 \text{ in } \lambda_j,$$

$$|b_1| \overset{E2}{\approx} |\widehat{h}_1| \overset{5.1}{=} |h_1| \overset{D1}{=} |\widehat{g}_1| \overset{4.1}{=} |g_1| \overset{B2}{>} \mu + 1 \text{ in } \lambda_k.$$

$$|b_2| \overset{E1}{=} |\widehat{h}_2| \overset{5.2}{=} |h_2| \overset{D2}{\approx} |\widehat{g}_2| \overset{4.2}{>} \mu + 1 \text{ in } \overline{T}_k,$$

$$|b_1| \overset{E2}{\approx} |\widehat{h}_1| \overset{5.1}{>} \mu + 1 \text{ in } \overline{T}_j,$$

$$|b_2| \overset{E1}{=} |\widehat{h}_2| \overset{5.2}{=} |h_2| \overset{D2}{\approx} |\widehat{g}_2| \overset{4.2}{=} |g_2| \overset{B3}{>} \mu \text{ in } \overline{R}_j,$$

$$|b_1| \overset{E2}{\approx} |\widehat{h}_1| \overset{5.1}{=} |h_1| \overset{D1}{=} |\widehat{g}_1| \overset{4.1}{=} |g_1| \overset{B3}{>} \mu \text{ in } \overline{R}_k,$$

so taking into account (as can be seen in Figure 6) that,

$$bZ \subset \bigcup_{m=1}^{n} \lambda_m \cup \overline{T}_m \qquad \overline{Z} \setminus Y \subset \bigcup_{m=1}^{n} \overline{R}_m \cup \overline{T}_m$$

if the approximations (B1), (D2), (E2) are strong enough, we can take $\widetilde{f} = b$. $\square$

2.5 Proof of Theorems 1 and 2

Using all the ideas and results in the last three sections, we can prove Theorems 1 and 2. In fact, we prove stronger versions of both:

Theorem 2.5.1 (Alarcón-Fonstneric-López)**.** *If X is an open Riemann surface, $S \subset X$ is an admissible Runge set and $f : S \to \mathbb{C}^3$ a generalised legendrian map, f can be approximated uniformly in S by legendrian, proper embeddings without constant component $g = (x, y, z)$ such that $(x, y) : X \to \mathbb{C}^2$ is also proper.*

Proof. By 2.4.1 and 2.3.2, there is an open set M_0 with smooth boundary of which S is a deformation retract and we can approximate f in S by a legendrian embedding f_0 without constant components, defined in a neighbourhood of $\overline{M}_0$. Since the component maps are non-constant, their zeros are isolated so after a slight edition of the boundary of M_0, we can assume that there is a $\mu > 0$ such that $\max\{|x_0|, |y_0|\} \geq \mu$ in bM_0. Note that M_0 is Runge
Let M_k be an exhaustion by open sets with smooth boundary as in Idea 1. This is:

a) M_k is Runge in X

b) $M_k \subset\subset M_{k+1}$

c) M_k is a deformation retract of M_{k+1} or $M_k \cup \alpha_k$ is a deformation retract of M_{k+1} where α_k is a curve with endpoints in M_k.

Given $\varepsilon > 0$, with $\varepsilon < \frac{\mu}{2}$, we will construct inductively holomorphic maps f_k defined in some neighbourhood of $\overline{M}_k$ and an ε_k in such a way that:

(1_k) $\|f_k - f_{k-1}\|_{\overline{M}_{k-1}} \leq \frac{1}{2^k} \min\{\varepsilon, \varepsilon_0, , \ldots, \varepsilon_{k-2}\}$.

(2_k) f_k is an embedding in $\overline{M}_k$ and if $\|g - f_k\|_{\overline{M}_{k-1}} \leq \varepsilon_k$ then g is an embedding in $\overline{M}_{k-2}$.

(3_k) $\max\{|x_k|, |y_k|\} > \mu + k$ in bM_k.

(4_k) $\max\{|x_k|, |y_k|\} > \mu + k - 1$ in $\overline{M}_k \setminus M_{k-1}$.

The map f_0 already satisfies the conditions, so we prove the inductive step, considering two cases:

If M_k is a deformation retract of M_{k+1}, we use, in this order, theorems 2.4.3, 2.3.1 and 2.3.2, we obtain an embedding f_{k+1} approximating f_k in $\overline{M_k}$. If the approximation given by 2.3.2 and 2.3.1 are strong enough, the conclusions i) y ii) of 2.4.3 are still preserved and therefore f_{k+1} can be taken to satisfy (3_{k+1}) and (4_{k+1}). Since f_{k+1} approximates f_k, we also obtain (1_k).

If $M_k \cup \alpha_k$ is a deformation retract of M_{k+1} and p, q are the endpoints of α, by (4_k) we have

$$f_k(p), f_k(q) \in \{(x, y, z) : |x| > \mu + k \text{ o } |y| > \mu + k\},$$

which is open and connected so they can be joined by a smooth map which is legendrian in these two points, so by 2.4.2, we can extend f_k to $M_k \cup \alpha_k$, which is admissible. If we use 2.4.1 in M_{k+1}, we obtain a legendrian map $\widetilde{f}_{k+1}$ defined in M_{k+1} that approximates f_k in $\overline{M}_k$ and verifies

$$\max\{|\widetilde{x}|, |\widetilde{y}|\} > \mu + k \text{ en } bM_k \cup \alpha_k.$$

Since this it is a continuous map, there is an open set W such that $\overline{M}_k \cup \alpha_k \subset W$, W is a deformation retract of M_{k+1} and the previous inequality also occurs in bW. As before, if we use 2.4.3, 2.3.1 and 2.3.2 in the given order, we obtain the desired legendrian map f_{k+1}a.
Note that in both cases, ε_k is the one given by 2.1.3.

By (1_k), the sequence f_k converges uniformly over compact subsets of X to a holomorphic map f. Since df_k converges to df by 2.1.1, f is legendrian. Furthermore,

$$\|f - f_{k+1}\|_{M_k} \leq \| \sum_{j=k+1}^{\infty} f_{j+1} - f_j\|_{M_k} \leq \sum_{j=k+1}^{\infty} \frac{\varepsilon_k}{2^{j+1}} \leq \varepsilon_k$$

So, due to (2_{k+1}), f is a legendrian embedding in $\overline{M}_{k-1}$, for all k. Since X is the union of all the M_k, it follows that f is an injective, legendrian immersion. Finally, by (1_k)

$$\|f - f_k\|_{\overline{M}_k} \leq \sum_{j=k}^{\infty} \|f_{j+1} - f_j\|_{\overline{M}_j} \leq \varepsilon \leq \frac{\mu}{2},$$

and thus, using (4_k),

$$\max\{|x|, |y|\} \geq \max\{|x_k|, |y_k|\} - \frac{\mu}{2} \geq k - 1 + \frac{\mu}{2} \geq k - 1 \text{ en } \overline{M}_k \setminus M_{k-1}$$

for all k. In particular, if $|x(p)|, |y(p)| \leq N$, $p \in \overline{M}_N$, which proves that (x, y) is a proper map and so is f then. Since it is a proper, injective, holomorphic immersion it is an embedding (as can be seen in proposition 4.22 in [Lee12]).
Finally, $\|F - f_0\|_{\overline{M}_0} \leq \varepsilon$, which was arbitrary, so since f_0 approximates f, F does so. □

As a corollary, we have an analogue of Whitney embedding theorem:

Corollary 2.5.2. *Any open Riemann surface can be properly embedded in* $\mathbb{C}^3$.

The proof of 2 follows from 2.3.2 and the Darboux theorems:

Theorem 2.5.3. *If X is a bordered Riemann surface, (M, Ψ) is a complex, contact manifold of dimension 3, then there is a legendrian embedding of X en M.*

Proof. Let R be a Riemann surface such that $X \subset\subset R$ with smooth boundary. The function $(0, 0, 0)$ is legendrian and therefore there is a legendrian embedding $f = (x, y, z) : \overline{X} \to \mathbb{C}^3$ by 2.3.2. Since $\overline{X}$ is compact, $f(X)$ is bounded. By 1.5.4, we can find an open set $U \subset M$ and Darboux coordinates $G : U \to \mathbb{C}^3$. After changing G by λG if necessary, we can assume that $f(X) \subset G(U)$ and therefore, $G^{-1} \circ f : X \to M$ is a legendrian embedding. □

We note that one cannot aim to embed any Riemann surface in any contact manifold. For example, if $M \subset \mathbb{C}^3$ is a bounded open set with the standard contact structure, we cannot embed $\mathbb{C}$ in M because the component functions would be holomorphic an bounded, and therefore constant by Liouville theorem.

Appendix A

Functional analysis

A complex Banach space is a vector space E over the complex numbers, equipped with a norm $\|.\| : E \to [0, +\infty)$ that makes it a complete metric space.
The prototype of a Banach space is $\mathcal{C}(K)$, the space of continuous functions $f : K \to \mathbb{C}$,with the norm

$$\|f\|_K = \sup\{|f(x)| : x \in K\}.$$

Other examples are the spaces $\mathcal{C}^r([0,1])$ of continuous, complex functions with contiunuous derivatives up to order r in $[0,1]$, with the norm

$$\|f\|_{\mathcal{C}^r} = \|f\|_{[0,1]} + \|f'\|_{[0,1]} + \ldots + \|f^{r)}\|_{[0,1]}$$

although $[0,1]$ can be substituted by any compact domain in $\mathbb{R}^n$ with smooth boundary.
A linear map $T : E \to \mathbb{C}$ is said to be a functional. We will say that it is bounded if there is some $C > 0$ such that

$$|Tx| \leq C\|x\|,$$

and the infumum of such C is the norm of T. The set of bounded functionals in E forms the dual space E'. The celebrated theorem of Hahn-Banach allows us to obtain linear functionals with diverse properties:

Theorem A.1 (Hahn-Banach theorem)**.** *Let $M \subset E$ be a linear subspace and $T \in M'$. There is a functional $R \in E'$ extending T and such that $\|R\| = \|T\|$.*

A simple argument that can be found after the proof of the Hahn-Banach theorem in [Rud86] gives the following useful criterion
If E is a Banach space and $M \subset E$ is a subspace, then $x \in \overline{M}$ if an only if for all $T \in E'$ such that $T(M) = 0$, $T(x) = 0$.
In particular, if $M \subset \mathcal{C}(K)$, f is a uniform limit of functions of the family M if and only if for all T in $\mathcal{C}(K)'$ such that $T(M) = 0$, we have $T(f) = 0$.
Is there a way to characterise $\mathcal{C}(K)'$? The answer is yes, and it is known as Riesz representation theorem

Theorem A.2 (Riesz representation Theorem)**.** *If K is a compact Hausdorff space, any linear functional in $T \in \mathcal{C}(K)'$ is represented by a unique regular complex Borel measure μ in K, in the sense that*

$$\int_K f d\mu = T(f)$$

for all $f : K \to \mathbb{C}$.

Recall that a measure μ is said to be orthogonal to a subset S of $\mathcal{C}(K)$ if $\int_K f d\mu = 0$ for all $f \in S$. With A.2 and A.1 in hand, and taking int account what was commented earlier, we obtain the following principle, which we use in the proof of 1.3.7:

If $\mathcal{F} \subset \mathcal{C}(K)$ is a family of functions, then $f \in \mathcal{C}(K)$ belongs to the closure of $\mathcal{F}$ (in other words, there is a sequence f_n contained in $\mathcal{F}$ converging uniformly to f) if and only if for any Borel complex regular measure μ in K which is orthogonal to $\mathcal{F}$, $\int_K f d\mu = 0$.

A complex measure in K is a function $\mu : \mathcal{M} \to \mathbb{C}$, where $\mathcal{M} \subset \mathcal{P}(K)$ is a σ-algebra, which is countably additive for absolutely convergent sums. If $\mu(\mathcal{M}) \subset [0, +\infty)$, μ is said to be positive. Given a complex measure, it is always possible ot find a positive measure $|\mu|$ such that $|\mu|(A) \geq |\mu(A)|$, called the total variation of μ.
There is a reasonable way to define the integral of a measurable function $f : X \to \mathbb{C}$ with respect to a complex measure, but we do not describe it. It can be found in [Rud86].

If K is a topological space, we say that a measure μ is a Borel measure if $\mathcal{M}$ is the smallest σ-algebra containing the open sets. In this case, continuous functions are measurable, so

$$f \mapsto \int_K f d\mu$$

defines a bounded linear functional in $\mathcal{C}(K)$. We say that μ is regular if

$$|\mu|(A) = \inf\{|\mu|(V) : A \subset V, V \text{ open}\} \quad |\mu|(B) = \sup\{|\mu|(K) : K \subset B, K \text{ compact}\},$$

for all $A \in \mathcal{M}$ and all open B. Complex measures can be added and multiplied by scalars or by bounded functions to obtain new complex measure, as we do in the proof of Mergelyan theorem.
A common example of a Borel measures arises when K is a compact submanifold of a Riemann surface and μ is given by the integration with respect to a $(1,1)$-form . We use the theorem of Fubini in repeated occasions. Recall that a function is in $L^1(\mu)$ if $\int_X |f| d|\mu| < \infty$.

Theorem A.3 (Fubini Theorem)**.** *Let μ y λ be Borel measures in the compact spaces X and Y. If $\mu \times \lambda$ is its product measure (which is again a Borel measure), $f : X \times Y \to \mathbb{C}$ is measurable, and we define*

$$f_x(y) = f(x,y) \qquad f_y(x) = f(x,y),$$

then

a) $|\mu \times \lambda| = |\mu| \times |\lambda|$

b) If f, λ, μ are positive, and we define

$$G(x) = \int_Y f_x(y) d\mu(y) \text{ and } H(y) = \int_Y f_y(x) d\lambda(x),$$

G and H are measurable and

$$\int_{X \times Y} f d(\mu \times \lambda) = \int_X G(x) d\mu(x) = \int_Y H(y) d\lambda(y).$$

c) If $f \in L^1(\mu \times \lambda)$ then the functions G and H exist almost everywhere, are in $L^1(\mu)$ and $L^1(\lambda)$ respectively, and

$$\int_{X \times Y} f d(\mu \times \lambda) = \int_X G(x) d\mu(x) = \int_Y H(y) d\lambda(y).$$

The proofs of all of these results can be found along chapters 1,2,5,6,7 and 8 of [Rud86], as well as a great amount of important theorems about measure theory which can be used in complex analysis.

Appendix B

Transversality

Let X, Y be smooth manifolds without border n y m, respectively, and let $Z \subset Y$ be a submanifold of dimension k.
A smooth map $F : X \to Y$ is said to be traversal to Z if for all $p \in X$ one of the following two things happen:

- $F(p) \notin Z$.
- $F(p) \in Z$ y $T_{F(p)}Y = D_pF(T_pX) \oplus T_{F(p)}Z$.

Informally, F is transversal if $F(X)$ intersects Z "not tangently". For example, if $f, g : \mathbb{R} \to \mathbb{R}^2$ are given by

$$f(t) = (t, \sin t) \quad \text{y} \quad g(t) = (t, t^2)$$

then f is transverse to $\mathbb{R} \times \{0\}$ but g is not. On the other hand, submersions are always transverse to any submanifold of Y. The reason why transversality is a desirable property is the following proposition:

Proposition B.1. *If X is allowed to have boundary and both $iF : int(X) \to Y$ and $bF : bX \to Y$ are transverse to Z and $F^{-1}(Z)$ is nonempty, $F^{-1}(Z)$ is a submanifold of X with codimension $m - k$ having $f^{-1}(Z) \cap bX$ as boundary.*

Proof. This is a direct consequence of the regular value theorem. $\square$

The key to finding transverse maps is the following theorem:

Theorem B.2. *[Parametric transversality] If A is a smooth manifold without border, we let X to have border and $F : X \times A \to Y$ is smooth and both , then for almost all $s \in B$, $iF_s = iF(\cdot, s)$ and $bF_s = bF(\cdot, s)$ are transverse to Z, and in particular B.1 applies*

Here, almost all is in the sense of measure theory. We recall that it makes sense to speak about sets of measure 0, as it is explained in [Lee12]. IN particular, measure 0 sets have empty interior, so the conclusions of the previous theorem happen in a dense subset of A. The proof of this theorem uses Sard's theorem and can be found in [ORR20], or in [Lee12] in the particular case when X does not have boundary. The first reference also contain numerous examples, applications and generalisations of the theorem. In [EM02], this theorem is used to prove theorems concerning holonomic approximation and the h-principle.

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