1. Complex tori

X = Complex torus ~ compact, connected complex Lie group

 $g = dim_{C}(X)$ V≡T₀≺ X is commutative, so  $exp: \mathbb{C}^{3} \simeq \mathbb{T}_{0} \times \longrightarrow \times$ gives an isomorphism  $X \simeq C^{9}/\Lambda \simeq V/\Lambda$ where  $\Lambda \subseteq V$  is a lattice of nank Zg.

Cohomology of X

Singular cohomology:

 $H^1(X, \mathbb{Z}) \simeq Hom(\Lambda, \mathbb{Z})$ 

 $H^{(X,Z)} \simeq Alt^{k}(\Lambda,Z) = \Lambda^{k}Hom(\Lambda,Z)$ 

Poubleaut cohomology  $\Omega := Hom_{\mathbb{C}}(V,\mathbb{C})$  $\overline{\Omega} := Hom_{\overline{E}}(V, \mathbb{C})$  $\square$ , this means  $f(z,v) = \overline{z} f(v)$  $H^{P,q}(X) = H^{q}(X, \Omega^{P}) \simeq \Lambda^{P} \Omega \otimes \Lambda^{q} \overline{\Omega}$ Line bundles ou X From the exponential sequence, we obtain  $\begin{array}{c}
 & \int H^{*}(X, \partial_{X}) \\
 & H^{*}(X, ZL)
\end{array} \xrightarrow{\varphi} H'(X, \partial_{X}^{*}) \xrightarrow{\zeta_{1}} H^{2}(X, ZL) \cap H^{1/}(X) \\
 & \downarrow
\end{array}$  $H^{1}(X, O_{X}^{*}) = P_{ic}(X) = \begin{cases} lim bundler \\ ou X \end{cases}$ H<sup>2</sup>(X, Z) ∩ H'''(X) = Néron-Severi group NS(X)

An element of H<sup>2</sup>(X, Z) is the same  $E: \forall \times \forall \longrightarrow \mathbb{R}$ ( E(v,u)=0 such that  $E(\Lambda, \Lambda) \in \mathbb{Z}$ . If H(v,w) = E(iv,w) + i E(v,w)Lemma  $E \in H^{1,1}(X) \iff H$  is Hermitian So  $NS(X) = \begin{cases} H: V \times V \longrightarrow C & Hermitian \\ uch that In <math>H(\Lambda, \Lambda) \in \mathbb{Z} \end{cases}$ The fibers of c1

Let  $H \in NS(X)$ . A semichanacter for His  $\chi : \Lambda \longrightarrow S^1$  such that

 $\chi(\lambda + \mu) = \chi(\lambda) \times (\mu) \exp(\pi i \operatorname{Im} H(\lambda, \mu)).$ 

Define

 $: \triangle \times \lor \longrightarrow \mathbb{C}^*$ a(H,x)  $(\lambda, v) \longmapsto \chi(\lambda) e^{\pi H(v, v) + \frac{\pi}{2} H(\lambda, \lambda)}$ 

$$a_{(H,x)}(\lambda+\mu,v) = a_{(H,x)}(\lambda,\mu+v) a_{(H,x)}(\mu,v)$$

So 
$$a_{(H,\chi)}$$
 is a factor of automorphy, and  
definer a line bundle  $L_{(H,\chi)}$ :  
sections of  $L_{(H,\chi)} = \begin{cases} f: V \longrightarrow C \quad s.t. \\ f(v+\chi) = a_{(H,\chi)}(v,\chi) f(v) \end{cases}$ 

We have said that 
$$Pic(X) = H'(X, O_X^*)$$
, but  
line bundles on V are trivial, so  
 $H'(X, O_X^*) = H'(\pi_1(X), H^o(V, O_X^*)) =$   
 $= \begin{cases} a: i > V \longrightarrow C^* s.t. \\ a(x,y_1,v) = a(x,y_1+v)a(y_1,v) \end{cases}$ 

map & from the exponential sequence

 $\phi: H'(X, \Theta_X) \xrightarrow{} ker(c_1)$   $SI \qquad SI$   $\overline{\Omega} \xrightarrow{} \cdots \xrightarrow{} Hom(\Lambda, S^4)$ satisfies  $\phi(\xi) = e^{2\pi i \langle \xi, \cdot \rangle} e Hom(\Lambda, \xi')$   $\overline{\Omega} = Hom_{\overline{e}}(V, C)$   $Wall \qquad I$ 

With kernel

 $\overline{\Lambda} := \{ \xi \in \overline{\Omega} : \operatorname{Im}(\xi(\Lambda)) \subseteq \mathbb{Z} \}^{2} + \operatorname{Hom}(\Lambda, \mathbb{Z})$ The dual complex form is  $\ker(e_{4}) = \operatorname{Pric}^{0}(\times) \simeq \overline{\Gamma}_{\Lambda}^{2}.$ 

2. Line bundes & cohomology

Let  $L = L_{(H, x)}$  be a line bundle on X. Since H is Hermitian, it gives a map  $\sqrt{\frac{H(\cdot,-)}{52}}, \overline{52}$ such that  $H(\Delta, -) \subseteq \overline{\Delta}$  and so, an homomorphism  $\mathfrak{P}_{\underline{}} \colon X \longrightarrow X^{\vee}$ One can check that  $\mathcal{P}_{L}(x) = t_{x}^{*} \bot \otimes \bot^{-'}$ . Since H is Hermitian, it can be diagonalized with real eigenvalues. (# positive eigenvalues, # negative eigenvalues) is the signature of H. - We say that L is non-degenerate it PL is sujective (=) H is non-degenerate ⇐) it is of type (g-i, i)

- The number i is the index of L, i(L). If i(L) = 0, we say L is ponitive - Im (H) is a symplectic form, so it can be written as

$$\left(\begin{array}{c|c} 0 & p \\ \hline -p & 0 \end{array}\right)$$

over  $\mathbb{Z}$ , where  $D = diag(d_1, \dots, d_{r+s}, 0, \dots, v)$ where (r, s) is the signature of H,  $di | d_{i+1}$ ,

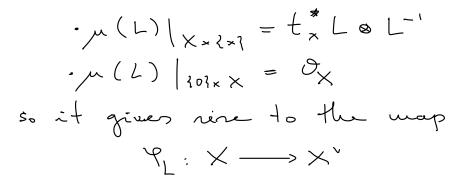
Theorem (Cohomology of l.b.) If L is non-degenerate,  $h^{i}(X, L) = \begin{cases} Pf_{2}(L) & \text{if } i = i(L) \\ 0 & \text{otherwise} \end{cases}$ 

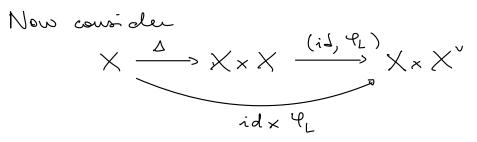
In general, if (r,s) is the signature,  $h^{i}(X, L) = \begin{cases} \begin{pmatrix} g-r-s \\ i-s \end{pmatrix} & Pfr(L) & if s \leq i \leq g-r \\ and & L \mid ker(q_{L})_{o} & is trivial \\ O & otherwise \end{cases}$ 

 $\longrightarrow$  Note that LE Pic<sup>o</sup>(X) iff ker ( $P_{L}$ ) = X, so if LEPic°(X), 203, h'(X,L)=0 Vi. -> If L is ample (in the sense that L<sup>en</sup> embeds X in IP<sup>N</sup> for some n) then L is positive. In fact, the converse is also true...  $\longrightarrow \chi(L) = \begin{cases} (-1)^{i(L)}, Pf(L) & \text{if } L \text{ is non-} \\ degenerate \\ 0 & \text{othewise} \end{cases}$ And so, deg  $(\Psi_L) = \mathcal{X}(L)^{L}$ . The Poincaré bundle Since X' is a fine moduli space, there is line bundle P- X × X such that the unique  $P|_{0\times X^{\vee}} \simeq O_{X^{\vee}} \& P|_{X \times \{L\}} \simeq L$ Given L, consider the Muntord bundle

$$\mu(L) = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

on XXX, where m, ps, p2: XXX -> X are the multiplication & projections. Then,

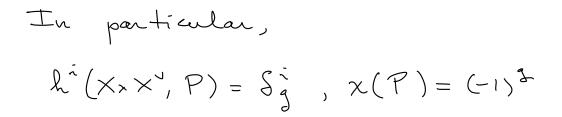




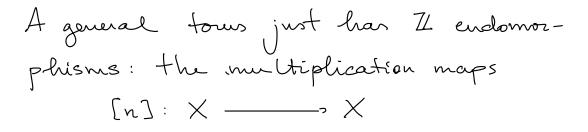
Then  $(id \times \Psi_{L})^{*} P \simeq \Delta^{*}(\mu(L)) \simeq [2]^{*} L \otimes L^{2}$ where  $[2]: X \longrightarrow X$  $\propto \longmapsto x + \infty$ 

We will see later that [2]\*L & L<sup>-2</sup> = = L & [-1]\*L

Lemma: Pis a non-degenerate brendle of index g and type (1,..., 1) Proof: Let  $(\forall \times \overline{\Omega}) \times (\forall, \overline{\Omega}) \longrightarrow \tilde{C}$ H :  $(v_1, \xi_1), (v_2, \xi_2) \longrightarrow \overline{\xi_2(v_1)} + \xi_1(v_2)$  $\chi : \land \times \overline{\land} \longrightarrow \mathfrak{S}'$ (v, 5) - ein Im (3(v)) Then L(H, x) satisfies the conditions of P. If V1, ..., Vg is a baris of V and is a basis of 52, and in this basis,  $H = \begin{pmatrix} 0 & \pm 2 \\ \pm 4 & 0 \end{pmatrix} = \pm 4 \otimes \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$ so Spec (H) = 21, 3, 1, -1, 3, -1). Moreover, In (H) is mimodular by definition of A, so it has type (1, ---, 1) D



3. Torsion elements



a 1 + . . . + a

An element a e ker (En]) is a torsion element. X [n] = ker ([n]),

$$\longrightarrow$$
 X[n]  $\simeq (Z/nZ)^{2g}$ 



Proof: We have

 $[n]^{*} L(H, \chi) \simeq L(n^{2}H, n\chi)$  k  $L(H, \chi) \simeq L(kH, k\chi)$   $L(H_{1}, \chi_{1}) \otimes L(H_{2}, \chi_{2}) \simeq L(H_{1} + H_{2}, \chi_{1}\chi_{2})$ 

In particular,

## $(id \times \Psi_L)^* P \simeq [z]^* L \otimes L^2 \simeq L \otimes [I]^* L.$

4. Abelian varieties

We say that X is abelian if it has a positive line bundle.

If it does,  $L^{\circ 3}$  embeds X in projective space, so X is algebraic. An homomorphism  $\Psi: X \longrightarrow X^{\circ}$  of the form  $\Psi_{L}$  for L ample is a polarization,

Theorem (Riemann bilinear relations) If  $X = C^3/\Lambda$ , and  $v_{1,...,v_{2g}}$ are a baris of  $\Lambda$ ,  $\Pi = (v_1) \dots |v_{2g}\rangle \in M_{g \times 2g}(C)$ is a period matrix of  $X \dots X$  is abelian iff  $\exists A \in Sp(2g, Z)$ :

$$\cdot \Pi A^{-1} \Pi^{t} = 0$$

$$\cdot \iota \Pi A^{-1} \Pi^{t} > 0$$

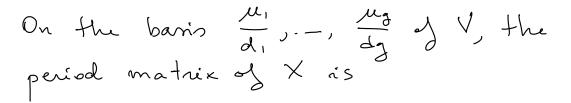
Proof. Let E: V × V -> Z have matrix A. Then

• 
$$E(iu, iv) = E(u, v) \iff \Pi A^{-1} \Pi^{+} = D$$
  
let  $H = E(i \cdot, \cdot) + i E(\cdot, \cdot)$ . Then  
•  $H(u, v) = 2i u^{+} (\Pi A^{-1} \Pi^{+})^{-1} \overline{J}, \infty$   
H is positive iff  $i \Pi A^{-1} \Pi > 0$ .  
Finally, any Hermitian form has a  
semicharacter. Let  $z_{1}, \dots, z_{2g} \in S'$ , and  
define  $\chi(v_{i}) = z_{i}$  and extend by linearity:  
 $\chi(\Sigma a_{i} \overline{J}_{i}) = (\Pi z_{i}^{a_{i}}) \cdot e^{\pi i} (\sum_{i=1}^{n} a_{i} i \prod_{j=1}^{n} \Pi A^{-1} (z_{i}, z_{j}))$ 

A principal polarization is  $4: X \longrightarrow X^{\vee}$ that is an isomorphism

5. Analytic moduli

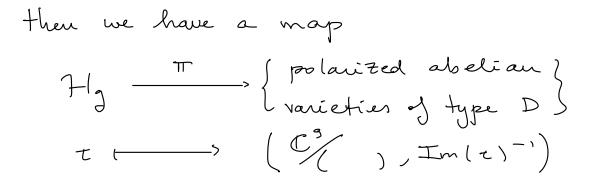
Let  $X = \frac{1}{2}$  and  $H \in NS(X)$  a polanization of type  $D = (d_1, ..., dg)$ Let  $\lambda_1, ..., \lambda_g, \mu_1, ..., \mu_g$  be a symplectic basis for  $\Lambda$  wr.t. Im H.



$$\Pi = (Z | D)$$

$$Z = Z^{-1}, Im(Z) > 0 \text{ and } Im(Z)^{-1}$$
is the matrix of H on the basis  
of V.  
Proof: Use Riemann bilinear relations.

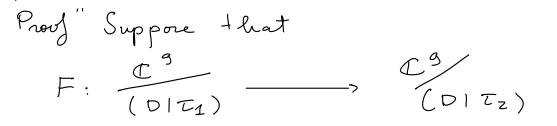
 $\mathcal{H}_{g} = \{ \tau \in M_{g}(\mathbb{C}) : \tau = \tau^{-1}, \operatorname{Im}(\tau) > o \}$ 



Let

 $G_{D} = \left\{ M \in S_{p_{2g}}(\mathbb{R}) \middle| M^{t} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^{g} \leq \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \mathbb{Z}^{g} \right\}$ act on Flg by  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} T = (\alpha T + \beta) (\delta T + \gamma)^{-1}$ 

Lemma An isomorphism n(I2) ~ n(I2) is the same as a matrix MEGD such that  $M_{\circ}T_{1} = T_{2}$ .

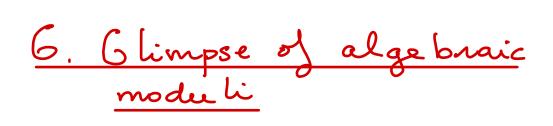


is the isomorphism, encoded by  
matrices AE Mg(C), RE M2g(Z) s.t  
(1) A (DIT<sub>1</sub>) = (DIT<sub>2</sub>) R  
(2) R<sup>t</sup> (
$$_{-DO}^{OD}$$
) R = ( $_{-DO}^{OD}$ )

.

If 
$$N = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{-1} R \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{+}$$
  
Hen A can be recovered form N:  
 $A = T_{1} \gamma^{+} + \delta^{+}$   
and  $T_{2} = T_{2}^{+} = (\alpha T_{1} + \beta)(\gamma T_{2} + \delta)^{-1}$ .  
Moreover,

Rintegral 
$$\mathcal{E}$$
  $R^{+} \begin{pmatrix} 0 \\ -0 \\ 0 \end{pmatrix} R = \begin{pmatrix} 0 \\ -0 \\ 0 \end{pmatrix}$   
N  $\mathcal{E}$   $\mathcal{G}_{D}$   $\mathcal{D}$ 



An abelian scheme is X -> S proper group scheme, flat & with geometrically integral fibers. A polarization is  $\Psi: \times \longrightarrow \times = \operatorname{Pic}_{\times 15}^{\circ}$ s.t. over any seS, P is a polarization of abelian varieties. I is a finite map If d is invertible in k, define Ag,d/ = {(X - S, Q) · Q is a polai-? zation of constant degree d

Theorem Ag, d is a smooth DMstack over Spec Z[4] of dimen-

Sion 
$$\frac{g(g+n)}{2}$$
. It is ineducible.  
(Proof ": Algebricity and use Hilbert schemen  
 $\left( (id \times \Psi)^* P \right)^{\otimes n}$   
embeds  $X \longrightarrow S$  into  $P^N$ 

Finite stabilizers: Serve's lemma: if  
X is a polarized abelian variety, and  
$$\phi: X \longrightarrow X$$
 is such that  $\phi|_{X[3]} = id$   
then  $\phi = id$ .

2) Deforming X as an abelian variety is the same as deforming it as an abstract variety (Grothendiech) b) Deformations of X are unobstructed c) Deformations of (X, L) have dimension g(g+i). Idea of b):  $R' \rightarrow R$  a thickening given by  $I \subseteq R'$ ,  $X' \rightarrow Spec(R')$ abelian scheme extending  $X \rightarrow Spec(R)$ . The obstruction class

 $\Pi \in H^{2}(X, T_{X} \otimes I) \cong (\Pi \wedge \widehat{\Pi}) \otimes \Omega \otimes I$ satisfier  $[-1]^{*} D = D$ , but  $E_{1}I^{*}acds$ as -1 on  $(\Pi \wedge \widehat{\Omega}) \otimes \Omega$ (this works over characteristic  $\pm 2$ ). Idea of c): If L is a line bundle on X,  $H^{'}(T_{X})$   $Def(X, L) = ku \left(c_{1}(L): Def(X) \longrightarrow H^{2}(\Omega_{1})\right)$ 

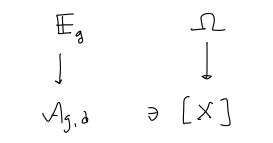
But  $c_1(L)$ :  $\Pi \otimes \overline{\Pi} \longrightarrow \overline{\Pi} \wedge \overline{\Pi}$  is

 $v \otimes \mathfrak{Z} \longrightarrow (\mathcal{Y}_{L})_{o} (v) \wedge \mathfrak{Z}$ Since d is invertible,  $(\mathcal{Y}_{L})_{o}$  is an isomorphism on tangent space,

So

Def (X,L) a ker (I&I)  $\simeq Sym^2(\overline{\Omega})$ Ο

Note that the vector bundle



is the Hodge bundel. In particular,  $T_{A_{5,d}} \simeq Sym^2 E_g^{\vee}$