Graded rings and Projective Varieties

Candidate number 1061176

University of Oxford

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Chapter 1

Introduction

Hilbert's Nullstellensatz sweeps under the rug the complicated structure of non-reduced schemes. For finite schemes, for instance, reduced schemes over an algebraically closed field are unions of $\operatorname{Spec} k$, but a non-reduced scheme will look like $Z = \operatorname{Spec} B$ where B is a finite dimensional k-algebra, whose dimension is the degree of Z, $\deg(Z)$.
Any possibility of classification breaks down when $\dim_k B > 6$. For that reason, it is more interesting to study finite schemes embedded into other varieties, usually $\mathbb{P}^n$.
Such an embedding makes allows us to write $Z = \operatorname{Proj} A$ where A is a graded k-algebra. The dimension of the d-th graded part of A is given by the Hilbert function of A, denoted by H_A. This function is strictly increasing until it reaches the value $\deg(Z)$. There is in fact a full characterisation of what functions can appear as the Hilbert function of such algebras.
However, this is almost everything we are certain about. If we write $A = k[x_0, \ldots, x_n]/I$, we have

$$H_A(d) = \binom{d+n}{d} - \dim_k I_d,$$

so H_A gives information on the space of degree d- linear forms that contain $\operatorname{Spec} A$. This is directly related to several interpolation problems. For example, if

$$I = \langle x_1^2, x_2 \rangle \cap \langle x_2^2, x_0 x_2, x_0^2 \rangle,$$

$\dim_k I_d$ is the number of linearly independent, degree d curves that go through $(1 : 0 : 0)$ with tangent line $\mathbb{V}(x_2)$, and are singular at $(0 : 1 : 0)$. One wonders:

- What is the smallest d such that $\dim_k I_d > 0$? (this is denoted by $\alpha(Z)$)
- What is the smallest d such that $\dim_k I_d$ is expected

  $$\dim_k I_d = \binom{d+n}{d} - \deg(Z)$$

  (this is denoted by $\operatorname{reg}(Z)$).

The answer to both questions is unknown in general, but when the ideal I is of the form $I_1^{m_1} \cap \ldots \cap I_s^{m_s}$ (in which case Z is said to be a fat-point scheme) tight bounds for both constants have been found, and it is conjectured that the expected function

$$d \mapsto \min \left\{ \binom{d+n}{d}, \deg(Z) \right\}$$

is indeed the function of A for generic fat points if $n = 2$. This is the Segre-Harbourne-Gimigliano-Hirschowitz conjecture (SHGH).

A somewhat different approach is to study flat families of finite schemes. For example, we might want so see how the coefficients $\operatorname{reg}(Z)$ and $\alpha(Z)$ vary inside such families.
In this context the next class of schemes containing reduced schemes are the ones that can be obtained as a limit of such. These are the smoothable schemes.
It turns out that a family like that is the same as a morphism into the Hilbert scheme of r points, which is a projective variety that parametrizes all finite subschemes $Z \subset \mathbb{P}^n$ with $\deg(Z) = r$, and for instance, the smoothability of finite points translates into the study of the irreducible components of the Hilbert scheme.

The dissertation is structured as follows: In the second chapter the main tools, which are Hilbert functions, Hilbert polynomials, flat families, as well as some theory of monomial ideals are introduced. In the third chapter the main structural results about the Hilbert function of a finite scheme are explained and, in particular, the characterisation mentioned earlier is proved, in a direct way, for subschemes of $\mathbb{P}^2$. Moreover, the proofs of some bounds of $\operatorname{reg}(Z)$ for some classes of fat point schemes that are given in the literature are simplified after reinterpreting $\operatorname{reg}(Z)$. In the fourth chapter the Hilbert scheme is introduced, as well as how it is related to smoothability. A simple proof of a classical theorem of Fogarty, which says that the Hilbert scheme of r points in $\mathbb{P}^2$ is smooth and irreducible is presented here. This implies that all subschemes of $\mathbb{P}^2$ are in fact smoothable. Finally, the statement of the SHGH is explained, and a flat family parametrizing fat point schemes, together with Fogarty's theorem, is used to give an argument motivating the conjecture.

It should also be noted that all schemes are assumed to be separated, of finite type and Noetherian, and the main reference for facts about schemes is Hartshorne's book [Har77], Chapters II, III and V. The main references followed were [CH13] for fat points and the SHGH conjecture, and [Mac07] for Hilbert schemes.

Chapter 2

Hilbert polynomials and other tools

2.1 Hilbert functions

Let's fix some notation first; If R is a graded ring and M a graded R-module, M_d denotes the subgroup of degree d elements in M, and $M[n]$ is the same module with the grading given by $M(n)_d = M_{d+n}$. We will say that R is a standard k-algebra if $R_0 = k$, and R is finitely generated over k by elements in R_1. In this case, R is Noetherian (Proposition 10.7 in [AM69]).
On the other hand, if X is a scheme over a field, $\mathcal{L}$ is a very ample line bundle on X and $\mathcal{F}$ any coherent sheaf on X, $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$. Note that if $X = \operatorname{Proj} R$, M is a graded R-module, R is a standard k-algebra and $\mathcal{L} = \mathcal{O}(1) = R(1)^{\sim}$, then $M^{\sim}(d) = M(d)^{\sim}$.

Definition 2.1. *Let X be a projective scheme over a field k, $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{L}$ a very ample line bundle on X. The Hilbert function of $\mathcal{F}$ relative to $\mathcal{L}$ is*

$$h_{\mathcal{F}}(d) = h^0(X, \mathcal{F}(d)).$$

Note that $h_{\mathcal{F}}(d) < \infty$ because X is projective (see Serre's Theorem on the next page).

If $\mathcal{F} = \mathcal{O}_X$, we will say that $h_{\mathcal{F}}$ is the Hilbert function of X (relative to $\mathcal{L}$) and denote it by h_X

Despite its simple definition, the Hilbert function is very hard to compute. This is because, even if $X = \operatorname{Proj} R$ and $\mathcal{F} = M^{\sim}$, the natural map $M_d \mapsto H^0(X, M(d))$ need not be injective nor surjective. For example, if $M_d = N_d$ for $d > d_0$, the sheaves $M^{\sim}$ and $N^{\sim}$ are isomorphic, and therefore $H^0(X, M^{\sim}(d)) = H^0(X, N^{\sim}(d))$ for all d, even $d \leq d_0$.

Proposition 2.2. [Ex. II.5.19 in [Har77]] *Suppose $X = \operatorname{Proj} R$, where R is a standard k-algebra. Then the functors*

$$\mathcal{F} \mapsto \bigoplus_d H^0(X, \mathcal{F}(d)) \qquad M \mapsto M^{\sim}$$

give an equivalence of categories between coherent sheaves on X and finitely generated, graded R-modules up to torsion ($M = N$ up to torsion if $M_d = N_d$ for large d).

Therefore, if $\mathcal{F} = M^{\sim}$ is a coherent sheaf on X, then $h_{\mathcal{F}}(d) = \dim M_d \; d >> 0$. Moreover, suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact. Then $0 \to \mathcal{F}'(d) \to \mathcal{F}(d) \to \mathcal{F}''(d) \to 0$ is exact for all d because $\mathcal{L}$ is invertible, but $H^0(X, \cdot)$ is only left exact so the Hilbert function is not in general an additive function. In fact, we obtain a sequence

$$0 \to H^0(X, \mathcal{F}'(d)) \to H^0(X, \mathcal{F}(d)) \to H^0(X, \mathcal{F}''(d)) \to H^1(X, \mathcal{F}'(d)) \to \dots$$

and, if we recall Serre's Theorem:

Theorem 2.3. [Theorem II.5.2 in [Har77]] *Let X be a projective scheme and $\mathcal{L}$ a very ample line bundle on X. Then for any coherent sheaf $\mathcal{F}$:*

a) $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space for all i, whose dimension is $h^i(X, \mathcal{F})$.

b) There exists some d_0 such that $H^i(X, \mathcal{F}(d)) = 0$ for all $i > 0$ and $d \geq d_0$.

We see that, $H^1(X, \mathcal{F}'(d)) = 0$ for $d >> 0$, so the Hilbert function if we twist by $\mathcal{L}$ enough times.
These two examples show that the long-term behaviour of the Hilbert function appears to behave nicely. This long-term behaviour is encoded in the Hilbert polynomial.

Definition 2.4. *Let X be a projective scheme over a field k, $\mathcal{F}$ a coherent sheaf on X and $\mathcal{L}$ a very ample sheaf on X. The Hilbert polynomial of $\mathcal{F}$ is*

$$p_{\mathcal{F}}(d) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F}(d)).$$

Note that $p_{\mathcal{F}}(d)$ is finite thanks to Serre's Theorem a) and Grothendieck's Vanishing Theorem (Theorem II.2.7 in [Har77]), which says that all cohomology groups vanish for $i > \dim(X)$. $p_{\mathcal{O}_X}$ is also denoted p_X.

Theorem 2.5. *Let X and $\mathcal{L}$ be as above.*

a) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, then $p_{\mathcal{F}} = p_{\mathcal{F}'} + p_{\mathcal{F}''}$.

b) $p_{\mathcal{F}}(0)$, the Euler characteristic of $\mathcal{F}$, does not depend on $\mathcal{L}$.

c) $p_{\mathcal{F}}(d) = h_{\mathcal{F}}(d)$ for all large d.

d) $p_{\mathcal{F}}$ is a polynomial of degree $\dim \mathcal{F}$.

Proof. a) follows from the long exact sequence in cohomology and the definition of p. b) is because $p_{\mathcal{F}}(0)$ does not depend on any twisting. c) is a consequence of Serre's Theorem.

For d) we induct in $\dim \mathcal{F}$. We can assume that k is infinite using a base change. If $\dim \mathcal{F} = 0$, $\mathcal{F}$ then for all $p \in X$, $\mathcal{F}(d)_p = \mathcal{F}_p \otimes \mathcal{O}_p = \mathcal{F}_p$ because $\mathcal{L}$ is invertible, and so $\mathcal{F}(d)$ is a skyscraper sheaf for all d. In particular, it is acyclic and its global sections are the direct sum of its stalks, so $p_{\mathcal{F}}$ is constant.

Now let $\dim \mathcal{F} \geq 1$ and suppose we have an embedding $X \to \mathbb{P}^n_k$ relative to $\mathcal{L}$. Then, since the associated points of $\mathcal{F}$ are a finite set, there is a hyperplane h in $\mathbb{P}^n$ not going through them (this is the same as having a hyperplane that intersects transversely $\operatorname{Supp} \mathcal{F}$). The restriction of h to X gives a section of $\mathcal{L}$ and so we have an exact sequence

$$0 \to \mathcal{F}(-1) \stackrel{\cdot h}{\to} \mathcal{F} \to \mathcal{G} \to 0,$$

where injectivity follows form the choice of h. Furthermore, $\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap \mathbb{V}(h)$, so by Krull's Haupiensatz (Corollary 11.17 in [AM69]), $\dim \mathcal{G} = \dim \mathcal{F} - 1$, and we finish the induction step because if $f, g : \mathbb{Z} \to \mathbb{Z}$, g comes form a polynomial of degree d and $f(x+1) - f(x) = g(x)$ then f comes from a polynomial of degree $d+1$. □

Even though $p_{\mathcal{F}}$ only attains integer values, it will often have rational coefficients. Such polynomials are called numerical polynomials. Since $d \mapsto \binom{d}{0}, \dots, d \mapsto \binom{d}{n}$ forms a basis over $\mathbb{Z}$ of the numerical polynomials of degree $\leq n$ we see that if $f(d) = a_n n^d + \dots$, then $n! a_n \in \mathbb{Z}$. If $f = p_{\mathcal{O}_X}$, $n! a_n$ is the degree of X. Since $p_{\mathcal{O}_X}(d) = h_X(d)$ for large d, $\deg(X) > 0$.

Example 2.6.

a) If $R = k[x_0, \ldots, x_n]$, and $X = \operatorname{Proj} R = \mathbb{P}^n_k$ then (Prop. 5.13 in [Har77]) $S_d \to H^0(X, \mathcal{O}_X(d))$ is an isomorphism for all d, so $h_{\mathbb{P}^n}(d) = \binom{n+d}{d} = p_{\mathbb{P}^n}(d)$ is already a numerical polynomial, so $\deg \mathbb{P}^n = 1$.

b) If now $X = \mathbb{P}^n$ is embedded in $\mathbb{P}^N$ via the v-th Veronese embedding, $\mathcal{L} = \mathcal{O}(k)$ so $h_X(d) = \dim_k H^0(X, \mathcal{O}_X(vd)) = \binom{n+vd}{n}$ and $\deg X = v^n$.

c) Suppose $i : X \to \mathbb{P}^n$ is a closed embedding and $\mathcal{L} = i^*\mathcal{O}(1)$. Then thanks to the projection formula,

$$H^0\left(\mathbb{P}^n, i_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(d)\right) = H^0\left(\mathbb{P}^n, i_*(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^n}(d))\right) = H^0\left(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}\right),$$

the Hilbert function of $\mathcal{F}$ in X is the same as the Hilbert function of $i_*\mathcal{F}$ in $\mathbb{P}^n$, and so the Hilbert polynomials also agree. This allows one to compute p_X using the classical short exact sequence:

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X \to 0$$

and additivity of the hilbert polynomial, getting $p_X(d) = \binom{n+d}{n} - p_{\mathcal{I}_X}(d)$.

d) If $i : X = V(f) \to \mathbb{P}^n$, where $f \in R$ has degree r. Then the earlier formula says that $p_X(d) = \binom{n+d}{n} - \binom{n+d-r}{n}$ so $\deg(X) = r$.

e) If $Z \to \mathbb{P}^n$ is a 0-dimensional scheme, $\deg(Z)$ does not depend on the embedding. This is because, being a 0-dimensional scheme, p_Z is constant, but $p_Z(0)$ does not depend on the embedding of Z. In fact, we can write $Z = \operatorname{Spec} B$ for a 0-dimensional k-algebra B, and so $\deg(Z) = h^0(Z, \mathcal{O}_Z) = \dim_k(B)$.

Corolary 2.7. [Bezout's Theorem, 18.6.K in [Vak17]] *Suppose $X \to \mathbb{P}^n$ is an projective variety of positive dimension m and H is a hypersurface of $\mathbb{P}^n$ not containing any associated points of X. If $X \cap H \to \mathbb{P}^n$ is their scheme theoretic intersection,* $\deg(X \cap H) = \deg(X)\deg(H)$

Proof. Let $a = \deg X$, $b = \deg H$ and form the fibre square

$$\begin{array}{ccc} X \cap H & \xrightarrow{\ l\ } & H \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\ i\ } & \mathbb{P}^n \end{array}.$$

If we go back to the proof of 2.5, we are in the same situation, with $\mathcal{F} = \mathcal{O}_X$. If we write $H = \mathbb{V}(f)$ for a homogeneous polynomial of degree b we obtain the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-b) & \xrightarrow{\cdot f} & \mathcal{O}_X & \longrightarrow & i^* j_* \mathcal{O}_H & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & k_* \mathcal{O}_{X \cap H} & & \end{array}.$$

Therefore, identifying $p_{X \cap H}$ with $p_{k_* \mathcal{O}_{Z \cap H}}$ as in 2.6

$$\begin{aligned} p_{X\cap H}(d) &= p_X(d) - p_X(d-b) = \\ &= \frac{a}{m!} d^m + c d^{n-1} + \ldots - \left[\frac{a}{m!}(d-a)^m + c(d-a)^{n-1} + \ldots\right] = \frac{abn}{n!} d^{n-1} + \ldots \end{aligned}.$$

□

We will also work with Hilbert functions of graded rings and modules.

Definition 2.8. *Suppose R is a standard k-algebra and M is a finitely generated, graded R-module. The Hilbert function of M is $H_M(d) = \dim_k M_d$.*

The Hilbert function on modules behaves better than on sheaves, because taking the d-th graded part is exact. On top of that, we have:

Theorem 2.9. [Hilbert] *Let M be a finitely generated, graded module over a standard algebra R. Then H_M is a polynomial for $d >> 0$.*

Proof. If $\mathcal{F} = M^\sim$ in $X = \operatorname{Proj} R$, then by 2.2, for large d, $H_M(d) = h_{\mathcal{F}}(d)$ but the latter is also a polynomial for large d by 2.5. □

Are the Hilbert functions of M and $\mathcal{F}$ the same? The answer in general is not. It is equivalent to asking when the functor

$$\Gamma_*(\mathcal{F}) = \bigoplus_n H^0(X, \mathcal{F}(d))$$

from 2.2 is the inverse of $\sim$ without needing torsion. It turns out that this is simple to characterise for the ideal sheaves of projective space:

Definition 2.10. *Let $I \subset R = k[x_0, \ldots, x_n]$ be a homogeneous ideal. Its saturation is defined to be the set of all polynomials f such that for all j, $x_j^N f \in I$ for N big enough, and is denoted by I^{sat}. Equivalently, if $R_+ = \langle x_0, \ldots, x_n \rangle$,*

$$I^{sat} = \bigcup_{n>0} (I : R_+^n).$$

We say that I is saturated if $I = I^{sat}$.

Proposition 2.11. [Problem II.5.9 in [Har77]] *Let $X = \mathbb{P}^n$. Then:*

a) $\Gamma_*(\mathcal{O}_X) = k[x_0, \ldots, x_n]$.

b) $\Gamma_*(I^\sim)$*, as a submodule of* $\Gamma_*(\mathcal{O}_X)$*, is* I^{sat}.

c) There is an equivalence between closed subschemes of X and saturated ideals of $k[x_0, \ldots, x_n]$.

In particular, if I is a saturated ideal and $\mathcal{I} = I^\sim$, $H_I = h_\mathcal{I}$, as we desired.

Remark 2.12. Let Z be a 0-dimensional subscheme of $\mathbb{P}^n$, corresponding to the homogeneous ideal I. Then we have two associated Hilbert functions, h_Z and $H_{R/I}$, which are in general different but eventually equal the Hilbert polynomial of Z. This makes sense, since h_Z only depends on $\mathcal{L}$ but $H_{R/I}$ depends on the surjection $\mathcal{O}_Z^{n+1} \to \mathcal{L}$ we choose for the embedding into $\mathbb{P}^n$. In fact, $h_Z = p_Z$ is constant because Z is finite, so $H_{R/I}$ is more interesting to study. We will denote it by H_Z.

2.2 Flatness

Hilbert polynomials turn out to have a strong relationship with "good" families of schemes. The make sense of the term "good" one has to introduce flatness:

Definition 2.13. *Let $\phi : Y \to S$ be a morphism of schemes and $\mathcal{F}$ a quasi-coherent sheaf on Y. We say that $\mathcal{F}$ is flat (relative to ϕ) if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,\phi(x)}$-module. We say that ϕ is flat if $\mathcal{O}_Y$ is flat.*
A family of projective schemes over S is a flat and projective [1] *morphism $\phi : Y \to S$.*

It is a standard fact from commutative algebra that if M is an A-module, M is flat if and only if $M_\mathfrak{p}$ is flat over each $A_\mathfrak{p}$.
We think of a morphism as a good family of schemes (the fibres $Y_s = Y \times_S k(s)$) parametrized by the points of S. If $Y \to S$ is projective, each fibre Y_s embeds into $\mathbb{P}^r_{k(s)}$ so we for the Hilbert polynomials:

$$p(\mathcal{F}_s, d) := p_{\mathcal{F}_s}(d), \qquad p(Y_s, d) := p_{\mathcal{O}_{Y_s}}(d).$$

The semicontinuity theorem is the main reason why we desire flat families.

[1] in the sense of [Har77], i.e., it factors through a closed immersion $Y \to \mathbb{P}^r_S$

Theorem 2.14. [Theorem III.12.8 in [Har77]] *Let $\phi : Y \to S$ be a projective morphism, and $\mathcal{F}$ a coherent sheaf on Y, flat over S. Then for all $i \geq 0$, the function*

$$s \mapsto \dim_{k(s)} H^i(Y_s, \mathcal{F}_s)$$

is upper-semicontinuous on S.

Hilbert polynomials turn out to be very useful to characterise flat families:

Theorem 2.15. [Theorem III.9.9 in [Har77]] *Let $\phi : Y \to S$ be a projective morphism, where S is connected and reduced, and $\mathcal{F}$ a coherent sheaf on Y. Then $\mathcal{F}$ is flat over S if and only if $p(\mathcal{F}_s, \cdot)$ is independent of s.*

In particular, the degree of the embedding $Y_s \to \mathbb{P}^r_{k(s)}$ is independent of s if $Y \to S$ is flat.

2.3 Monomial ideals

Monomial ideals turn out to be very useful to do computations and combinatorial arguments. Throughout this section, S denotes $k[x_1, \ldots, x_n]$ or $k[x_0, \ldots, x_n]$. Note that all the arguments preserve the grading. A monomial ideal is an ideal $I \subset S$ that can be generated (and therefore finitely generated) by monomials. Any monomial is written as $x_1^{c_1} \ldots x_n^{c_n}$, notation that we will simplify as $\mathbf{x^c}$, where $\mathbf{c} = (\mathbf{c_1}, \ldots, \mathbf{c_n}) \in \mathbb{N}^\mathbf{n}$. In this way, the set of all monomial ideals is in bijection with $\mathbb{N}^n$

Definition 2.16. *A monomial order is an well- order $<$ in the set of all monomials, having 1 as its smallest element, such that $m < m'$ implies $mn < m'n$ for all monomials n.*
If $f = \sum_\mathbf{c} a_\mathbf{c} \mathbf{x^c}$ is any polynomial, and $<$ is a monomial order, $\mathrm{in}_<(f)$ is defined to be $a_{\mathbf{c}^*} \mathbf{x}^{\mathbf{c}^*}$, where c^* is the biggest such that $a_\mathbf{c}$ is nonzero. If I is any ideal, its initial ideal with respect to $<$ is defined to be $\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) : f \in I \rangle$

Example 2.17.

a) If we prescribe and ordering $x_{a(1)} > \ldots > x_a(n)$, then the lex order with respect to the function a is the one in which $\mathbf{x^c} <_\mathbf{lex} \mathbf{x^{c'}}$ if for the smallest i such that $c_{a(i)} \neq c_{a(i)}$, $c_{a(i)} > c_{a(i)}$.

b) If $\lambda : \mathbb{N}^n \to \mathbb{N}$ is a linear function (given by the dot product with some vector $v \in \mathbb{N}^n$), then we can define the partial order $<_\lambda$ declaring that $\mathbf{x^c} <_\lambda \mathbf{x^{c'}}$ if $\lambda(\mathbf{c}) < \lambda(\mathbf{c'})$. This, however, is not a well-order, but it will be useful. It can be corrected with some of the lex orderings, saying that $\mathbf{x^c} <_{\lambda,\mathbf{lex}} \mathbf{x^{c'}}$ if $\lambda(\mathbf{c}) < \lambda(\mathbf{c'})$ or if $\lambda(\mathbf{c}) = \lambda(\mathbf{c'})$ and $\mathbf{x^c} <_{\mathbf{lex}} \mathbf{x^{c'}}$

If I is any ideal of S, since monomials spam S/I, some subset of them will be a basis for it as a k-vector space, but more can be said:

Theorem 2.18. [Macaulay] *If $>$ is a monomial order and I is any ideal in S, the set of monomial ideals not in $\mathrm{in}_<(I)$ forms a basis for S/I.*

Proof. If m_i are monomials and $\sum a_i m_i \in I$, then $\mathrm{in}_<(\sum a_i m_i) \in \mathrm{in}_>(I)$ so one of them has to be in $\mathrm{in}_>(I)$.
If J is the linear spam of the set of monomials not in $\mathrm{in}_<(I)$, if $J + I \neq S$, there is a polynomial f not in $J + I$ with minimal initial term, but this is a contradiction because its initial term bust be in J or be the initial term of some polynomial in I, so it can be substracted, contradicting minimality. □

On the other hand, we also have:

Proposition 2.19. *If $\mathcal{B}$ is a set of monomials such that $m \in \mathcal{B}$ and $n \mid m$ implies $n \in \mathcal{B}$, and I is the monomial ideal generated by all the monomials not in $\mathcal{B}$, then $\mathcal{B}$ is a basis for S/I over k.*

Proof. It is clear that $\mathcal{B}$ is a generating set for the quotient ring. If $\sum_i a_i m_i \in I$, since I is a monomial ideal, each m_i with $a_i \neq 0$ belongs to I, so $\mathcal{B}$ is also linearly independent. □

A finite generating system for a monomial ideal is minimal if no element of the generating set is divisible by the others. It is clear that such generating systems exist and are unique for any monomial ideal.

Remark 2.20. When working with monomial ideals in $k[x, y]$ it is convenient to use box diagrams. More concretely, there is a bijection between the set of all monomial ideals and all the unions of boxes in a lattice $\mathbb{N}^2$ such that whenever one box is, its adjacent boxes on the left and bottom are also, which we will refer to as a box diagram. The minimal generators of the ideal can be read off the corners of the diagram, the monomials not in the ideal are the cones contained in the boxes and the Hilbert function of $k[x, y]/I$ counts the number of elements of each diagonal that are in the interior of the diagram. See Figure 2.1.

Definition 2.21. *A generating set $g_1, \ldots, g_s$ of an ideal I in S is said to be a Grobner basis with respect to a monomial ordering $<$ if* $\text{in}_<(g_1), \ldots, \in_< (g_s)$ *forms a generating set of* $\text{in}_<(I)$.

Grobner basis always exist by the Hilbert basis theorem. In fact, any set whose initial terms generate $\text{in}_<(I)$ generate I, as a consequence of the following lemma:

Lemma 2.22. *If $I \subset J$ are ideals such that* $\text{in}_<(I) = \text{in}_<(J)$ *for some monomial order $<$, we must have $I = J$.*

Proof. It is clear considering $f \in J \setminus I$ such that $\text{in}_<(f)$ is minimal. □

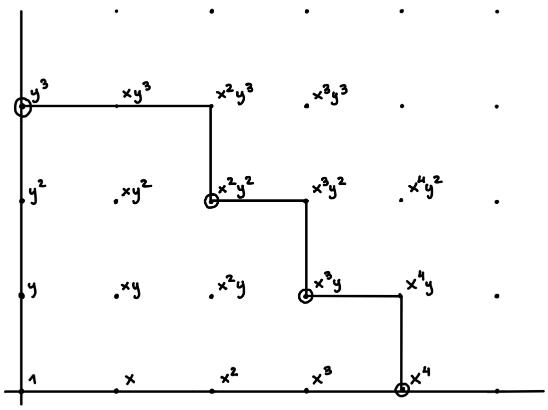

Figure 2.1: Box diagram associated to the ideal $\langle y^3, x^2y^2, x^3, x^4 \rangle$.

Remark 2.23. There are, in fact, algorithms to compute Grobner basis that do not involve the Hilbert basis theorem, but we are not interested in them in this project.

Remark 2.24. The notions of initial terms, initial ideals, or Grobner bases can be defined in an analogue way if $<$ is not a well-ordering but instead a weighted ordering: we define

$$\text{in}_\lambda \left(\sum_{\mathbf{c}} a_{\mathbf{c}} \mathbf{x}^{\mathbf{c}} \right) = \sum_{\lambda(\mathbf{c}) = \mathbf{t}^*} a_{\mathbf{c}} \mathbf{x}^{\mathbf{c}}, \text{ where } \mathbf{d}^* = \max\{\mathbf{d} : \lambda(\mathbf{c}) = \mathbf{d} \text{ and } \mathbf{a}_{\mathbf{c}} \neq \mathbf{0}\}$$

and the initial ideal in a similar fashion.

Initial ideals over weighted can be used to obtain initial ideals with respect to monomial orders:

Proposition 2.25. [Proposition 15.16 in [Eis95]] *If $>$ is a monomial order and $g_1, \ldots, g_s$ a Grobner basis for I with respect to λ, there is always a linear function $\lambda : \mathbb{N}^n \to \mathbb{N}$ such that the g_i also form a Grobner basis for I with respect to $>_\lambda$ and* $\mathrm{in}_>(I) = \mathrm{in}_\lambda(I)$.

The reason why weighted orderings are useful is because they allow for a nice family of ideals:
If $f = \sum_{\mathbf{c}} a_{\mathbf{c}} \mathbf{x}^{\mathbf{c}}$, let $\hat{f} = \sum_{\mathbf{c}} a_{\mathbf{c}} x^{\mathbf{c}} t^{d-\lambda(\mathbf{c})}$, and define $\hat{I} = \langle \hat{f} : f \in I \rangle$ as an ideal of $S[t]$, and so $\mathrm{Spec}(S[t]/\hat{I})$ (resp. $\mathrm{Proj}(S[t]/\hat{I})$) gives a family over $\mathbb{A}^1$.

Proposition 2.26. [Theorem 15.17 in [Eis95]] *For any ideal, $S[t]/\hat{I}$ is a free $k[t]$-module and*

$$S[t]/\hat{I} \otimes_{k[t]} k[t, t^{-1}] \cong S/I[t, t^{-1}]$$

$$S[t]/\hat{I} \otimes_{k[t]} k[t]/(t) \cong S/\mathrm{in}_\lambda(I)$$

In particular, the families described above are flat over $\mathbb{A}^1$.

Definition 2.27. *If we choose λ as in 2.25, then such a family is called a Grobner degeneration from I to* $\mathrm{in}_<(I)$.

Chapter 3

Hilbert functions and regularity

3.1 The Hilbert function of a 0-dimensional scheme

Throughout this section, Z will be a closed, 0-dimensional subscheme of $\mathbb{P}^n$, corresponding to the saturated ideal $I \subset k[x_0, \ldots, x_n] =: R$ where $I_d = H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$. By 2.12, we are interested in the Hilbert function of the ring R/I, which we will denote by $H_{R/I}$ or H_Z. The degree of Z will be denoted by $\deg(Z)$. And we have an exact sequence

$$0 \to I \to R \to R/I \to 0,$$

which implies that $\binom{n+d}{d} = H_R(d) = H_I(d) + H_{R/I}(d)$.

Proposition 3.1. *$H_{R/I}$ is a strictly increasing function until it reaches $\deg(Z)$.*

Proof. Consider a linear form f of $\mathbb{P}^n$ not vanishing through any point of Z, and form the exact sequence

$$0 \longrightarrow \frac{R}{I} \xrightarrow{\cdot f} \frac{R}{I}(1) \longrightarrow \frac{R}{I + \langle f \rangle}(1) \longrightarrow 0,$$

where right exactness occurs because f does not go through any of the associated primes of R/I. Therefore,

$$H_{R/I}(d+1) - H_{R/I}(d) = H_{R/I+\langle f \rangle}(d) \geq 0,$$

But $R/I + \langle f \rangle$ is a standard k-algebra, so any element of degree $d+1$ can be recovered as sums of products of elements of degree 1 and degree d. Therefore, if $H_{R/(I,f)}(d) = 0$, $H_{R/I+\langle f \rangle}(d+1) = 0$. The result follows because for large d we know that $H_{R/I}(d) = \deg(Z)$ □

There is an old theorem by Macaulay which completely characterises which functions $H : \mathbb{N} \to \mathbb{N}$ appear as Hilbert functions of graded algebras:

Definition 3.2. *If $h, d \in \mathbb{N}$, there are unique $m_d \geq m_{d-1} \geq \ldots \geq m_j \geq j$ such that*

$$h = \sum_{i=j}^{d} \binom{m_i}{i},$$

and this is called the d-nomial expansion of h. To obtain them, one chooses the biggest element of the form $\binom{m}{d}$ that is less than h, subtract it of h and repeat for $d-1$. If

$$h^{\langle d \rangle} = \sum_{i=j}^{d} \binom{m_i + 1}{i + 1},$$

a sequence h_d is said to be an O-sequence if $h_0 = 1$ and $h_{d+1} \leq h_d^{\langle d \rangle}$ for all $d \geq 0$.

Example 3.3.

a) If $h_d = d+1$ then $h_{d+1} \leq d+2$: In this case, $h_d = \binom{d+1}{d}$ is the d-nomial expression, so $h_d^{\langle d \rangle} = \binom{d+2}{d+1} = d+2$.

b) If $h_d \leq d$ for some d, then $h_{d+1} \leq h_d$: In this case, $h_d = \binom{d}{d} + \ldots + \binom{j}{j}$ is the d-nomial expression, so $h_d^{\langle d \rangle} = \binom{d+1}{d+1} + \ldots + \binom{j+1}{j+1} = h_d$.

Theorem 3.4. [Theorem 2.2. in [Sta78]] *For a function $H : \mathbb{N} \to \mathbb{N}$, the following are equivalent:*

a) $H = H_A$ for some standard k-algebra A.

b) $\{H(d)\}_{d \geq 0}$ is an O-sequence.

Proof. Suppose we write $A = R/I$. Let $<$ be a monomial order and let $\mathcal{B}$ be the set of monomials of R not contained in $\mathrm{in}_<(I)$. By 2.18, we have that the images of $\mathcal{B}$ in A for a k-basis for A. Also, we proved in 2.19 that such sets B are characterised by the property that $m \in \mathcal{B}$ and $m' \mid mB$ implies $m' \in \mathcal{B}$.
Therefore, we have to prove that $\{H(d)\}_{d \geq 0}$ is an O-sequence if and only if there is a set of monomials $\mathcal{B}$ with the above property, such that $H(d) = \mathrm{card}(\mathcal{B} \cap \{\deg m = d\})$. The difficult part is the "if" part (see [Sta75]); for the "only if" part, let $n+1 = H(1)$, and let $\mathcal{B}_d$ be the first (in lexicographic order) $H(d)$ monomials in $x_0, \ldots, x_n$ of degree d . Then a counting argument shows that $H(d+1) \leq H(d)^{\langle d \rangle}$ implies that all monomials of degree d dividing monomials in $\mathcal{B}_{d+1}$ must be in $\mathcal{B}_d$. □

Remark 3.5. In the language of box diagrams, the above proof is equivalent to showing that, if we choose the first $H(d)$ points (starting form below) in in the d-th diagonal, we obtain a box diagram if and only if $H(d+1) \leq H(d)^{\langle d \rangle}$. See Figure 3.1.

If $\{h_d\}$ is an O-sequence, by 2.5 it will agree with some polynomial for large d, and the degree of such polynomial is said to be the dimension of the sequence.

Example 3.6. *Consider the* 0*-dimensional* O*-sequence* $\{1, 2, 3, 2, 2, \ldots\}$*. Using the construction in the proof of 3.4, we find that it correspond to the ideal* (x^3, x^2y)*. This ideal is, of course, non-saturated, because the sequence is not increasing. In particular, the saturation of* (x^3, x^2y) *is* (x^2)*.*

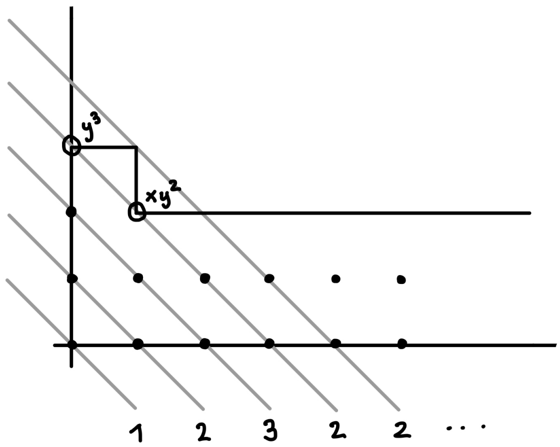

Figure 3.1: Box diagram associated to the Hilbert function $\{1, 2, 3, 2, 2, \ldots\}$

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Example 3.7. [Closed subschemes of $\mathbb{P}^1$] *(Assuming* $k = \overline{k}$*) Any* 0*-dimensional subscheme of* $\mathbb{P}^1$ *is the one corresponding to the ideal*

$$I = \bigcap_{i=1}^{s} I_{P_i}^{m_i},$$

where I_{P_i} *is the ideal of a closed point* $P_i \in \mathbb{P}^1$ *and* m_i *is its multiplicity. In other words, if* $P_i = (a_i : b_i)$*,* $I_{P_i} = \langle x_0 b_i - x_1 a_i \rangle$*. In this case, if* $r = \sum m_i$*,*

$$H_{R/I}(d) = \begin{cases} d & \text{if } d < r \\ r & \text{if } d \geq r \end{cases}$$

The proof is simple: after removing a point not in Z, one has to classify subschemes of $\mathbb{A}^1$, which is a simple task because $k[\mathbb{A}^1]$ is a PID. Then, if $\langle f \rangle$ is the affine ideal of Z in $k[x]$, f is a polynomial of degree l and $H_{R/I}(d)$ is equal to the number of polynomials in $k[x]/\langle f \rangle$ of degree $\leq d$.

In higher dimensions, however, the coordinate ring is not a PID, and so the picture gets more complicated. The analogue subschemes to the ones of this example are called fat point schemes.

Definition 3.8. *A fat point scheme* $Z = m_1P_1 + \ldots + m_sP_s \subset \mathbb{P}^n$ *is the closed subscheme corresponding to the (saturated) ideal*

$$I_Z = \bigcap_{i=1}^{s} I_{P_i}^{m_i},$$

where I_{P_i} *is the homogeneous prime ideal corresponding to the point* P_i*. Note that its degree is*

$$\deg(Z) = \sum_{i=1}^{s} \binom{m_i + n - 1}{n}.$$

We want to characterise the sequences coming from 0-dimensional subschemes. To do so, if one goes back to the proof of 3.1, we have that

$$\Delta H_{R/I}(d) := H_{R/I}(d+1) - H_{R/I}(d) = H_{R/I+\langle f \rangle}(d)$$

is the Hilbert function of a standard k-algebra, so it is an O-sequence. The converse is also true, as a consequence of the following result:

Theorem 3.9. [Theorem 3.2 in [GMR83]] *A sequence* h_d *is the Hilbert function of some* d*-dimensional subscheme of* $\mathbb{P}^n$ *if and only if it is a* d*-dimensional* O*-sequence such that its first difference* $(\Delta h)_d = h_d - h_{d-1}$ *is also a* O*-sequence and* $h_1 \leq n+1$*. In fact, one can pick a reduced scheme for any such sequence.*

We can prove this directly for $d = 0$ and $n = 2$. In this case, $e = \Delta h$ starts with $\{1, 2, \ldots\}$. I claim that if e is a 0-dimensional sequence, e is strictly increasing, with difference 1, and then decreases (non-strictly) until it reaches 0. This is a consequence of what was said in example 3.3. Therefore, the following theorem suffices:

Theorem 3.10. [Problem 5.3. in [CH13]] *Let* $r_1 > \ldots > r_s > 0$ *and pick* s *distinct lines in* $\mathbb{P}^2$*. Then choose* r_i *distinct points in line* i*, which do not lie on the other lines. If* Z *be the subscheme of* $\mathbb{P}^2$ *consisting of those points, with ideal* I*,* $\Delta H_{R/I}(d)$ *is the sequence*

$$\{1, \ldots, (s-1), \underbrace{s, \ldots s}_{r_s \text{ times}}, \underbrace{(s-1), \ldots, (s-1)}_{r_s - r_{s-1} - 1 \text{ times}}, \underbrace{(s-2), \ldots, (s-2)}_{r_{s-1} - r_{s-2} - 1 \text{ times}}, \ldots, 0, 0, \ldots\}.$$

Proof. Let $L_k = \mathbb{V}(f_k)$ be the lines, and define inductively $Z_0 = Z$, $Z_k = Z_{k-1} \setminus L_k$ until $Z_{s+1} = \emptyset$. Let I_k be the corresponding ideals and $\mathcal{I}_k$ the ideal sheaves. Note that $I_{k+1} = I_k : \langle f_{k+1} \rangle = I_k \cap \langle f_{k+1} \rangle$, so for each k we form an exact sequence

$$0 \longrightarrow I_{k+1} \xrightarrow{\cdot f_{k+1}} I_k \longrightarrow \frac{I_k}{I_k \cap \langle f_{k+1} \rangle} \longrightarrow 0.$$

If we sheafify the inclusion $f_k I_{k+1} \subset I_k$ we obtain $\mathcal{I}_{k+1}(-1) \subset \mathcal{I}_k$, so $\mathcal{I}_k/\mathcal{I}_{k+1}(-1)$ is the sheafification of $I_k/f_{k+1}I_{k+1} = I_k/I_k \cap \langle f_{k+1}\rangle = I_k + \langle f_{k+1}\rangle/\langle f_{k+1}\rangle$, which is the inclusion of $\mathcal{I}_{Z_k \cap L_k, L_k}$. Therefore, we obtain short exact sequences:

$$0 \longrightarrow \mathcal{I}_{k+1}(t-1) \xrightarrow{\cdot f_k} \mathcal{I}_k(t) \longrightarrow i_* \mathcal{I}_{Z_k \cap L_{k+1}, L_{k+1}}(t) \longrightarrow 0. \qquad \text{(Eq. 1)}$$

For any t and k. For $k = n$, they look like

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(t-1) \longrightarrow \mathcal{I}_{s-1}(t) \longrightarrow i_* \mathcal{I}_{Z_k \cap L_s, L_s}(t) \longrightarrow 0. \qquad \text{(Eq. 2)}$$

Note that $\mathcal{I}_{Z_k \cap L_{k+1}, L_{k+1}}(t)$ is the ideal corresponding to r_{k+1} different points of the line L_{k+1}, so it is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(t - r_{k+1})$

$$h^0(\mathbb{P}^2, i_* \mathcal{I}_{Z_k \cap L_{k+1}, L_{k+1}}(t)) = \begin{cases} 0 & \text{if } t < r_{k+1} \\ t - r_{k+1} + 1 & \text{if } t \geq r_{k+1} \end{cases} = \max\{0, t - r_{k+1} + 1\}.$$

Therefore, by taking global sections in (Eq. 1) and (Eq. 2) repeatedly, for $t = d, d-1, \dots, d-s$ we see that

$$h^0(\mathbb{P}^2, \mathcal{I}_k(d)) \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-(s+1))) + \sum_{k=0}^{s-1} h^0(\mathbb{P}^2, i_* \mathcal{I}_{Z_k \cap L_{k+1}, L_{k+1}}(d-k)),$$

so

$$H_Z(d) \geq \binom{d+2}{2} - \binom{d-s+2}{2} - \sum_{k=0}^{s-1} h^0(\mathbb{P}^2, i_* \mathcal{I}_{Z_k \cap L_{k+1}, L_{k+1}}(d-k)).$$

Since $\binom{a}{2} - \binom{b}{2}$ is the sum of all positive integers between b and $a-1$, the last expression reduces to

$$\sum_{k=0}^{s-1} \max\{0, d-k+1\} - \max\{0, d-k-r_{k+1}+1\}.$$

Each summand can be expressed as

$$\begin{cases} 0 & \text{if } d-k < 0 \\ d-k+1 & \text{if } 0 \geq d-k < r_{k+1} - 1 \\ r_{k+1} & \text{if } r_{k+1} - 1 \geq d-k \end{cases},$$

and so its difference function is the characteristic function of the set $\{k, \dots, k + r_{k+1} - 1\}$. The sum of all these is precisely the sequence in the statement of the theorem, so it is enough to show that taking global sections in the equations (Eq 1.) and (Eq 2.) is right exact when $t = d-k-1$. Part of the long exact sequences of homomology from equations with $t = d - k$ (Eq 1.) and (Eq 2.) are

$$\xrightarrow{\alpha_k} H^0(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d-k-r_{k+1})) \longrightarrow H^1(\mathbb{P}^2, \mathcal{I}_{k+1}(d-k-1)) \to H^1(\mathbb{P}^2, \mathcal{I}_k(d-k)) \longrightarrow H^1(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d-k-r_{k+1})) \qquad \text{(Eq. 3)}$$

for $k < s-1$ and

$$\xrightarrow{\alpha_{s-1}} H^0(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d+1-s-r_{s+1})) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-s) \to H^1(\mathbb{P}^2, \mathcal{I}_{s-1}(d-s+1)) \longrightarrow H^1(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d+1-s-r_s)) \qquad \text{(Eq. 4)}$$

for $k = s-1$. Using the cohomology of $\mathcal{O}_{\mathbb{P}^1}(l)$ (Theorem 5.1. in [Har77]), if k be the smallest such that $H^0(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d-k-r_{k+1}))$ is not 0. Then $d-k-r_{k+1} \geq 0$, so if $j > k$, $d-j-r_{j+1} \geq -1$ because the sequence $r_1, r_2, \ldots$ is strictly increasing and thus $H^1(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d-j-r_{j+1})) = 0$. Not only that, but also $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-s)) = 0$, so by exactness of (Eq. 4) we have $H^1(\mathbb{P}^2, \mathcal{I}_{s-1}(d-s+1)$, and then by successive applications of exactness of (Eq. 3), $H^1(\mathbb{P}^2, \mathcal{I}_j(d-j) = 0$ for all $j > k$. Therefore, α_j is surjective for all $j \geq k$, and if $j < k$, by the choice of k, $H^0(\mathbb{P}^2, i_*\mathcal{O}_{\mathbb{P}^1}(d-j-r_{j+1})) = 0$ so α_j is clearly surjective. □

3.2 Regularity of fat points

If A is a standard 1-dimensional k-algebra the regularity index of A, $\operatorname{reg}(A)$ is the smallest d after which $H_A(d)$ becomes constant. If Z is a finite subscheme of $\mathbb{P}^n$ with ideal I, its regularity index is defined as the regularity index of R/I.
By proposition 3.1, it follows that $H_{R/I}(d) = \deg(Z)$ for all $d \geq \operatorname{reg}(Z)$. The regularity index is easier to calculate than the whole Hilbert function, but still has a relationship with other geometrical properties:

Example 3.11. *For any Z, $\operatorname{reg}(Z) \leq \deg(Z) - 1$, and equality holds if and only if Z is reduced and lies on a line.*

Proof. The Hilbert function of Z strictly increases until it reaches $\deg(Z)$. Therefore, $\operatorname{reg}(Z) \leq \deg(Z) - 1$, with equality if and only if its Hilbert function is

$$\{1, 2, \ldots, \deg(Z), \deg(Z), \ldots\}.$$

Note that $H_{R/I}(1) = 2$ if and only if I contains $n-1$ linearly independent 1-forms. This shows that Z is contained in a line, so it must be also reduced. The result follows from 3.7. $\square$

It is easy to see that $\operatorname{reg}(Z') \geq \operatorname{reg}(Z)$ if $Z \subset Z'$. This indicates that the more points of Z that lie on a small linear subvariety, the higher the regularity of Z will be:

Definition 3.12. *Let $Z = m_1P_1 + \ldots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$. The Segre bound of Z is*

$$\operatorname{Seg}(Z) = \left\{ \left\lfloor \frac{\dim L - 2 + \sum_{P_i \in L} m_i}{\dim L} \right\rfloor : L \subset \mathbb{P}^n \text{ linear and } \dim L > 0 \right\}.$$

The fact that $\operatorname{reg}(Z) \leq \operatorname{Seg}(Z)$ was proven 5 years ago in [NT16], but the methods escape the scope of this work. We will present the proofs of this result in two particular cases: When $n = 2$ and when n is arbitrary but the points are in general position.

3.3 Linear systems and separating directions

We will reinterpret regularity in terms of linear systems first. This point of view has will allow us to give original but simple proofs of the Segre bound in the cases mentioned above.

If $Z = m_1P_1 + \ldots + m_sP_s$, $H^0(\mathbb{P}^n, \mathcal{I}_Z(d))$ is the set of all d-linear forms that vanish at the points P_i to order $m_i - 1$. In a more concrete way, if the linear for is F and $P_i = (1 : 0 : \ldots : 0)$, after dehomogeneizing with respect to x_0, we are asking that all the terms of F of order at most $m_i - 1$ vanish.
Taking global sections in the short exact sequence $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^n} \to i * \mathcal{O}_Z \to 0$ and using the projection formula, we obtain an injection

$$\alpha_Z : \frac{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))}{H^0(\mathbb{P}^n, \mathcal{I}_Z(d))} \longrightarrow H^0(Z, i^*\mathcal{O}_{\mathbb{P}^n}(d)).$$

Note that the dimension of the LHS is the Hilbert function $H_Z(d)$, whereas the dimension of the RHS is $\deg(Z)$ because Z is finite. Thus, $d \geq \operatorname{reg}(Z)$ if and only if α_Z is surjective. Note that

$$i^*\mathcal{O}_{\mathbb{P}^n}(d) \cong \bigsqcup_i \operatorname{Spec} \frac{k[y_1, \ldots, y_n]}{\langle y_1, \ldots, y_n \rangle^{m_i}},$$

where the isomorphism is obtained after dehomogeneizing with respect to some linear form not containing P_i, for each P_i separately. For example, if $P = (1 : 0 : \ldots : 0)$, for any homogeneous $f \in k[x_0, \ldots, x_n]$, $\alpha_Z(f)$ gives a section

$$\alpha_Z(f) : \{P_1, \ldots, P_s\} \longrightarrow \bigoplus_i \frac{k[y_1, \ldots, y_n]}{\langle y_1, \ldots, y_n \rangle^{m_i}},$$

whose restriction to P_1 is obtained setting $x_0 = 1$ and then reducing modulo $\langle x_1, \ldots, x_n \rangle^{m_1}$. We say that forms of degree d separate directions of Z at P_1 if for any monomial m of degree $< m_1$ there is some f of degree d such that the restriction of $\alpha_Z(f)$ to P_1 described above yields m, and its restriction to the rest of the P_i is 0 (or, in other words, it vanishes to order m_i at each different P_i). Similarly one can define what separating directions at P_i means (and it does not depend on the choice of a dehomogeneization).

Finding the regularity of Z is therefore related to the interpolation problem:

Proposition 3.13. *$d \geq \operatorname{reg}(Z)$ if and only if forms of degree d separate directions of Z at each P_i.*
In particular, if $m_i = 1$ for all i, $d \geq \operatorname{reg}(Z)$ if and only if for any P_i there is a form of degree d vanishing at all P_j except at P_i

If Z' is a subscheme of Z, and $j : Z' \to \mathbb{P}^n$, from the exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & H^0(Z, i^*\mathcal{O}_{\mathbb{P}^n}(d)) \\
 & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{I}_{Z'}(d)) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & H^0(Z', j^*\mathcal{O}_{\mathbb{P}^n}(d))
\end{array}$$

we form

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
0 & \longrightarrow & \frac{H^0(\mathbb{P}^n, \mathcal{I}_{Z'}(d))}{H^0(\mathbb{P}^n, \mathcal{I}_Z(d))} & \xrightarrow{\beta} & H^0(Z, \mathcal{I}_{Z'/Z}(d)) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \frac{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))}{H^0(\mathbb{P}^n, \mathcal{I}_Z(d))} & \xrightarrow{\alpha_Z} & H^0(Z, i^*\mathcal{O}_{\mathbb{P}^n}(d)) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \frac{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))}{H^0(\mathbb{P}^n, \mathcal{I}_{Z'}(d))} & \xrightarrow{\alpha_{Z'}} & H^0(Z', j^*\mathcal{O}_{\mathbb{P}^n}(d)) \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

where β is the restriction of α_Z. Now, by the Snake lemma, α_Z is surjective if and only if $\alpha_{Z'}$ and β is. If $Z = m_1 P_1 + \ldots + m_s P_s + (a+1)P$, $Z' = m_1 P_1 + \ldots + m_s P_s + aP$, then the group $H^0(Z, \mathcal{I}_{Z'/Z}(d))$ can be identified as before with monomials of degree a in $x_1, \ldots, x_n$.

We are now ready to prove the Segre bound:

Proposition 3.14. [Proposition 5 in [CTV93]] *Let $P_1, \ldots, P_s, P$ be different points of $\mathbb{P}^n$ in general position, and let $Z = m_1P_1 + \ldots + m_sP_s + (a+1)P$, $Z' = m_1P_1 + \ldots + m_sP_s + aP$. If $d \geq m_2 + m_1 - 1$ and $nd \geq a + \sum_i m_i$ and $d \geq \operatorname{reg}(Z')$, then $d \geq \operatorname{reg}(Z)$.*

Proof. We may assume $P = (1 : 0 : \ldots : 0)$, $P_1 = (0 : 1 : 0 : \ldots : 0), \ldots P_n = (0 : \ldots : 1)$ because the points are in general position. If $s \leq n$, and m is a monomial of order a in $x_1, \ldots x_n$, since $d \geq m_1 + a$, $f := x_0^{m_1} x_0^{d-a-m_1} m$ is a form of degree d that vanishes to order $m_1 \geq m_i$ at each P_i, so it is in $\mathcal{I}_{Z'}(d)$ and $\alpha_Z(f) = m$.
If $s > n$, let $m = x_1^{c_1} \ldots x_n c_n$ with $c_1 + \ldots + c_n = a$. Note that m vanishes to order $m - c_i$ at each P_i for $i \leq n$, so if $t = d - a$, it is enough to show that We can find t linear forms f_i not vanishing at P, but vanishing to order $m_i + c_i - a$ and to order m_i at P_i for $i \leq n$ and $i > n$, respectively, for then $f := f_1 \ldots f_l m$ is in $I_{Z'}(d)$ and $\alpha_Z(f) = m$. Note that

$$n(t + a) \geq a + \sum m_i \quad \text{and} \quad l + a \geq m_1 + m_2 - 1,$$

so

$$nt \geq (n-1)a + \sum m_i = \sum_{i \leq n} (m_i + c_i - a) + \sum_{i > n} m_i \quad \text{and} \quad l \geq m_1.$$

Therefore it is enough to show that

Lemma 3.15. *[Lemma 4 in [CTV93]] If $a_1 \geq \ldots \geq a_s$ and $P_1, \ldots, P_s, P$ are different points in general position in $\mathbb{P}^n$, there exist t linear forms vanishing to order a_i at each P_i and not vanishing at P if $nt \geq \sum_i a_i$ and $t \geq a_1$.*

□

Proof of the lemma. This is trivial if $s \leq n$. If $s > n$, since $nt \geq a_1 + \ldots + a_{n+1} \geq (n+1)a_{n+1}$, and $t \geq a_1$,

$$t - 1 \geq \max\{a_1 - 1, \ldots, a_n - 1, a_{n+1}, \ldots, a_s\},$$

and it is clear that $n(t-1) \geq (a_1 - 1) + \ldots + (a_n - 1) + a_{n+1} + \ldots + a_s$ so we can argue by induction, using a linear form vanishing at the points $P_1, \ldots, P_n$. □

In the proof of this proposition we only used that the points are in general position for the lemma. For the case of points in $\mathbb{P}^2$ that are not in general position, we have an analogue:

Lemma 3.16. [Lemma 3 in [Thi99]] *If $a_1 \geq \ldots \geq a_s$ and $P_1, \ldots, P_s, P$ are different points in $\mathbb{P}^2$, there exist t linear forms vanishing to order a_i at each P_i and not vanishing at P if $2t \geq \sum_i a_i$, $t \geq \sum_{P_i \in L} a_i$ for all lines L containing P.*

Proof. If all the points are on a line L, there is noting to prove by 3.11.
Now let $L_1, \ldots, L_k$ be all the lines and let $b_j = \sum_{P_i \in L_j} a_i$, and assume $b_1 \geq b_2 \geq \ldots$. Note that $\sum_i a_i = \sum_j b_j$.
If we pick different points P_{i_1}, P_{i_2} from lines L_1 and L_2 respectively, then:

- If $2b_1 \geq \sum b_i$ (so $t \geq b_1$) then $b_3 < b_1$, so $t - 1 \geq \max\{b_1 - 1, b_2 - 1, b_3, \ldots, b_k\}$ and clearly $2(t-1) \geq (b_1 - 1) + (b_2 - 2) + b_3 \ldots + b_k$.
- If $\sum b_i \geq 2b_1$, then $t \geq 2b_1$, and so $t - 1 \geq b_1 \geq \max\{b_1 - 1, b_2 - 1, b_3, \ldots, b_k\}$, and again clearly $2(t-1) \geq (b_1 - 1) + (b_2 - 2) + b_3 \ldots + b_k$

So in both cases we can apply induction using a linear form vanishing at P_{i_1} and P_{i_2}. □

Proposition 3.17. [Lemma 4 in [Thi99]] *Let $P_1, \ldots, P_s, P$ be different points of $\mathbb{P}^2$ and $Z = m_1P_1 + \ldots + m_sP_s + (a+1)P$, $Z' = m_1P_1 + \ldots + m_sP_s + aP$. If $d \geq a + \sum_{P_i \in L} m_i$ whenever L is a line through P, $2d \geq a + \sum_i m_i$ and $d \geq \operatorname{reg}(Z')$, then $d \geq \operatorname{reg}(Z)$.*

Proof. If all the points are on a line then this is trivial by 3.11.
If not, we may assume $P = (1:0:0), P_1 = (0:1:0), P_2 = (0:0:1)$. As in the proof of 3.14, it is enough to show that, if $c_1 + c_2 = a$, there are $d - a$ linear forms vanishing to order $m_1 - c_2$ at P_1, $m_2 - c_1$ at P_2 and m_i at P_i for the rest of the points. This can be done using the lemma because $d - a \geq \sum_{P_i \in L} m_i$ and

$$2d \geq a + m_1 + \ldots + m_s = 2a + (m_1 - c_2) + (m_2 - c_1) + \ldots + m_s.$$

□

Theorem 3.18. *If $Z = m_1P_1 + \ldots + m_sP_s \subset \mathbb{P}^n$ and the points are in general position, or if $n = 2$, then* $\operatorname{reg}(Z) \leq \operatorname{Seg}(Z)$.

Proof. This follows from repeated applications of the previous propositions and using the fact that

$$\left\lfloor \frac{q + n - 1}{n} \right\rfloor = \min\{d \mid nd \geq q\}.$$

□

In fact, the Segre bound is attained:

Proposition 3.19. [Proposition 7 in [CTV93]] *If the points P_i lie on a rational normal curve X (and therefore are in general position), and if $Z = m_1P_1 + \ldots + m_sP_s$ with $m_1 \geq \ldots \geq m_s$, then*

$$\operatorname{reg}(Z) = \max\left\{m_1 + m_2 - 1, \left\lfloor \frac{n - 2 + \sum m_i}{n} \right\rfloor\right\} = \operatorname{Seg}(Z).$$

Proof. First of all, the fact that $\max\left\{m_1 + m_2 - 1, \left\lfloor \frac{n-2+\sum m_i}{n} \right\rfloor\right\} = \mathrm{Seg}(Z)$ follows because the points are in general position, and so for any k-plane can contain at most k points, and $-1 + m_1 + \ldots + m_k \le k(m_1 + m_2 - 1)$.
It is not difficult to see that if $Z' = m_1P_1 + m_2P_2$, $\mathrm{reg}(Z) \ge m_1 + m_2 - 1$, so we may assume that $\mathrm{Seg}(Z) = \left\lfloor \frac{n-2+\sum m_i}{n} \right\rfloor$.
Suppose that X is given by the n-th Veronese embedding and that $P_s = (1 : 0 : \ldots : 0)$
Suppose that $d \ge \mathrm{reg}(Z)$, so there is a hypersurface H of degree d such that $\alpha_Z(H)$ is $x_1^{m_s-1}$ at P_s and 0 at the other points. By Bezout theorem 2.7, if X and H meet transversely, $-1 + \sum_i m_i \le \deg(X \cap H) = dn$, so we must have $d \ge \mathrm{Seg}(Z)$ or X is contained in H. H can be written as $x_0^{d-m_s-1}x_1^{m_s-1} + F$, so

$$0 = H(1 : a : \ldots : a^n) = a^{m_s-1} + F(1 : a : \ldots : a^n).$$

But F vanishes to order m_s at P_s, so $F(1 : a : \ldots : a^n)$ is divisible by a^{m_s}, a contradiction. □

Chapter 4

Families of zero-dimensional schemes

4.1 The Hilbert scheme

The Hilbert scheme is the standard technique to deal with families of schemes in great generality. Note that we will be mainly interested in the case $X = \mathbb{P}^n$.

Definition 4.1. *Let X be any scheme. The Hilbert scheme of r points on X is a scheme* $\mathrm{Hilb}_r(X)$ *such that morphisms $S \to \mathrm{Hilb}_r(X)$ are in a natural correspondence with closed subschemes of $X \times S$ such that the projection onto S is flat and the fibers are all finite schemes of degree r.*

Note that $\mathrm{Hilb}_r(X)$, if it exists, is unique up to isomorphism.

Example 4.2.

a) Taking $S = \operatorname{Spec} k$ in the definition, we see that k-points of $\mathrm{Hilb}_r(X)$ are in correspondence with finite subschemes of X. If $Z \subset X$ is a finite scheme, the corresponding scheme in the Hilbert scheme is denoted by $[Z]$.

b) Taking $S = \mathrm{Hilb}_r(X)$, there is a subscheme $\mathcal{U}_r \subset \mathrm{Hilb}_r(X) \times X$, corresponding to the identity, which we call the universal family. Note that the fibre of $\mathcal{U}_r \to \mathrm{Hilb}_r(X)$ over $[Z]$ is Z.

Theorem 4.3. [Theorem 3.2 in [Gro61]] *If X is (quasi-)projective over a field k,* $\mathrm{Hilb}_r(X)$ *exists and is also (quasi-)projective over k.*

Any finite scheme $Z \subset X$ of degree r a union of schemes $Z_1, \ldots, Z_l$ supported at different points $P_1, \ldots, P_l$ with degrees $d_1, \ldots, d_l$, such that $r = d_1 + \ldots + d_l$, so we might want to associate to Y the uple

$$(\underbrace{P_1,\dots,P_1}_{d_1 \text{ times}},\underbrace{P_2,\dots,P_2}_{d_2 \text{ times}},\dots,\underbrace{P_l,\dots,P_l}_{d_l \text{ times}}) \in X^r$$

but this is only well defined up to permutation of the entries. This suggests the introduction of the "quotient" of X^r by the natural action of S_r, $X^{(r)}$, and we will call it the r-th symmetric product of X. By a quotient we mean the following:

Definition 4.4. *If X is a variety and G is a finite group acting on X, the quotient X/G is a scheme with a projection $X \to X/G$ such that points of X/G correspond to orbits of the action and any G-invariant morphism $X \to X'$ factors through X/G.*

Proposition 4.5. [Lecture 10 in [Har92]] *The r-th symmetric product of X exists, it is a projective variety of dimension nr and its k-points are in correspondence with the 0-cycles $r_1P_1 + \ldots + r_sP_s$ where the $P_i \in X$ are all distinct and $r_1 + \ldots + r_s = r$ (not to be confused with fat points).*

One can then define the Hilbert-Chow map

$$\rho : \mathrm{Hilb}_r(X)_{red} \to X^{(r)},$$

sending Z to the 0-cycle $\sum_P h^0(Z, \mathcal{O}_{Z,P})P$, but the fact that ρ is actually a morphism of varieties is non-trivial (see Section 5.4 in [MF82]).

If $U \subset X^r$ is the open subset consisting of different points, then U is S_r-invariant, and so U/G is an open subset of $X^{(r)}$. Its preimage under the Hilbert-Chow morphism, which is denoted by $\mathrm{Hilb}^\circ_r(X)$, is formed by all the points $[Z]$ such that Z is reduced, and ρ is an isomorphism when restricted to $\mathrm{Hilb}^\circ_r(X)$.

Definition 4.6. *The smooth locus of degree r is $\mathrm{Hilb}^\circ_r(X)$, and $\mathrm{Hilb}^{sm}_r(X) = \overline{\mathrm{Hilb}^\circ_r(X)}$ with the reduced induced structure, is called the smoothable component.*

Since U is irreducible of dimension nr, the same is true for $\mathrm{Hilb}^{sm}_r(X)$.

4.2 Smoothability

We will use the notion of smooth morphism given in Chapter III.10 [Har77], and will assume that k is algebraically closed to simplify the arguments.
Therefore, in light of Example III.10.0.3 and Theorem III.10.2 in [Har77], X is smooth over k if X is regular and has constant dimension, and a morphism $X \to Y$ is smooth if and only if it is flat and all the fibres have the same dimension and are non-singular.

In particular, a finite scheme is smooth if and only if it is reduced.
Since the localization of a regular ring is again regular, we see that smoothness is an open condition: If $\phi : X \to Y$ and all the fibres have the same dimension, X_y non-singular is equivalent to the existence of an open neighbourhood U of y such that $\phi : \phi^{-1}(U) \to U$ is smooth.

Definition 4.7. *Let Z be a finite scheme over k. An abstract smoothing of Z is a flat family $\phi : Y \to T$ of schemes over k such that:*

1. *T is irreducible with generic point η.*
2. *Y_η is smooth over k (or equivalently there is an open subset $U \subset T$ such that $\phi_{|\phi^{-1}(U)}$ is smooth).*
3. *There is a k-rational point t such that $Y_t \cong Z$.*

If Z is embedded in $\mathbb{P}^n$, such a family will be said to be an embedded smoothing if Y is a closed subscheme of $\mathbb{P}^n \times T$, ϕ is the projection and the isomorphism in 3. is induced by the morphism $X \to \mathbb{P}^n \times T$.

The first thing to be noted is that, since $\mathbb{P}^n$ is smooth, the notions of abstract and embedded smoothings are equivalent (Theorem 3.16 in [BJ17]), so we will just refer to them as smoothings, and a scheme Z for which a smoothing exists as smoothable.
It is easy to see that if $Z = Z_1 \sqcup Z_2$ then X is smoothable if and only if both Z_1 and Z_2 are (Corollary 3.14 in [BJ17]), so smoothability only depends on the local properties of Z and not on the configuration of the points that form Z, contrary to what happened in the previous chapter.

Proposition 4.8. [Proposition 5.6. in [BJ17]] *A finite scheme $Z \subset \mathbb{P}^n$ of degree r is smoothable if and only if $[Z]$ lies on the smoothable component of* $\mathrm{Hilb}_r(\mathbb{P}^n)$.

Proof. An embedded smoothing $Y \subset \mathbb{P}^n \times T$, with notation as in the definition of smoothing corresponds, via the universal property of the Hilbert scheme, to a morphism $\phi : T \to \mathrm{Hilb}_r(Z)$ such that the image of some k-point s is a k-point representing a smooth scheme and the image of t is $[Z]$. Since T is irreducible, $\phi(s)$ and $[Z]$ must lie on the same connected component.
On the other hand, if $[Z] \in \mathrm{Hilb}_r^{sm}(\mathbb{P}^n)$, then the restriction of the universal family $\mathcal{U}_r \subset \mathrm{Hilb}_r(\mathbb{P}^n) \times \mathbb{P}^n$ to $\mathrm{Hilb}_r^{sm}(\mathbb{P}^n) \times \mathbb{P}^n$ is still a flat family over $\mathrm{Hilb}_r^{sm}(\mathbb{P}^n)$,which is irreducible, having Z as some fibre and smooth over the smooth locus defined in 4.6, so it gives a smoothing of Z. □

Remark 4.9. We have reduced the study of smoothability to the geometry of the Hilbert scheme. However, if $\dim(\mathbb{P}^n)$ is greater than 2, the Hilbert scheme will be in general non-reduced and reducible. Not only that, but $\dim(\mathrm{Hilb}_r(\mathbb{P}^3)) = O(r^{4/3})$ (See [BI78]). Since the smoothable component has dimension $3r$, we see that for large r the smoothable component is a very tiny part of the Hilbert scheme, and so most subschemes of $\mathbb{P}^3$ are not smoothable.

4.3 Fogarty's Theorem

We now give a complete proof of the fact that $\mathrm{Hilb}_r(\mathbb{P}^2)$ is smooth of dimension $2r$ using monomial ideals and Serre duality. To do so, we will first show that $\mathrm{Hilb}_r(\mathbb{P}^2)$ is connected, and then compute the dimension of the tangent space at its closed points.

4.3.1 Connectedness

To obtain connectedness, we have to show that one can get from any finite scheme to another using these flat families over irreducible schemes.
We will use Grobner degenerations, which, as we already showed, are flat families over $\mathbb{A}^1$. We can assume that our subschemes are in $\mathbb{A}^2$ and therefore identify them with their ideal in $k[x, y]$. Since any ideal has a Grobner basis, it can be connected to a monomial ideal. We also have:

Proposition 4.10. [Section 4.1. in [Mac07]] *Any monomial ideal representing a finite scheme of degree d can be connected to $\langle x^d, y\rangle$ using Grobner degenerations.*

Proof. Since it represents a finite scheme, the ideal I must have the form

$$\langle x^{u_1}, x^{u_2}y^{v_2}, \dots, x^{u_{l-1}}y^{v_{l-1}}, y^{u_l}\rangle,$$

where the generators are minimal. Consider the box diagram associated to it, which is represented in Figure 4.1

If I is not $\langle x^d, y\rangle$, then there must be some corner $x^{a+1}y^{b+1}$ with $b > 0$ such that $x^{a+1}y^b$, $x^a y^{b+1}$ are in I but $x^a y^b$ is not. We define

$$J = \langle x^a y^b - x^{u_1}, x^{u_2}y^{v_2}, \dots, x^{u_{l-1}}y^{v_{l-1}}, y^{u_l}\rangle$$

If we consider the lexicographic order where $x > y$, it is clear then that, since $u_l \geq u_i$ for all i, I is the initial ideal of J with respect to this ordering. In particular, R/J has length d.

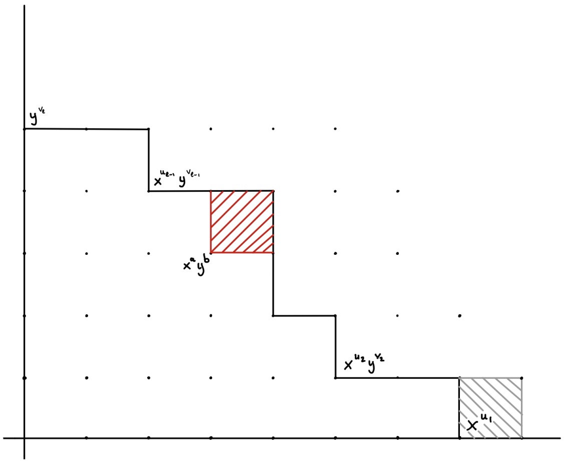

Figure 4.1: The algorithm in 4.10

On the other hand, if we consider the lexicographic order where $y > x$, we will have that $x^a y^b$ and all the monomials $x^{u_i}y^{v_i}$ for $i \geq 2$ are in the initial ideal of I. Also, by the way we chose a and b, $x^{a+1}y^b$ is in the ideal generated by the $x^{u_i}y^{v_i}$ for $i \geq 2$ and therefore in J, but then $x^{u_l+1} = x^{a+1}y^b - x(x^a y^b - x^{u_l})$ is also in the initial ideal of J. Note that the ideal generated by $x^a y^b$, x^{u_l+1} and the $x^{u_i}y^{v_i}$ for $i \geq 2$ has length d and is contained in the initial ideal of J. therefore it is the initial ideal of J with respect to this ordering.

As a conclusion, we have seen that I can be connected with Grobner degenerations with an ideal I' with the same form but with $u'_l > u_l$. In Figure 4.1, the box diagram for I' is obtained removing the red box and adding the grey box to the box diagram of I.

Since any monomial ideal of degree d having x^d as a minimal generator has to be $\langle x^d, y \rangle$. □

Corolary 4.11. [Theorem 5.8 in [Har66]] $\mathrm{Hilb}_r(\mathbb{P}^2)$ *is connected.*

4.3.2 Tangent space

Proposition 4.12. *The tangent space of* $\mathrm{Hilb}_r(X)$ *at* $[Z]$ *is naturally isomorphic to* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, i_*\mathcal{O}_Z)$*, where* $i : Z \to X$ *is the inclusion.*

Sketch of a proof. By the universal property of the Hilbert scheme, a tangent vector at $[Z]$ is equivalent to a closed subscheme $W \subset X \otimes k[\varepsilon]/(\varepsilon^2)$ that is flat over $k[\varepsilon]/(\varepsilon^2)$ and reduces to Z over $\varepsilon = 0$. We can work locally becasue Z is finite. If $X = \operatorname{Spec} A$ and I

is the ideal of Z then we want to fit an ideal J in the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{\cdot\varepsilon} & J & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\cdot\varepsilon} & A[\varepsilon] & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A/I & \xrightarrow{\cdot\varepsilon} & B & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

such that B is flat over $k[\varepsilon]$. But flatness of B is equivalent to injectivity of the $\cdot\varepsilon$ map (because (ε) is the only ideal in $k[\varepsilon]/(\varepsilon^2)$), and implies that the map $I \to J$ is also injective. In this situation, to have commutativity on the left, the image of J in $A[\varepsilon]$ is completely determined by the image of $J/\varepsilon I$ in $A[\varepsilon]/\varepsilon I = A + \varepsilon A/I$, and to have commutativity on the right the first component of this map has to be the inclusion. Therefore all the freedom lies in the choice of an A-homomorphiusm $I \to A/I$. □

Proposition 4.13. *If $Z \subset \mathbb{P}^2$, is a finite scheme then $T_{[Z]} \operatorname{Hilb}_r(\mathbb{P}^2)$ has dimension at most $2\deg(Z)$.*

Proof. The long exact sequence we get when applying $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(-, i_*\mathcal{O}_X)$ to $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_Z \to 0$ is

$$0 \to \operatorname{Hom}(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) \xrightarrow{\alpha} \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^2}, i_*\mathcal{O}_Z) \to \operatorname{Hom}(\mathcal{I}_Z, i_*\mathcal{O}_Z) \xrightarrow{\beta} \operatorname{Ext}^1(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z)$$

Note that α is surjective because $i_*\mathcal{O}_Z$ is a sum of skyscraper sheaves, so β is injective. For any pair of sheaves $\mathcal{F}, \mathcal{G}$ on $\mathbb{P}^2$, let $\chi(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^{\infty} \dim_k \operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$. If $K_{\mathbb{P}^2}$ is the canonical sheaf, by Serre duality

$$\begin{aligned} \chi(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) &= \dim \operatorname{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) + \dim \operatorname{Ext}^1(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) + \dim H^0(\mathbb{P}^2, i_*\mathcal{O}_Z \otimes K_{\mathbb{P}^2})' = \\ &= 2n - \dim \operatorname{Ext}^1(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z), \end{aligned}$$

So it suffices to show that $\chi(i_*\mathcal{O}_Z, i_*\mathcal{O}_Z) = 0$. Since χ is clearly additive on short exact sequences on its first entry, taking a resolution of $i_*\mathcal{O}_Z$ by locally free sheaves:

$$0 \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to i_*\mathcal{O}_X \to 0,$$

it will be enough to compute $\chi(\mathcal{E}, i_*\mathcal{O}_Z)$. Note that, by Proposition III.6.9 in [Har77] we can calculate $\mathrm{Ext}^i(\mathcal{E}, i_*\mathcal{O}_Z(d))$ as the global sections of $\mathcal{E}xt^i(\mathcal{E}, i_*\mathcal{O}_Z(d))$ for $d >> 0$, and by Proposition III.6.5. in [Har77], $\mathcal{E}xt^i(\mathcal{E}, \mathcal{G}) = 0$ for any $\mathcal{G}$ and any $i > 0$. Since $i_*\mathcal{O}_Z(d) \cong i_*\mathcal{O}_Z$ because Z is finite, we conclude that $\chi(\mathcal{E}, i_*\mathcal{O}_Z) = \dim \mathrm{Hom}(\mathcal{E}, i * \mathcal{O}_Z)$ and, because $\mathcal{E}$ is locally free, this is $\deg(Z) \operatorname{rank}(\mathcal{E})$. Finally, since Z is finite, looking at the stalks at some point we have $\operatorname{rank}(\mathcal{E}_0) - \operatorname{rank}(\mathcal{E}_1) + \operatorname{rank}(\mathcal{E}_2) = 0$.

□

4.3.3 Proof of the theorem and generalisations

Theorem 4.14. [Theorem 2.4 in [Fog68]] $\mathrm{Hilb}_r(\mathbb{P}^2)$ *is smooth, irreducible of dimension* $2r$.

Proof. $\mathrm{Hilb}_r^{sm}(\mathbb{P}^2)$ is an irreducible component of dimension $2r$, by 4.5. If there were more irreducible components, by connectedness, some of them would have to intersect $\mathrm{Hilb}_r^{sm}(\mathbb{P}^2)$ at some $[Z]$, but then $\dim T_{[Z]} \mathrm{Hilb}_r(\mathbb{P}^2) > 2r$, contradicting 4.13. Therefore $\mathrm{Hilb}_r(\mathbb{P}^2) = \mathrm{Hilb}_r^{sm}(\mathbb{P}^2)$ is irreducible and has dimension $2r$. Since the tangent space has dimension at most $2r$, it is also smooth. □

Corolary 4.15. *Any finite scheme that can be embedded in* $\mathbb{P}^2$ *is smoothable.*

Remark 4.16. The proof of conectedness of $\mathrm{Hilb}_r(\mathbb{P}^2)$ that we gave in 4.10 can be quickly generalised to show that $\mathrm{Hilb}_r(\mathbb{P}^n)$ is connected.

The fibre of the Hilbert-Chow morphism over a closed point of $X^{(n)}$ represented by the 0-cycle $n_1P_1 + \ldots + n_sP_s$ is isomorphic to the product of the Hilbert schemes $\mathrm{Hilb}_{n_i}(n_iP_i)$ (here, n_iP_i is a fat point scheme). If X is smooth of dimension n, then the schemes nP is isomorphic to $n0 \subset \mathbb{A}^n$.
In the proof we gave in 4.10, all the ideals were contained in $\langle x^r, \ldots, y^r \rangle$. It shows that the Hilbert schemes $\mathrm{Hilb}_r(r0)$ are connected.

Theorem 4.17. *If X is a non-singular, connected projective variety,* $\mathrm{Hilb}_r(X)$ *is connected.*

Proof. Since X is connected, so is $X^{(n)}$. The fibres of the Hilbert-Chow morphism at closed points are also connected by the discussion above. We conclude using the well known fact that if $S \to T$, T is connected and the fibres of the morphism are connected, so is S. □

Remark 4.18. We have showed that if $A = k[x_1, \dots, x_n]/\langle x_1, \dots, x_n\rangle^r$, $\mathrm{Hilb}_r(\mathrm{Spec}\, A)$ is connected, but it is in fact connected for any connected local ring A of finite dimension (Theorem 2.2 in [Fog68]). This shows that for any connected X, $\mathrm{Hilb}_r(X)$ is connected.

Remark 4.19. In fact, Fogarty proved theorem [Fog68] with $\mathbb{P}^2$ replaced by a smooth surface X. By theorem 4.17, we know that $\mathrm{Hilb}_r(X)$ is connected, and the proof of 4.13 can be done on any smooth, projective surface replacing $\mathbb{P}^2$, so the same argument given in 4.14 shows that $\mathrm{Hilb}_r(X)$ is smooth of dimension $2r$ whenever X is a smooth surface.

4.4 The SHGH conjecture

4.4.1 Families of fat points

Is there a similar tool to the Hilbert scheme for fat points? It turns out that the answer is yes.

Theorem 4.20. [Theorem 1 in [Pax91]] *Let $k = \overline{k}$ and $m_1, \dots, m_s, n$ be fixed positive integers, there is a family of projective schemes $f : Y \to S$ such that S is a smooth, irreducible variety and the fibres Y_s, where s varies through the closed points of S parametrize all fat point schemes in $\mathbb{P}^n_k$ of the form $m_1P_1 + \ldots + m_sP_s$.*

Proof. The choice of S is very natural. If $D \subset (\mathbb{P}^n)^s$ is the closed subset consisting of uples with some repeated point, we let S be its complement. Clearly S is quasi-projective, integral and smooth, and since any set of s points is not contained in some hyperplane, it can be covered by open sets isomorphic to $(\mathbb{A}^n)^s$.
If $\mathcal{I}_j$ is the ideal of the closed subset of $S \times \mathbb{P}^n$ given by the equality of the j-th and the last coordinate, we let $Y \subset \mathbb{P}^n_S$ be the closed subset associated to the ideal sheaf $\mathcal{J} = \mathcal{I}_1^{m_1} \cap \ldots \cap \mathcal{I}_s^{m_s}$, and let $f : Y \subset \mathbb{P}^n_S \to S$ be the projection. Since S is locally $(\mathbb{A}^n)^s$, the morphism

$$\mathbb{P}^n_{k(s)} = \mathbb{P}^n \times S \times_S \mathrm{Spec}\, k(s) \to \mathbb{P}^n \times S$$

is locally given by the natural ring homomorphism

$$k[y_1, \dots, y_n, x_{11}, \dots, x_{sn}] \to k[y_1, \dots, y_n] \otimes_k \frac{k[x_{11}, \dots, x_{sn}]_{\mathfrak{p}}}{\mathfrak{p}} = k[y_1, \dots, y_n] \otimes_k k(s)$$

The ideal corresponding to $\mathcal{J}$ in the left hand side is

$$\bigcap_i (y_1 - x_{i1}, \dots, y_n - x_{i_n})^{m_i}$$

And its extension (which is the ideal corresponding to Y_s) is the ideal corresponding to the fat point scheme $m_1Q_1 + \ldots + m_sQ_s$ in $\mathbb{A}^n_{k(s)} \subset \mathbb{P}^n_{k(s)}$, where $Q_i = (x_{i1}, \ldots, x_{in})$.
From this we deduce from 2.15 that the family is flat, since all fat point schemes $m_1Q_1 + \ldots + m_sQ_s$ have the same degree independently of the ground field.
If $s = (P_1, \ldots, P_n)$ is a closed point, then $k(s) = k$ because k is algebraically closed, and the x_{ij} correspond to the coordinates of the P_i, so $Y_s \subset \mathbb{A}^n$ is precisely $m_1P_1 + \ldots + m_sP_s$. □

Remark 4.21. Since $Y \to S$ is a flat family of projective schemes, it gives naturally a morphism $\phi : S \to \mathrm{Hilb}_r(\mathbb{P}^n)$. *A priori*, the image of ϕ, which we denote by $\mathrm{Fat}(m_1, \ldots, m_s)$ is constructible by Chevalley's theorem (Problem II.3.19 in [Har77]), but in fact, $\phi(S)$ is locally closed in $\mathrm{Hilb}_r(\mathbb{P}^n)$ (see [Cop93]).

Corolary 4.22. [Propositions 1 and 2 in [Pax91]] *With notation as in the previous theorem,*

a) For each d, $s \mapsto H_{Y_s}(d)$ is lower-semicontinuous.

b) $s \mapsto \mathrm{reg}(Y_s)$ is upper-semicontinuous.

c) There is an open, dense subset $W \subset S$ such that $\mathrm{reg}(Y_s)$ is minimal whenever $s \in W$.

d) There is an open, dense subset $V \subset S$ such that for all d, $H_{Y_s}(d)$ is maximal whenever $s \in V$. Moreover, $V \subset W$.

Proof. Note that, for big d, $H_{\mathcal{I}_{Y_s}}(d) = H_{\mathbb{P}^n}(d) - \deg(Y_s)$ does not depend on d, so by 2.15, $\mathcal{I}$ is flat and so $d \mapsto H_{\mathcal{I}_{Y_s}}(d)$ i upper-semicontinuous. Then a) follows form the equality $H_{Y_s}(d) = H_{\mathbb{P}^n}(d) - H_{\mathcal{I}_{Y_s}}(d)$.
By a), if $H_{Y_s}(d) < \deg(Y_s)$, then $H_{Y_{s'}}(d) < \deg(Y_{s'})$ for all s' in a neighbourhood of s. Therefore, $\mathrm{reg}(Y_s) > d$ implies $\mathrm{reg}(Y_{s'} > d$ on a neighbourhood of s. In other words, $\mathrm{reg}(Y_\cdot)$ is upper-semicontinuous.
c) follows immediately because r is clearly bounded below and therefore attains a minimum, and the fact that W is dense follows form the irreducibility of S.
For d), note that

$$W = \bigcap_d \{s : H_{Y_s}(d) \text{ is maximal}\}$$

is an intersection of open sets, but since $H_{Y_s}(d)$ is constant for all $d > m_1 + \ldots + m_s - 1$, this is a finite intersection of open sets. All of them are nonempty because $H_{Y_s}(d) \leq \deg(Y_s)$, so for fixed d, $H_{Y_s}(d)$ attains its maximum. Since S is irreducible, all of them intersect in a nonempty open set, W, and it is clear that $W \subseteq V$. □

This gives another proof of the Segre bound:

Corolary 4.23. *With notation as in the previous corollary, for a generic choice of s (more concretely, for any $s \in V$),* $\mathrm{reg}(Y_s) \leq \mathrm{Seg}(Y_s)$.

Proof. By 3.19, we know that if the $Y_s = m_1P_1 + \ldots + m_sP_s$ and the P_i lie on a rational normal curve, $r(Y_s) = \mathrm{Seg}(Y_s)$, so the minimal value of r must be at most $\mathrm{Seg}(Y_s)$. □

Remark 4.24. It is known that for large values of m_i, $w(X_s)$ is not the optimal bound for $r(Y_s)$ (see Theorem 3.1. in [Tru94]).

Note that part d) of 4.22 says that for any given m_i, n, there is a Hilbert function $F_{m_1,\ldots m_s,n}$ such that for a generic choice of points $P_i \in \mathbb{P}^n$, $Z = m_1P_1 + \ldots + m_sP_s$ will have $F_{m_1,\ldots m_s,n}$ as its Hilbert function. Since

$$H_Z(d) = \binom{n+d+1}{d} - H_{I_Z}(d) \quad \text{and} \quad H_Z(d) = \sum_i \binom{m_i+n+1}{n} \text{ for } d >> 0,$$

The candidate to being the largest Hilbert function is what we will call the expected dimension function:

$$E(d) = \max\left\{\binom{n+d+1}{d}, \deg(Z)\right\}.$$

More arguments supporting this claim for $n = 2$ come from using the flat $\mathcal{U}_r \to \mathrm{Hilb}_r(\mathbb{P}^2)$ instead of $Y \to S$ in 4.22. Note that the expected dimension function is an O-sequence such that its first difference function is again a O-sequence. Therefore, by 3.10, there is at least one closed subscheme having E as its Hilbert function. Since $\mathrm{Hilb}_r(\mathbb{P}^2)$ is irreducible by Fogarty's theorem, this implies the following:

Proposition 4.25. *For a generic choice of* $[Z] \in \mathrm{Hilb}_r(\mathbb{P}^2)$, $H_Z(d) = E(d)$.

Corolary 4.26. *For a generic choice of* $[Z] \in \mathrm{Hilb}_r(\mathbb{P}^2)$, $\mathrm{reg}(Z) = \min\left(d : \binom{n+d+1}{d} \geq r\right)$.

Therefore, the question of when is $F_{m_1,\ldots m_s,n} = E$ is equivalent to asking when $\mathrm{Fat}(m_1, \ldots, m_s)$, which is locally closed, intersects U.
This, among other reasons, suggests that the failure of the equality $H_Z = E$ whenever Z is a fat point in $\mathbb{P}^2$ is special in some way that we will make precise in the next section:

4.4.2 The rational surface

Consider the fat point scheme $Z = m_1P_1 + \ldots + m_sP_s \subset \mathbb{P}^2$. The equality $H_Z(d) = E(d)$ is equivalent to saying that

$$h^0(\mathbb{P}^2, \mathcal{I}_Z(d)) = \min\left\{0, \binom{n+d+1}{d} - \deg(Z)\right\}.$$

Now let Y be the rational surface obtained by blowing up the points $P_1, \ldots, P_s$.

Cl denotes the group of Weil divisors modulo linear equivalence. An example of such a divisor is the pullback of a line in $\mathbb{P}^2$, which we denote by E_0, or the exceptional divisor E_i arising from each blowup. After repeated applications of Proposition V.3.2 in [Har77] we see that $\mathrm{Cl}(Y)$ is freely generated over $\mathbb{Z}$ by the $E_0, \ldots, E_s$. The connection between divisors and Hilbert functions is given by:

Proposition 4.27. [Proposition IV.1.1. in [Har10]] *In the above situation, for any $m_i \geq 0$, let $D = dE_0 - m_1E_1 - \ldots - m_sE_s$, and let $\mathcal{O}_Y(D)$ be the invertible sheaf associated to D. There is a natural isomorphism*

$$H_Z(d) \cong H^0(Y, \mathcal{O}_Y(D)).$$

Therefore we can use the theory of surfaces to understand the Hilbert function of Z. The main tool is Riemann Roch Theorem:

Theorem 4.28. [Theorem V.1.6 in [Har77]] *If Y is a smooth rational surface, there is a bilinear form in* $\mathrm{Cl}(Y)$*, usually denoted by ., such that for any divisor C,*

$$h^0(Y, \mathcal{O}_Y(D)) - h^1(Y, \mathcal{O}_Y(D)) + h^2(Y, \mathcal{O}_Y(D)) = \frac{D.D - D.K_Y}{2} + 1,$$

where K_Y is the canonical divisor.

In our case the canonical divisor is $K_Y = -3E_0 + E_1 + \ldots + E_s$ by several applications of Proposition V.3.3 in [Har77], and the intersection product is given by Proposition V.2.2. in [Har77]:

$$E_i.E_j = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j \neq 0 \\ 1 & \text{if } i = j = 0 \end{cases}.$$

If C and D are effective divisors that intersect properly, $C.D$ is the degree of their intersection, so it has to be ≥ 0.
By Serre duality, $h^2(Y, \mathcal{O}_Y(D)) = h^0(Y, \mathcal{O}_Y(K_Y - D)) = 0$ if D is as in 4.27, and so

$$h^0(Y, \mathcal{O}_Y(D)) - h^1(Y, \mathcal{O}_Y(D)) = \binom{d+2}{2} - \sum \binom{m_i+1}{2}$$

is the expected value of $h^0(Y, \mathcal{O}_Y(D))$.

Proposition 4.29. *With notation as in 4.27, if* $\frac{D.D-D.K_Y}{2}+1\geq 0$ *but there is some prime divisor* C *such that* $C.D=-m<-1$ *and* $C.C=C.K_Y=-1$ *then The Hilbert function of* $m_1P_1+\ldots+m_sP_s$ *is not the expected dimension function* $E(d)$*.*

Proof. Let $F=D-mC$ We use that $\mathbb{P}H^0(Y,\mathcal{O}_Y(D))$ is in bijection to the set of effective divisors linearly equivalent to D (Proposition II.7.7. in [Har77]). If $F\equiv B$ then clearly $D\equiv mC+E$, but on the other hand, if $D\equiv B$ and B is effective, $B.C=-m$ so B contains m copies of C and therefore $B=mC+B'$. Therefore, $h^0(Y,\mathcal{O}_Y(D))=h^0(Y,\mathcal{O}_Y(F))$, but then, by the Riemann-Roch theorem,

$$h^0(Y,\mathcal{O}_Y(D))=h^0(Y,\mathcal{O}_Y(F))\geq\frac{F.F-F.K_Y}{2}+1=$$
$$=\frac{D.D-K_Y.D}{2}+1-\frac{m^2-m}{2}>\frac{D.D-K_Y.D}{2}+1=E(d).$$

□

Such a curve is called an exceptional curve of the linear system $\mathcal{O}_X(D)$. The SHGH conjecture states that the existence of these exceptional curves is the only way that the Hilbert function can not be the expected one:

Conjecture 4.30. *Let* $D=dE_0-m_1E_1-\ldots-m_sE_s$ *be a divisor in the rational surface* Y*, obtained after blowing up* $\mathbb{P}^2$ *at* s *different points. Then*

$$h^0(Y,\mathcal{O}_Y(D))\neq\max\left\{0,\binom{d+2}{2}-\sum_i\binom{m_i+1}{2}\right\}$$

if and only if there is an exceptional curve C*.*

Remark 4.31. Proposition 4.27 also explains why fat points are of special interest among all finite schemes, since they are the ones whose Hilbert functions give information about the geometry of rational surfaces.

Remark 4.32. There are numerous conjectures under the name of SHGH, and they are all equivalent, and they are true for $s\leq 9$ (Theorem 9 in [Nag60]).

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