1. Complex tori

$X =$ Complex torus $\rightsquigarrow$ compact, connected complex Lie group

$$g = \dim_{\mathbb{C}}(X)$$

$$V \equiv T_0 X$$

X is commutative, so

$$\exp : \mathbb{C}^g \simeq T_0 X \longrightarrow X$$

gives an isomorphism $X \simeq \mathbb{C}^g/\Lambda \simeq V/\Lambda$ where $\Lambda \leq V$ is a lattice of rank $2g$.

Cohomology of X

Singular cohomology:

$$H^1(X, \mathbb{Z}) \simeq \operatorname{Hom}(\Lambda, \mathbb{Z})$$

$$H^n(X, \mathbb{Z}) \simeq \operatorname{Alt}^k(\Lambda, \mathbb{Z}) = \Lambda^k \operatorname{Hom}(\Lambda, \mathbb{Z})$$

Doubleaut cohomology

$$\Omega := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

$$\overline{\Omega} := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$$

↳ this means $f(z.v) = \overline{z} f(v)$

$$H^{p,q}(X) = H^q(X, \Omega^p_X) \simeq \Lambda^p \Omega \otimes \Lambda^q \overline{\Omega}$$

Line bundles on X

From the exponential sequence, we obtain

See the proof of Rieman's bilin rel's for the discussion around this

$$0 \to \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \xrightarrow{\phi} H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \cap H^{1,1}(X) \to 0$$

$$H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X) = \left\{ \begin{matrix} \text{line bundles} \\ \text{on } X \end{matrix} \right\}$$

$$H^2(X, \mathbb{Z}) \cap H^{1,1}(X) = \text{Néron-Severi group}$$

$$NS(X)$$

An element of $H^2(X,\mathbb{Z})$ is the same as

$$E : V \times V \longrightarrow \mathbb{R} \quad (E(v,v) = 0$$

such that $E(\Lambda,\Lambda) \subseteq \mathbb{Z}$. If

$$H(v,w) = E(iv,w) + i\,E(v,w)$$

Lemma $E \in H^{1,1}(X) \Longleftrightarrow H$ is Hermitian

$$\text{So } NS(X) = \left\{ \begin{array}{l} H : V \times V \longrightarrow \mathbb{C} \quad \text{Hermitian} \\ \text{such that } \operatorname{Im} H(\Lambda,\Lambda) \subseteq \mathbb{Z} \end{array} \right\}$$

The fibers of c_1

Let $H \in NS(X)$. A semicharacter for H is $\chi : \Lambda \longrightarrow S^1$ such that

$$\chi(\lambda+\mu) = \chi(\lambda)\,\chi(\mu)\exp(\pi i \operatorname{Im} H(\lambda,\mu)).$$

Define

$$a_{(H,\chi)} : \Lambda \times V \longrightarrow \mathbb{C}^*$$
$$(\lambda, v) \longmapsto \chi(\lambda)\, e^{\pi H(v,v) + \frac{\pi}{2} H(\lambda,\lambda)}$$

Then

$$a_{(H,x)}(\lambda+\mu, v) = a_{(H,x)}(\lambda, \mu+v)\, a_{(H,x)}(\mu, v)$$

so $a_{(H,x)}$ is a factor of automorphy, and defines a line bundle $L_{(H,x)}$:

$$\text{sections of } L_{(H,x)} = \left\{ \begin{array}{l} f : V \longrightarrow \mathbb{C} \text{ s.t.} \\ f(v+\lambda) = a_{(H,x)}(v,\lambda) f(v) \end{array} \right\}$$

We have said that $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$, but line bundles on V are trivial, so

$$H^1(X, \mathcal{O}_X^*) \simeq H^1(\pi_1(X), H^0(V, \mathcal{O}_X^*)) \simeq$$

$$\simeq \left\{ \begin{array}{l} a : \Lambda \times V \longrightarrow \mathbb{C}^* \text{ s.t.} \\ a(\lambda+\mu, v) = a(\lambda, \mu+v)\, a(\mu, v) \end{array} \right\}$$

Apell-Humbert theorem

$$\left\{ \begin{array}{l} (H, x) : H \in NS(X) \text{ and} \\ x \text{ is a semicharacter} \end{array} \right\} \longrightarrow \operatorname{Pic}(X)$$

is an isomorphism, and the induced map ϕ from the exponential sequence

$$\begin{tikzcd}
\phi : H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\wr"] & \ker(c_1) \arrow[d, phantom, "\wr"] \\
\overline{\Omega} \arrow[r, dashed] & \mathrm{Hom}(\Lambda, S^1)
\end{tikzcd}$$

satisfies $\phi(\xi) = e^{2\pi i \langle \xi, \cdot \rangle} \in \mathrm{Hom}(\Lambda, S^1)$

$$\hat{\overline{\Omega}} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$$

With kernel

$$\overline{\Lambda} := \{ \xi \in \overline{\Omega} . \mathrm{Im}(\xi(\Lambda)) \subseteq \mathbb{Z} \} \simeq \mathrm{Hom}(\Lambda, \mathbb{Z})$$

The dual complex torus is

$$\ker(c_1) = \mathrm{Pic}^0(X) \simeq \overline{\Omega} / \overline{\Lambda} .$$

2. Line bundles & cohomology

Let $L = L_{(H,\alpha)}$ be a line bundle on X.

Since H is Hermitian, it gives a map

$$V \xrightarrow{H(\cdot,-)} \overline{\Omega}$$

such that $H(\Lambda,-) \subseteq \overline{\Lambda}$ and so, an homomorphism

$$\varphi_L : X \longrightarrow X^\vee$$

One can check that $\varphi_L(x) = t_x^* L \otimes L^{-1}$.

Since H is Hermitian, it can be diagonalized with real eigenvalues.

(# positive eigenvalues, # negative eigenvalues) is the signature of H.

→ We say that L is non-degenerate if φ_L is surjective $\Longleftrightarrow$ H is non-degenerate $\Longleftrightarrow$ it is of type $(g-i, i)$

→ The number i is the index of L, $i(L)$. If $i(L)=0$, we say L is positive

→ $\mathrm{Im}(H)$ is a symplectic form, so it can be written as

$$\left(\begin{array}{c|c} 0 & D \\ \hline -D & 0 \end{array}\right)$$

over $\mathbb{Z}$, where $D = \mathrm{diag}(d_1, \dots, d_{r+s}, 0, \dots, 0)$ where (r,s) is the signature of H, $d_i \mid d_{i+1}$,

→ We define $\mathrm{Pfr}(L) = d_1 \cdot \ldots \cdot d_{r+s}$.

Theorem (Cohomology of l.b.)

If L is non-degenerate,

$$h^i(X, L) = \begin{cases} \mathrm{Pfr}(L) & \text{if } i = i(L) \\ 0 & \text{otherwise} \end{cases}$$

In general, if (r,s) is the signature,

$$h^i(X, L) = \begin{cases} \binom{g-r-s}{i-s} \cdot \mathrm{Pfr}(L) & \text{if } s \leq i \leq g-r \\ & \text{and } L|_{\ker(\varphi_L)_0} \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

→ Note that $L \in \mathrm{Pic}^0(X)$ iff $\ker(\varphi_L) = X$, so if $L \in \mathrm{Pic}^0(X) \setminus \{0\}$, $h^i(X, L) = 0$ $\forall i$.

→ If L is ample (in the sense that $L^{\otimes n}$ embeds X in $\mathbb{P}^N$ for some n) then L is positive. In fact, the converse is also true...

⇝ $$\chi(L) = \begin{cases} (-1)^{i(L)} \cdot \mathrm{Pf}(L) & \text{if } L \text{ is non-degenerate} \\ 0 & \text{otherwise} \end{cases}$$

And so, $\deg(\varphi_L) = \chi(L)^2$.

The Poincaré bundle

Since $X^\vee$ is a fine moduli space, there is the unique line bundle $P \to X \times X^\vee$ such that

$$P|_{0 \times X^\vee} \simeq \mathcal{O}_{X^\vee} \quad \& \quad P|_{X \times \{L\}} \simeq L$$

Given L, consider the Mumford bundle

$$\mu(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$$

on $X \times X$, where $m, p_1, p_2 : X \times X \longrightarrow X$ are the multiplication & projections.

Then,

- $\mu(L)|_{X \times \{x\}} = t_x^*L \otimes L^{-1}$
- $\mu(L)|_{\{0\} \times X} = \mathcal{O}_X$

so it gives rise to the map

$$\varphi_L : X \longrightarrow X^\vee$$

Now consider

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[rr, bend right, "id \times \varphi_L"'] & X \times X \arrow[r, "(id, \varphi_L)"] & X \times X^\vee
\end{tikzcd}$$

Then $(id \times \varphi_L)^* P \simeq \Delta^*(\mu(L)) \simeq [2]^*L \otimes L^{-2}$

where $[2] : X \longrightarrow X$
$x \longmapsto x + x$

We will see later that $[2]^*L \otimes L^{-2} \simeq$

$\simeq L \otimes [-1]^*L$

Lemma: P is a non-degenerate bundle of index g and type $(1, \dots, 1)$

Proof: Let

$$H : (V \times \bar{\Omega}) \times (V, \bar{\Omega}) \longrightarrow \mathbb{C}$$
$$(v_1, \xi_1), (v_2, \xi_2) \longmapsto \overline{\xi_2(v_1)} + \xi_1(v_2)$$

$$\chi : \Lambda \times \bar{\Lambda} \longrightarrow S^1$$
$$(v, \xi) \longmapsto e^{i\pi \operatorname{Im}(\xi(v))}$$

Then $L_{(H,\chi)}$ satisfies the condition of P. If $v_1, \dots, v_g$ is a basis of V and $\xi_1, \dots, \xi_g$ in a basis of Ω then $\{\bar{\xi}_1, \dots, \bar{\xi}_g\}$ is a basis of $\bar{\Omega}$, and in this basis,

$$H = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} = Id \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so $\operatorname{Spec}(H) = \{1, \overset{g}{\dots}, 1, -1, \overset{g}{\dots}, -1\}$. Moreover, $\operatorname{Im}(H)$ is unimodular by definition of $\bar{\Lambda}$, so it has type $(1, \dots, 1)$ $\square$

In particular,

$$h^i(X \times X^\vee, P) = \delta^i_g \quad , \quad \chi(P) = (-1)^g$$

3. Torsion elements

A general torus just has $\mathbb{Z}$ endomorphisms: the multiplication maps

$$[n] : X \longrightarrow X$$
$$a \longmapsto a + \dots + a$$

An element $a \in \ker([n])$ is a torsion element. $X[n] = \ker([n])$.

$$\longrightarrow X[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$$

Lemma $[n]^* L \simeq L^{\frac{n(n+1)}{2}} \otimes L^{\frac{n(n-1)}{2}}$

Proof: We have

$$[n]^* L_{(H,x)} \simeq L_{(n^2 H, nx)}$$

$$L^{k}_{(H,x)} \simeq L_{(kH, kx)}$$

$$L_{(H_1, x_1)} \otimes L_{(H_2, x_2)} \simeq L_{(H_1 + H_2, x_1 x_2)}$$

and $\begin{cases} a+b = n^2 \\ a-b = n \end{cases} \Longrightarrow \begin{cases} a = n(n+1)/2 \\ b = n(n-1)/2 \end{cases}$ $\square$

In particular,

$$(id \times \varphi_L)^* P \simeq [2]^* L \otimes L^{-2} \simeq L \otimes [-1]^* L .$$

4. Abelian varieties

We say that X is abelian if it has a positive line bundle.

If it does, $L^{\otimes 3}$ embeds X in projective space, so X is algebraic.

An homomorphism $\varphi : X \longrightarrow X^\vee$ of the form φ_L for L ample is a polarization,

Theorem (Riemann bilinear relations) If $X = \mathbb{C}^g/\Lambda$, and $v_1, \dots, v_{2g}$ are a basis of Λ,

$$\Pi = \left(v_1 \middle| \dots \middle| v_{2g} \right) \in M_{g\times 2g}(\mathbb{C})$$

is a period matrix of X. X is abelian iff $\exists A \in Sp(2g, \mathbb{Z})$:

- $\Pi A^{-1} \Pi^t = 0$
- $i \Pi A^{-1} \overline{\Pi}^t > 0$

Proof. Let $E : V \times V \longrightarrow \mathbb{Z}$ have matrix A. Then

- $E(iu, iv) = E(u,v) \iff \Pi A^{-1} \Pi^t = 0$

let $H = E(i\cdot, \cdot) + i E(\cdot, \cdot)$. Then

- $H(u,v) = 2i \; u^t (\Pi A^{-1} \Pi^t)^{-1} \bar{v}$, so

H is positive iff $i \Pi A^{-1} \Pi > 0$.

Finally, any Hermitian form has a semicharacter. Let $z_1, \dots, z_{2g} \in S^1$, and define $\chi(v_i) = z_i$ and extend by linearity:

$$\chi\left(\sum a_i v_i\right) = \left(\prod z_i^{a_i}\right) \cdot e^{\pi i \left(\sum_{i,j} a_i a_j \operatorname{Im} H(z_i, z_j)\right)} \qquad \square$$

A principal polarization is $\varphi : X \longrightarrow X^\vee$ that is an isomorphism

5. Analytic moduli

Let $X = V/\Lambda$ and $H \in NS(X)$ a polarization of type $D = (d_1, \dots, d_g)$
Let $\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g$ be a symplectic basis for Λ w.r.t. $\operatorname{Im} H$.

On the basis $\frac{\mu_1}{d_1}, \dots, \frac{\mu_g}{d_g}$ of V, the period matrix of X is

$$\Pi = (Z \mid D)$$

- $Z = Z^{-1}$, $\operatorname{Im}(Z) > 0$ and $\operatorname{Im}(Z)^{-1}$ is the matrix of H on the basis of V.

Proof: Use Riemann bilinear relations.

Let

$$\mathcal{H}_g = \{ \tau \in M_g(\mathbb{C}) : \tau = \tau^{-1}, \operatorname{Im}(\tau) > 0 \}$$

then we have a map

$$\begin{array}{ccc} \mathcal{H}_g & \xrightarrow{\quad \pi \quad} & \left\{ \begin{array}{l} \text{polarized abelian} \\ \text{varieties of type } D \end{array} \right\} \\ \tau & \longmapsto & \left(\mathbb{C}^g \big/ (\quad) , \operatorname{Im}(\tau)^{-1} \right) \end{array}$$

Let

$$G_D = \left\{ M \in Sp_{2g}(\mathbb{Q}) \,\middle|\, M^t \begin{pmatrix} \mathbb{1} & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^g \subseteq \begin{pmatrix} \mathbb{1} & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^g \right\}$$

act on $\mathcal{H}_g$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tau = (\alpha\tau + \beta)(\delta\tau + \gamma)^{-1}$$

Lemma An isomorphism $\pi(\tau_1) \simeq \pi(\tau_2)$ is the same as a matrix $M \in G_D$ such that $M \circ \tau_1 = \tau_2$.

"Proof" Suppose that

$$F : \frac{\mathbb{C}^g}{(D \mid \tau_1)} \longrightarrow \mathbb{C}^g \big/ (D \mid \tau_2)$$

is the isomorphism, encoded by matrices $A \in M_g(\mathbb{C})$, $R \in M_{2g}(\mathbb{Z})$ s.t.

(1) $A\,(D \mid \tau_1) = (D \mid \tau_2)\,R$

(2) $R^t \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$

If $N = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & D \end{pmatrix}^{-1} R \begin{pmatrix} \mathbb{1} & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t$

then A can be recovered form N:

$$A = \tau_1 \gamma^t + \delta^t$$

and $\tau_2 = \tau_2^t = (\alpha\tau_1 + \beta)(\gamma\tau_2 + \delta)^{-1}$.

Moreover,

$$\begin{gathered} R \text{ integral } \& \quad R^t \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} R = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \\ \Updownarrow \\ N \in G_D \end{gathered}$$

D

$$\text{Therefore, } \mathcal{A}_{g,\delta} \simeq \underset{\substack{\uparrow \\ \text{stack quotient.}}}{[}\,\mathcal{H}_g / G_D\,]$$

6. Glimpse of algebraic moduli

An abelian scheme is $X \longrightarrow S$ proper group scheme, flat & with geometrically integral fibers.

A polarization is

$$\varphi : X \longrightarrow X^\vee = \mathrm{Pic}^0_{X/S}$$

s.t. over any $s \in S$, φ is a polarization of abelian varieties. φ is a finite map

If d is invertible in k, define

$$\mathcal{A}_{g,d}/_{Sch/\mathbb{Z}[\frac{1}{d}]} = \left\{ \begin{array}{l} (X \to S, \varphi) \cdot \varphi \text{ is a polari-} \\ \text{zation of constant degree } d \end{array} \right\}$$

Theorem $\mathcal{A}_{g,d}$ is a smooth DM-stack over $\mathrm{Spec}\, \mathbb{Z}[1/d]$ of dimension $\frac{g(g+1)}{2}$. It is irreducible.

"Proof": Algebricity ⇝ use Hilbert schemes

$$\left((id \times \varphi)^* P\right)^{\otimes n}$$

embeds $X \to S$ into $\mathbb{P}^N$

Finite stabilizers: Serre's lemma: if X is a polarized abelian variety, and $\phi : X \to X$ is such that $\phi|_{X[3]} = id$ then $\phi = id$.

Smoothness:

a) Deforming X as an abelian variety is the same as deforming it as an abstract variety (Grothendieck)

b) Deformations of X are unobstructed

c) Deformations of (X, L) have dimension $\frac{g(g+1)}{2}$.

Idea of b): $R' \twoheadrightarrow R$ a thickening given by $I \subseteq R'$, $X' \longrightarrow \mathrm{Spec}(R')$ abelian scheme extending $X \longrightarrow \mathrm{Spec}(R)$. The obstruction class

$$\varpi \in H^2(X, T_X \otimes I) \simeq (\bar{\Omega} \wedge \bar{\Omega}) \otimes \Omega \otimes I$$

satisfies $[-1]^* \varpi = \varpi$, but $[-1]^*$ acts as -1 on $(\bar{\Omega} \wedge \bar{\Omega}) \otimes \Omega$ (this works over characteristic $\neq 2$).

Idea of c): If L is a line bundle on X,

$$\mathrm{Def}(X, L) = \ker\left(c_1(L) : \overset{H^1(T_X)}{\overset{\wr}{\mathrm{Def}(X)}} \longrightarrow H^2(\mathcal{O}_X) \right)$$

But $c_1(L) : \Omega \otimes \bar{\Omega} \longrightarrow \bar{\Omega} \wedge \bar{\Omega}$ is

$$v \otimes \xi \longmapsto (\varphi_L)_0 (v) \wedge \xi$$

Since d is invertible, $(\varphi_L)_0$ is an isomorphism on tangent spaces, So

$$\mathrm{Def}(X,L) \simeq \ker(\overline{\Omega}\otimes\overline{\Omega} \longrightarrow \overline{\Omega}\wedge\overline{\Omega})$$
$$\simeq \mathrm{Sym}^2(\overline{\Omega}) \qquad \square$$

Note that the vector bundle

$$\begin{tikzcd} \mathbb{E}_g \arrow[d] & \Omega \arrow[d] \\ \mathcal{A}_{g,d} & \ni [X] \end{tikzcd}$$

is the Hodge bundel. In particular,

$$T_{\mathcal{A}_{g,d}} \simeq \mathrm{Sym}^2 \mathbb{E}_g^{\vee}$$