Fourier transform for compact Jacobians ①

(after D. Arinkin).

§1 Abelian schemes.

B : sm / $\mathbb{C}$ $\quad \pi : A \longrightarrow B$: abelian sch. rel dim $= g$

$\rightsquigarrow \pi^\vee : A^\vee := \mathrm{Pic}^\circ_{A/B} \longrightarrow B$: dual abelian sch

$P \longrightarrow A \times_B A^\vee$: Poincaré line bundle
(trivialized along the unit)

$$\begin{tikzcd} & A \times_B A^\vee \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A \arrow[dr] \arrow[rr, phantom, "\square"] & & A^\vee \arrow[dl] \\ & B & \end{tikzcd}$$

$\mathcal{F} : D^b_{coh}(A) \longrightarrow D^b_{coh}(A^\vee)$, $\quad \mathcal{E}^\cdot \longmapsto Rp_{2*}(p_1^* \mathcal{E}^\cdot \otimes P)$

<u>Thm</u> (Mukai) $\mathcal{F} : D^b_{coh}(A) \xrightarrow{\cong} D^b_{coh}(A^\vee)$ / B

- Thm of cube
- Cohomology of Poincaré line bundle :

$$Rp_{2*} P \cong e_*(\det \mathbb{E})[-g] \quad \text{in } D^b_{coh}(A^\vee)$$

Taking Chern character $ch(P) \in CH^*(A \times_B A^\vee)_{\mathbb{Q}}$, ②

$$\mathcal{F} : CH^*(A)_{\mathbb{Q}} \xrightarrow{\cong} CH^*(A^\vee)_{\mathbb{Q}}.$$

$\mathcal{F}$ is an important tool to study Chow group $/\mathbb{Q}$.

$[N] : A \longrightarrow A$ "multiplication by N map"

$$CH^*_{(\omega)}(A) := \{ \alpha \in CH^*(A)_{\mathbb{Q}} : [N]^*\alpha = N^\omega \alpha \ \forall N \in \mathbb{Z} \}.$$

Thm [Beauville, Deninger-Murre]

1/ $\mathcal{F} : CH^p_{(\omega)}(A) \xrightarrow{\cong} CH^{p-\omega+g}_{(2g-\omega)}(A)$

2/ $CH^*(A)_{\mathbb{Q}} = \bigoplus_{\omega=0}^{2g} CH^*_{(\omega)}(A)$ multiplicative.

- Multip. splitting of relative Chow motive $h(A/B)$
- Motivic Lefschetz decomposition [Künnemann].

Main Questions :

1/ What would be a generalization of abelian scheme?

2/ Can we extend Fourier-Mukai transform?

3/ Which properties can be extended?

Today : $C \longrightarrow B$ irred. $\overset{\text{loc.}}{\text{planar}}$ curve $J_C \longrightarrow B$: comp. Jac.

following Arinkin.

§2. Compactified Jacobians.

$p : C \to B$: irred. curve, loc planar sing, of genus $= g$.
(with section)

$J_C \to B$: relative Jacobian (deg $= 0$ line bundles)

$\pi : \overline{J}_C \to B$: compactified Jacobian
(rank 1. torsion free sheaves, deg $= 0$)

$\Rightarrow$ · π is proper surjective.

[Altman-Kleiman] · $\overline{J}_{C_b}$ irred, at worst lci singularity

- δ-regularity :

$A \to B$ · finite type com. group scheme

[Chevalley] $0 \to \underbrace{\mathrm{Aff}_b}_{\text{affine}} \to A_b \to \underbrace{G_b}_{\text{abelian variety}} \to 0$

$\delta : B \to \mathbb{Z}_{\geq 0}$ $\quad b \longmapsto \dim(\mathrm{Aff}_b)$

<u>Def.</u> $A \to B$ is called δ-regular if

$$\operatorname{codim} \overline{\{ b \in B : \delta(b) \geq k \}} \geq k \qquad \forall k \geq 0$$

<u>Example</u> · $J^{[0]}_{g,n} \to \overline{M}_{g,n}$ is δ-regular.
(locus of k-self nodes has codim $= k$)

· [Diaz-Harris] $\mathcal{C} \to B$: semiuniversal deform C/k
$\uparrow$ loc planar

$J_{\mathcal{C}} \to B$ is δ-regular

(see also [Maulik-Yun])

[Fantechi–Göttsche–van Straten] : $p: C \to B$ irred plane curve

$\bar{J}_C$ is smooth $\Rightarrow$ $J_C \to B$ is δ-regular.

As a generalization of abelian schemes, we consider a tuple : $C \to B$

$$\Big(\bar{J}_C \xrightarrow{\pi} B.\ \ \underbrace{J_C \longrightarrow B}_{\delta\text{-regular}}.\ \ \mu: J_C \times_B \bar{J}_C \longrightarrow \bar{J}_C\Big).$$

<u>Rmk</u> : Degenerate abelian scheme [Arinkin–Fedorov].

- C : sm $\Rightarrow$ J_C is principally polarized $(J_C \xrightarrow{\simeq} J_C^\vee)$.

§3. Outline of Arinkin's Fourier transform.

C/k : integral curve, loc planar singularity. ($p_0 \in C^{sm}$)

$p : C|_{J_C \times \bar{J}_C} \longrightarrow J_C \times \bar{J}_C$. L_1, F_2 : univ sheaves (triv. along p_0)

$$P := \langle L_1, F_2 \rangle := \det Rp_*(L_1 \otimes F_2) \otimes \det Rp_*(L_1)^\vee \otimes \det Rp_*(F_2)^\vee \otimes \det Rp_*(\mathcal{O})$$

C : sm $\Rightarrow$ $\langle -, - \rangle$: Deligne pairing (recovers Poincaré line bdl.)

Thm (Arinkin) $J_C \times \bar{J}_C \xrightarrow{p_1} J_C$. Then

$$Rp_{1*} P \cong k(e)[-g] \quad \text{in } D^b_{Coh}(J_C)$$

$$\begin{tikzcd} P \arrow[d, dash] & & ? \arrow[d] \\ J_C \times \bar{J}_C \cup \bar{J}_C \times J_C \arrow[rr, hook, "i"] & & \bar{J}_C \times \bar{J}_C \end{tikzcd}$$

codim of complement $= 2$

Argument in two steps

Step 1 : Extend P to maximal CM sheaf $\bar{P} \in Coh(\bar{J}_C \times \bar{J}_C)$

Step 2 : Consider $\mathcal{C} \to B$ where $J_{\mathcal{C}} \to B$ is δ-regular

$\Rightarrow$ dimension bound of $\mathrm{Supp}(\bar{P}^{-1} \circ \bar{P})$

Bound + CM $\Rightarrow$ extend auto-equivalence.

§4 Cohen-Macaulay sheaf

Def (C.A). R : loc ring M . finite R-mod. M is CM if

$$\dim M = \operatorname{depth} M.$$

It is maximal CM if CM + $\dim_R M = \dim R$

$\hookrightarrow$ $\mathcal{F} \in \mathrm{Coh}(X)$: CM.

Lemma F : maximal CM on X. $Z \hookrightarrow X$, codim $Z \geq 2$.
$j : X \setminus Z \hookrightarrow X$. Then

$$F \xrightarrow{\cong} j_* j^* F$$

$\omega_X^\bullet \in D^b_{\mathrm{Coh}}(X)$: dualizing complex

$\rightsquigarrow \mathbb{D} : D^b_{\mathrm{Coh}}(X) \xrightarrow{\cong} D^b_{\mathrm{Coh}}(X) \quad \mathcal{E}^\bullet \longmapsto R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \omega_X^\bullet).$

Example X : Gorenstein $\Leftrightarrow$ $\omega_X^\bullet$ is a line bundle.

Def ($\mathbb{D}$). $F \in \mathrm{Coh}(X)$ is CM of codim $= d$ if and only if

$$\mathcal{H}^i(\mathbb{D}F) = 0 \quad \text{if } i \neq d.$$

Lemma Suppose $\mathcal{E}^\bullet \in D^b_{\mathrm{Coh}}(X)$ satisfies

i) $\operatorname{codim}(\operatorname{supp} \mathcal{E}^\bullet) \geq d$

ii) $\mathcal{H}^i(\mathcal{E}^\bullet) = 0 \quad i > 0$

iii) $\mathcal{H}^i(\mathbb{D}\mathcal{E}^\bullet) = 0 \quad i > d$

Then $\mathcal{E}^\bullet$ is a CM sheaf of codim $= d$.

§5. Abel map.

$p_0 \in C$, sm pt. $C^{[n]}$ = Hilbert scheme of n points

We define the Abel map

$$\alpha : C^{[n]} \longrightarrow \bar{J}_C, \quad D \subset C \longmapsto I_D^{\vee}(-np_0).$$

If $n \geq 2g-1$, α is smooth & surj (proj bundle).

Idea) 1/ Construct maximal CM sheaf Q on $C^{[n]} \times \bar{J}_C$ extending $(\alpha \times \mathrm{id})^* P$.

2/ Maximal CM gives a descent data along

$$\alpha \times \mathrm{id} : C^{[n]} \times \bar{J}_C \longrightarrow \bar{J}_C \times \bar{J}_C$$

3/ fppf descent $\Rightarrow \exists\ \bar{P} \in \mathrm{Coh}(\bar{J}_C \times \bar{J}_C)$, $(\alpha \times \mathrm{id})^* \bar{P} \cong Q$

$$\iota^* \bar{P} \cong P \quad \text{and} \quad \bar{P} \text{ is flat} / \bar{J}_C$$

Baby case: C: sm proj.

$$\begin{aligned}
(\alpha \times \mathrm{id})^* P &\cong \langle \mathcal{O}_C(D - np_0), L_2 \rangle \\
&\cong \langle \mathcal{O}_C(D), L_2 \rangle \qquad \} \ L|_{p_0} \cong \mathcal{O}_C \\
&\cong \det R p_* ((L_2 - \mathcal{O}_C)|_D)
\end{aligned}$$

$\Rightarrow$ Fiber of $(\alpha \times \mathrm{id})^* P$ over (D, L_2)

$$\cong \Lambda^n H^0(L_2|_D) \otimes (\det H^0(\mathcal{O}_D))^{\vee}.$$

§6. CM extension of Poincaré line bundle

$C \overset{2}{\hookrightarrow} S$: sm surface (Altman–Kleiman).

$\rightsquigarrow\ C^{[n]} \times \bar{J}_C \overset{2}{\hookrightarrow} S^{[n]} \times \bar{J}_C$ ← construct a sheaf here

<u>1st attempt</u>:

$$S^{[n]'} := \{ D \subset S : D \subset (\text{sm curve}) \} \overset{\circ}{\hookrightarrow} S^{[n]}.$$

$$\Updownarrow$$

$$\mathcal{O}_D \cong \prod k[t]/t^{n_i}$$

"Curvilinear part" $\operatorname{codim}(S^{[n]} \setminus S^{[n]'}) \geq 2$.

$$\mathrm{Flags}^{n\,'}_S = \{ \phi = D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_n : D_n \in S^{[n]'} \}$$

$$\boxed{\begin{array}{ccccc} S^{[n]'} & \xleftarrow[\text{finite flat}]{\psi} & \mathrm{Flags}^{n\,'}_S & \xrightarrow[\Sigma_n\text{-equiv}]{\sigma} & S^n \xleftarrow{\iota} C^n \\ & & \circlearrowleft & & \circlearrowleft \\ & & \Sigma_n & & \Sigma_n \end{array}}$$

⇒ For each $F \in \bar{J}$, we consider

$$\underbrace{\left(\psi_* \sigma^* \iota_* F^{\boxtimes n} \right)^{\mathrm{sgn}}}_{\text{anti-inv } \Sigma_n} \otimes \det \underbrace{A^\vee}_{\text{taut bundle } H^0(\mathcal{O}_D)}$$

- Supp on $C^{[n]} \cap S^{[n]'}$.
- Compare to baby case $(\alpha \times \mathrm{id})^* P|_{(D,L)} \cong \Lambda^n H^0(L|_D) \otimes \det H^0(\mathcal{O}_D)^{\vee}$

2nd attempt:

- Haiman's isospectral Hilbert scheme.

$$\begin{tikzcd} \widetilde{S}^{[n]} \arrow[r, "\sigma"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\square^{red}"] & S^n \arrow[d] \\ S^{[n]} \arrow[r, "HC"] & S^{(n)} = S^n/\Sigma_n \end{tikzcd}$$

$\square^{red}$ = reduced fiber product

- ψ is finite flat.
- $\widetilde{S}^{[n]}$ is normal Gorenstein.
- $$\begin{tikzcd} S^{[n]'} \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathrm{Flag}_S^{n'} \arrow[l] \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & S^n \arrow[d, equal] \\ S^{[n]} & \widetilde{S}^{[n]} \arrow[l, "\psi"'] \arrow[r, "\sigma"] & S^n \\ & \circlearrowleft \Sigma_n & \circlearrowleft \Sigma_n \end{tikzcd}$$

On $C^n \times \overline{J}_C$

$$F_n := p_{1,n+1}^* F \otimes \cdots \otimes p_{n,n+1}^* F$$

$$S^{[n]} \times \overline{J}_C \xleftarrow{\psi} \widetilde{S}^{[n]} \times \overline{J}_C \xrightarrow{\sigma} S^n \times \overline{J}_C \xhookleftarrow{\iota} C^n \times \overline{J}_C$$

$$Q := (\psi_* \sigma^* \iota_* F_n)^{sgn} \otimes \det A^\vee \quad \text{on } S^{[n]} \times \overline{J}_C.$$

Prop Q is a CM sheaf supp on $C^{[n]} \times \overline{J}_C$.

($\because$ $\widetilde{S}^{[n]}$: Gorenstein, σ : finite Tor dim $\Rightarrow$ $\sigma^* \iota_* F_n$ is CM.
ψ : finite flat $\Rightarrow (\psi_* \sigma^* \iota_* F_n)^{sgn}$ is CM)

ThmA $\exists!$ maximal CM sheaf $\overline{P}$ on $\overline{J}_C \times \overline{J}_C$ extending Poincaré line bundle

Sketch) • We have

$$Q \cong (\alpha \times id)^* P$$

outside codim ≥ 2 locus.

• Q is maximal CM.

$\Longrightarrow$ desent data extends across codim ≥ 2.

$\underset{\text{fppf}}{\Longrightarrow}$ $\exists$ $\overline{P} \in Coh(\overline{J}_C \times \overline{J}_C)$ is maximal CM.

• Complement of

$$J_C \times \overline{J}_C \cup \overline{J}_C \times J_C \xhookrightarrow{i} \overline{J}_C \times \overline{J}_C$$

has codim $= 2$.

$$\overline{P} \cong \iota_* P$$

so it is unique. $\square$

§7. Arinkin's autoduality.

If $\overline{P}$ induces an auto-equivalence, the inverse kernel should be:

$$\overline{P}^{-1} = \overline{P}^{\vee} \otimes p_2^* \omega_\pi [g], \quad \overline{P}^{\vee} := \mathcal{H}om(\overline{P}, \mathcal{O}).$$

- $\Psi := Rp_{13*}(p_{12}^* \overline{P}^{\vee} \otimes p_{23}^* \overline{P}) \in D^b_{Coh}(\overline{J}_C \times \overline{J}_C)$.

  To show $\overline{P}^{-1} \circ \overline{P} \cong id$, it is enough to show:

Thm B $\Psi \cong \mathcal{O}_\Delta \otimes \det \mathbb{E}[-g]$ in $D^b_{Coh}(\overline{J}_C \times \overline{J}_C)$.

↳ $\Delta : \overline{J}_C \longrightarrow \overline{J}_C \times \overline{J}_C$.

Idea) 1/ Compute codim(Supp Ψ) by spreading out C to a δ-regular family

2/ Use coherent duality to show Ψ is CM sheaf

3/ Use extension property to extend isomorphism.

For $F \in \overline{J}_C$, let

$$P_F := P|_{J \times \{F\}} \longrightarrow J_C$$

: line bundle on J_C

Prop $(F_1, F_2) \in \operatorname{Supp} \Phi$, then $P_{F_1} \cong P_{F_2}$.

Pf) $H^i(\overline{J}_C, \overline{P}_{F_1}^\vee \otimes \overline{P}_{F_2}) \neq 0$, $\exists i$.

Let $T \xrightarrow{\mathbb{G}_m} J_C$ assoc. to $P_{F_1}^\vee \otimes P_{F_2}$.

T is a comm group scheme $\curvearrowright \overline{P}_{F_1}^\vee \otimes \overline{P}_{F_2}$

$T \curvearrowright H^i(\overline{J}_C, \overline{P}_{F_1}^\vee \otimes \overline{P}_{F_2})$

Schur's lemma $\Rightarrow$ $\chi : T \longrightarrow \mathbb{G}_m$. s.t. $\chi|_{\mathbb{G}_m} = \mathrm{id}$

$\Rightarrow$ $T \simeq J_C \times \mathbb{G}_m$ $\boxtimes$

Pull back along Abel map : $\alpha : C \to \overline{J}_C$

(*) $(F_1, F_2) \in \operatorname{Supp} \Phi \Rightarrow F_1|_{C^{sm}} \cong F_2|_{C^{sm}}$

Cor Let $\tilde{g} = g(\tilde{C}) < g$. $\tilde{C} \to C$: normalization. Then

$$\dim(\operatorname{Supp} \Phi) < 2g - \tilde{g}.$$

Pf) $\mu \times \mathrm{id} : J_C \times \overline{J}_C \times \overline{J}_C \longrightarrow \overline{J}_C \times \overline{J}_C$: sm. rel dim $= g$.

Enough to show :

$$\dim (\mu \times \mathrm{id})^{-1} \operatorname{Supp} \Phi < 3g - \tilde{g}.$$

Consider projection :

$$(\mu \times \mathrm{id})^{-1} \operatorname{Supp} \Phi \xrightarrow{p_{12}} \overline{J}_C \times \overline{J}_C$$

(*)

$\Rightarrow$ dim of fibers $\leq g - \tilde{g}$ $\boxtimes$

(strict inequality involves more work).

Proof of Thm B). $\Psi \cong \mathcal{O}_\Delta \otimes \det \mathbb{E}[-g]$.

- Take $C \to B$ s.t. $J_C \to B$ is δ-regular

$$\Psi_{univ} := Rp_{13*}(p_{12}^* \bar{P}^\vee \otimes p_{23}^* \bar{P}) \in D^b_{Coh}(\bar{J}_C \times_B \bar{J}_C).$$

Step 1) codim Supp $\Psi_{univ} = g$.

$B_{(\tilde{g})} := \{ C \in B : g(\tilde{C}) = \tilde{g} \}$

δ-regular $\Rightarrow$ codim $B_{(\tilde{g})} \geq g - \tilde{g}$.

Cor $\Rightarrow$ codim Supp $\Psi_{univ} \geq g$ (over $B_{(g)}$, $=$)

In fact all codim $= g$ components meet $\pi^{-1}(B_{(g)})$ with Δ. $\boxtimes$

Step 2) $\Psi_{univ}[g]$ is a CM sheaf of codim $= g$

We use Lemma after Def ($\mathbb{D}$).

- $\mathcal{H}^i(\Psi_{univ}) = 0 \quad i > g \quad (\because$ rel dim $p_{13} = g)$
- $\mathcal{H}^i(\mathbb{D}\Psi_{univ}) = 0 \quad i > 0 \quad (\because$ coherent duality along $p_{13})$

Step 3) $\Psi_{univ}[g] \cong \mathcal{O}_\Delta \otimes \det \mathbb{E}$

- Supp$(\Psi_{univ}) = \Delta$
- By previous work of Arinkin, two sides are isom on $J_C \times_B J_C$
- Both sides are maximal CM sheaf on Δ

Apply extension property. $\boxtimes$

§8. What's next?

- Generalization to reducible plane curve

  C/k : reduced reducible planar curve

Let $\varepsilon_1, \varepsilon_2$: two non-degenerate st. conditions

$\mathcal{C} \to B$: miniversal family.

Thm [Melo-Rapagnetta-Viviani] $\exists !$ $\overline{P}$ maximal CM on $\overline{J}_{\mathcal{C}}^{\varepsilon_1} \times_B \overline{J}_{\mathcal{C}}^{\varepsilon_2}$

$$\mathcal{F}_i = - \otimes \overline{P} : D^b_{Coh}(\overline{J}_C^{\varepsilon_1}) \xrightarrow{\cong} D^b_{Coh}(\overline{J}_C^{\varepsilon_2}).$$

- Properties of $h(\overline{J}_{\mathcal{C}}/B)$.

Beauville-Deninger-Murre gives multiplicative splitting of $h(A/B)$

Thm [Maulik-Shen-Yin] There exists "multiplicative filtration" (orthogonal projectors) on $h(\overline{J}_{\mathcal{C}}/B)$ whose homological realization gives perverse filtration on $H^*(\overline{J}_{\mathcal{C}}, \mathbb{Q})$.

Rmk On the other hand $\nexists$ multiplicative splitting for general $\overline{J}_{\mathcal{C}}/B$. [B-Maulik-Shen-Yin].

$\underline{Q}$ What if $\overline{J}_{\mathcal{C}} \xrightarrow{\pi} B$ is Lagrangian fibration?

- Extending to other degenerate abelian schemes.

  $(M \to B,\ A \to B,\ \mu : A \times_B M \to M)$

  $\uparrow$ δ-regular.

[Arinkin-Fedorov]. Poincaré line bundle on $M \times_B \mathrm{Pic}^{\circ}_{M/B}$.

Q Can we extend P to a "CM object"?

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Reference

[Altman-Kleiman] Compactifying the Picard scheme, Adv. Math. (1980)

[Arinkin] Autoduality of compactified Jacobians with plane singularities, J.A.G. (2013).

[Arinkin-Fedorov] Partial Fourier-Mukai transform for integrable systems with applications to Hitchin fibration. Duke (2016).

[Deninger-Murre] Motivic decomposition of abelian schemes and the Fourier transform. J. Reine Angew. Math (1991)

[Diaz-Harris] Ideals associated to deformations of singular plane curves, Trans. AMS (1988).

[Maulik-Shen-Yin] Perverse filtrations and Fourier transforms, arXiv 2308.13160.