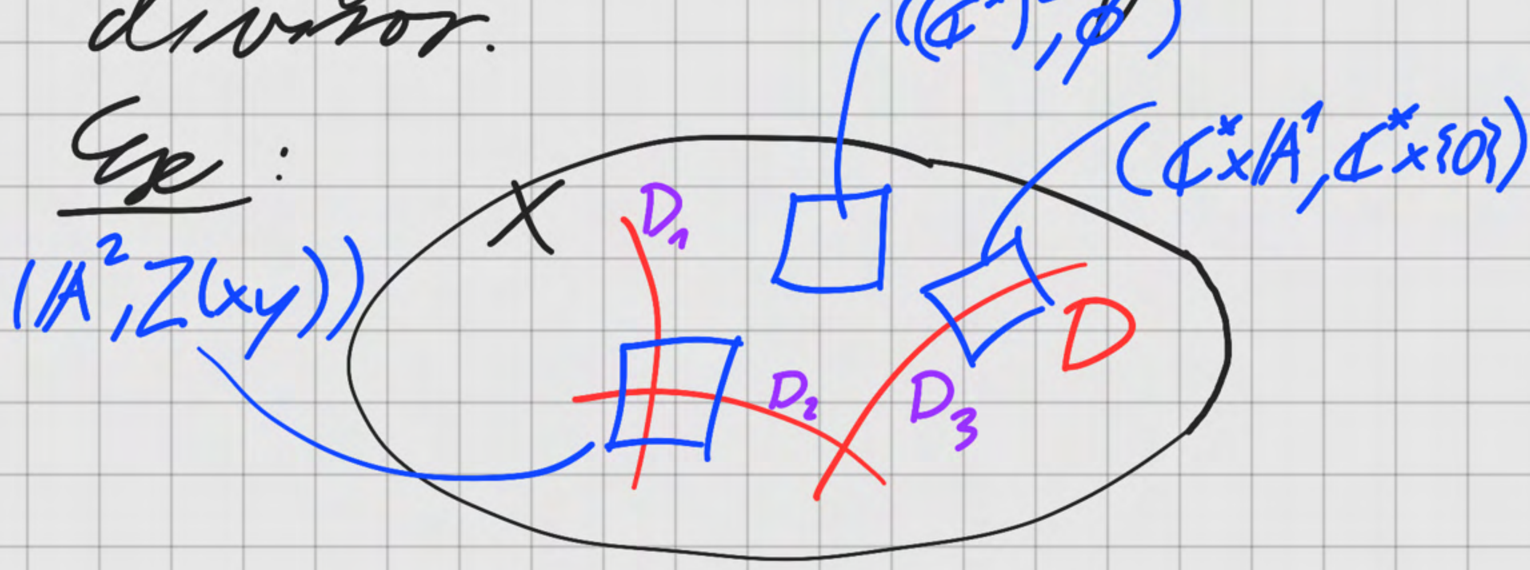
Toroidal compactifications and their cone stacks

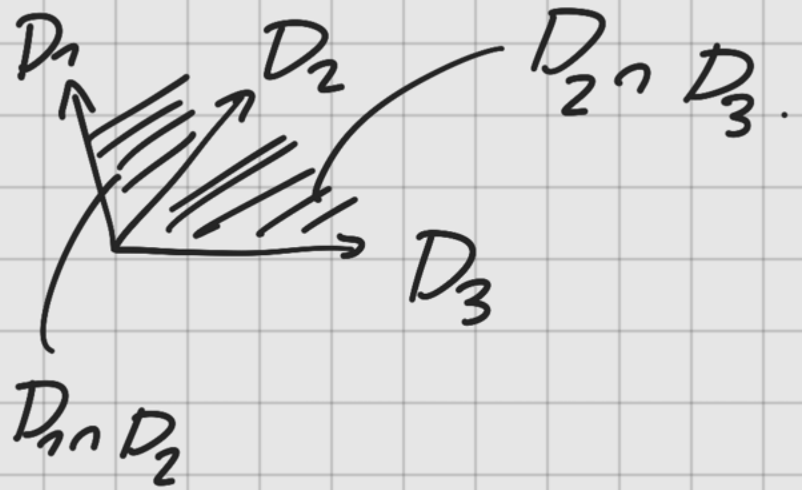
1 Toroidal embeddings / cone stacks

Def: A toroidal embedding is a pair (X, D) of a normal variety X and an effective divisor D which étale locally looks like a toric variety with its distinguished divisor.

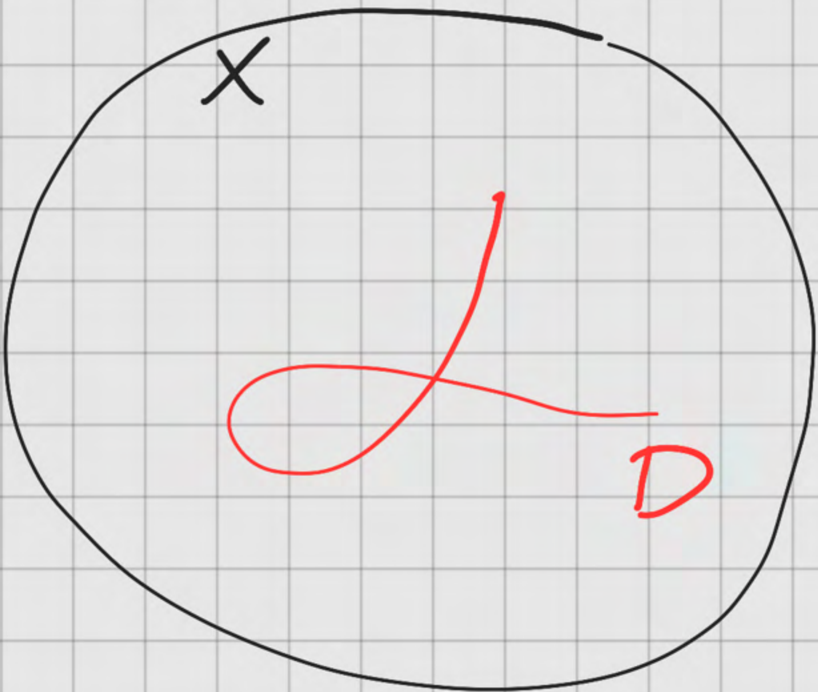
Ex:

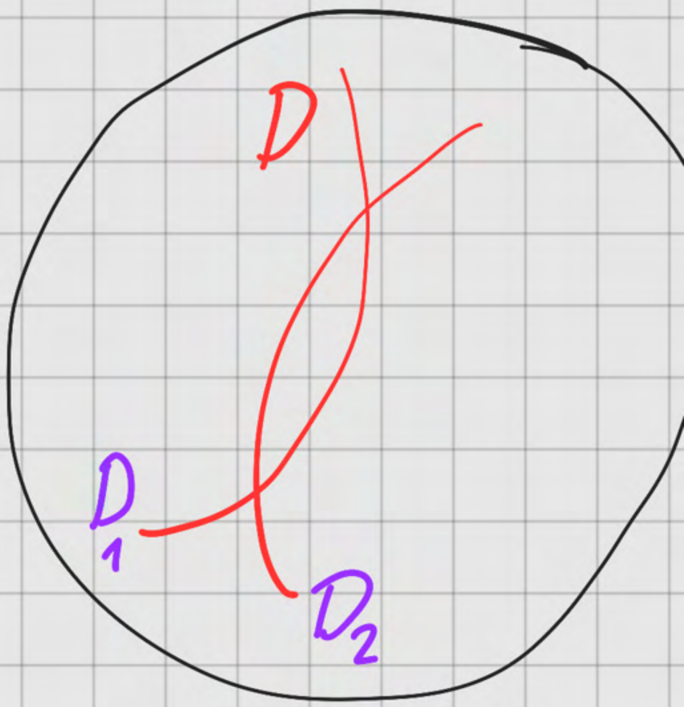


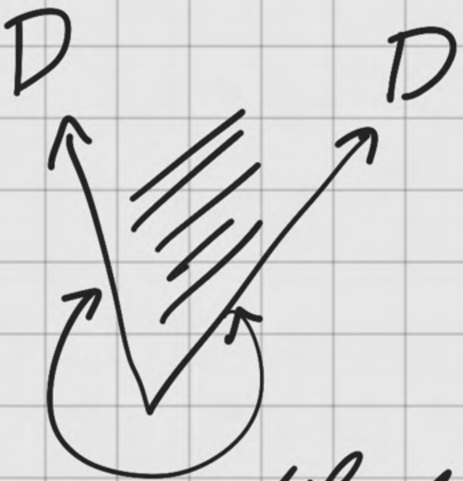
We can glue the cones of the charts:



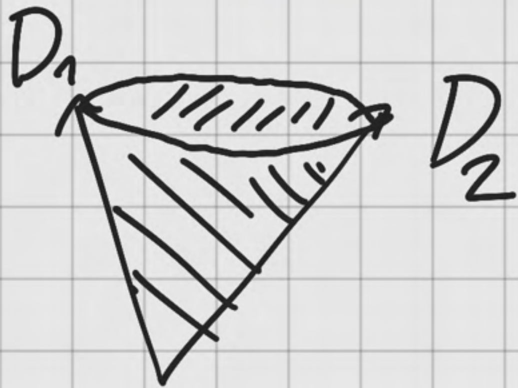
What kind of an object should this be? Can it always be realized as a fan in a vector space?
No! There are 2 problems:







identify these



$\rightsquigarrow$ We get 2 types of waffle cones, that cannot be embedded as fans in any vector space.

Def: An RPC is a pair (σ, N) where N is a finitely generated free group and $\sigma \subseteq \mathbb{R} \otimes N \cong \mathbb{R}^{\mathrm{rk} N}$ is an RPC in $\mathbb{R}^{\mathrm{rk} N}$ with $\mathrm{span}(\sigma) = \mathbb{R} \otimes N$.

Ex: If σ is an RPC in $\mathbb{R}^n$ with $\mathrm{span}\, \sigma = \mathbb{R}^n$, then $(\sigma, \mathbb{N}^n)$ is an RPC.

We can upgrade RPC to a category, by letting the homs from (σ, N_σ) to (τ, N_τ) be the set of group morphisms $N_\sigma \to N_\tau$ s.t. the image of σ under $N_\sigma \otimes \mathbb{R} \xrightarrow{\varphi} N_\tau \otimes \mathbb{R}$ lies in τ. We say it is a face morphism, if φ is an iso. of σ onto a face of τ. Clearly, the composition of face morphisms is a face morphism, so we get a subcategory RPC^f s.t. $Ob(RPC^f) = Ob(RPC)$ and the morphisms are face morphisms.

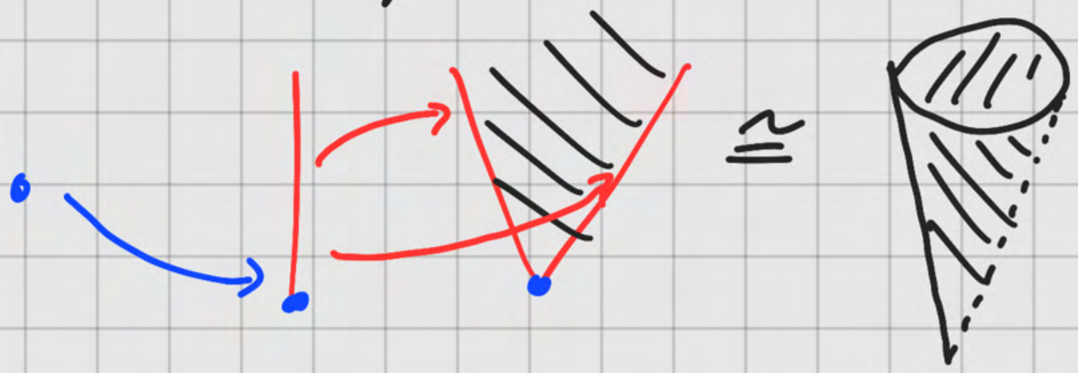
Def: A rational polyhedral cone ~~complex~~ space Σ is a subcategory of RPC^f satisfying:

~~1)~~ Every face of a cone is the image of exactly one morphism in Σ.

~~2)~~ There is at most one morphism between any 2 cones in Σ.

Expl: (cone complex)

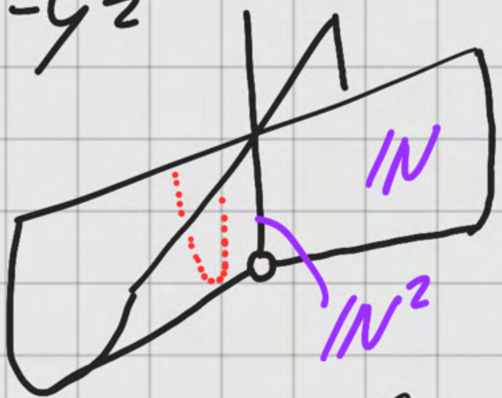
Expl: (cone space)



Hence cone spaces solve the 2 problems from before. The problem is that I lied. There are 3 problems (the third isn't visible in the 2d case).

Punctured Whitney umbrella

$x^2 - y^2 z$



$=: D \subseteq X := \mathbb{A}^3 \setminus \{0\}$

[Pandharipande-Ranganathan-Schmitt-Spelier]

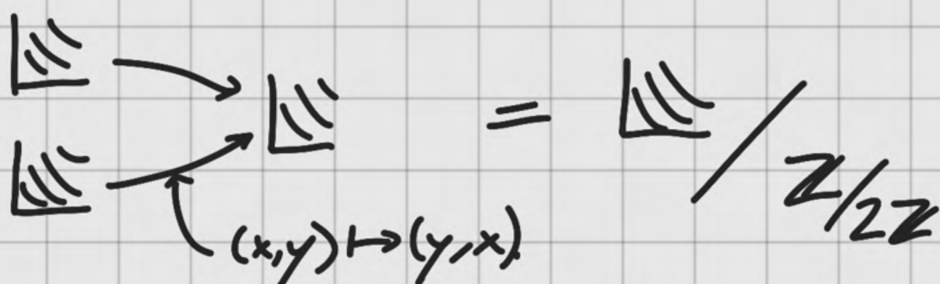
If I walk up and down the central ray pointing along one of the branches (red dots), I end up in the opposite branch.

What does this mean for the cone space?

Locally at $(0,0,z)$ for $z \neq 0$, we have:

$$(X, D) \cong (\mathbb{A}^3, Z(xy)).$$

We can cover $0 \times 0 \times \mathbb{C}^*$ by 2 charts s.t. on each chart a choice of branch is well-defined. We then get a presentation:



Hence we are naturally forced to consider <u>cone stacks</u>:

<u>Def</u>: [Cavalieri–Chan–Ulirsch–Wise]
A (combinatorial) cone stack is a category Σ fibered in groupoids over RPC^f: $\sigma: \Sigma \to RPC^f$. Fibered in groupoids means:

(i): for each $\alpha \in \Sigma$ and face inclusion $\tau \to \sigma_\alpha$ there is a morphism $\beta \to \alpha$ over $\tau \to \sigma_\alpha$.
(pullbacks exist)

(ii): For any diagram:

$$\begin{tikzcd} \beta \arrow[dr] & \\ & \alpha \\ \gamma \arrow[ur] & \end{tikzcd} \quad \text{in } \Sigma$$

such that $\sigma_\gamma \to \sigma_\alpha$ factors through $\sigma_\beta \to \sigma_\alpha$, there exists a unique morphism $\gamma \to \beta$ lying over the factorization $\sigma_\gamma \to \sigma_\beta$ and making

$$\begin{tikzcd} \beta \arrow[dr] & \\ & \alpha \\ \gamma \arrow[uu] \arrow[ur] & \end{tikzcd} \quad \text{commute.}$$

<u>Rem</u>: It would be natural to add the following two axioms to the definition:

1) The diagonal of Σ is representable by cone spaces

2) Σ has a cover by cones.

However, for cone stacks, this is actually automatic. For example, for (2), We get a cover:

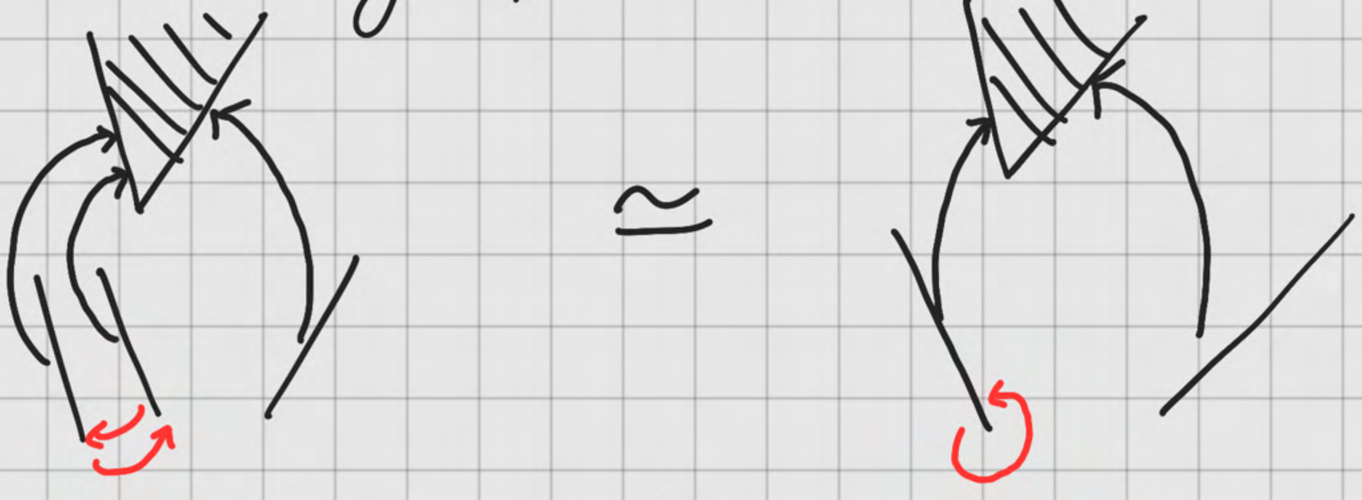
$$\coprod_{\alpha \in \Sigma} \sigma_\alpha \to \Sigma.$$ (this is strict, representable by cone complexes, ...)

<u>Claim</u>: $\Sigma/_\alpha \cong \sigma_\alpha$.

<u>Proof</u>: Every $\beta \to \alpha$ lies over $\tau \hookrightarrow \sigma_\alpha$ (and every $\tau \to \sigma_\alpha$ has a preimage). If $\beta \to \alpha$, $\gamma \to \alpha$ lie over the same $\tau \to \sigma_\alpha$, they are isomorphic via a unique iso. by (ii).

<u>Expl</u>: A cone stack is a cone complex iff its category is equivalent to a partially ordered set.

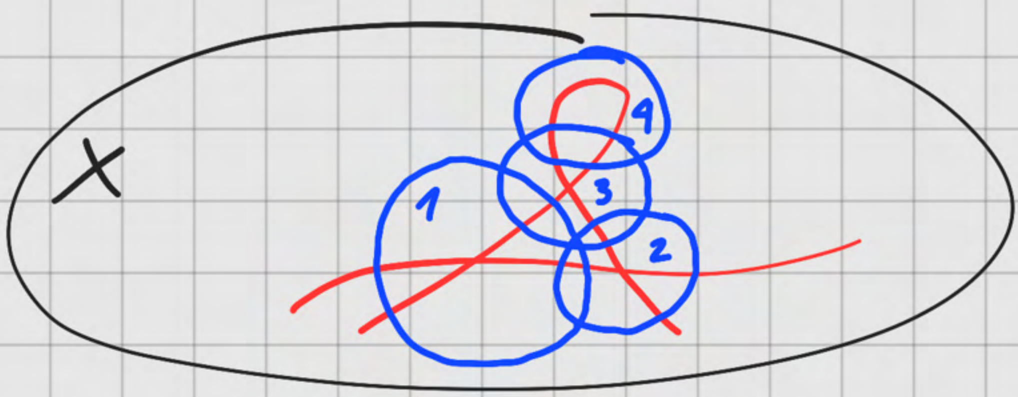
<u>Expl</u>: Now, a face can be the image of multiple cones:

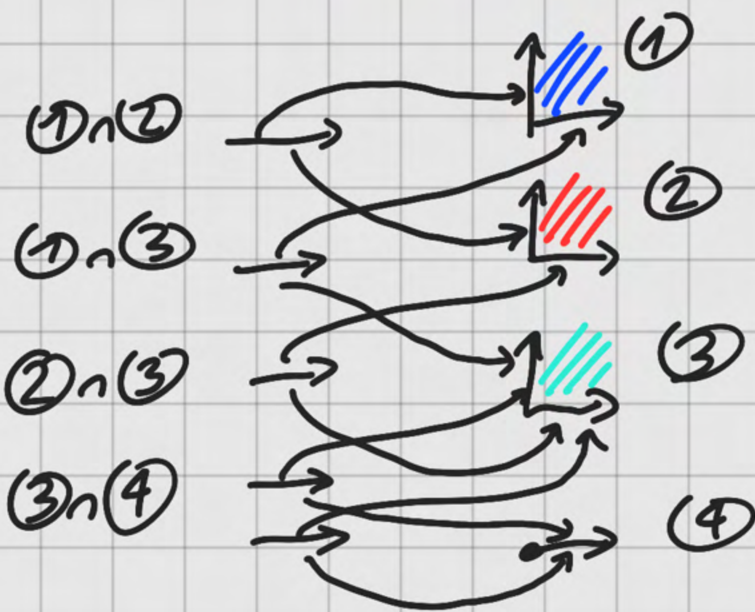


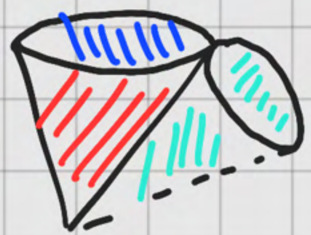
But any two such cones must be isomorphic by property (ii) (red arrows).

E.g., for the punctured Whitney umbrella:

How can we construct a cone stack from an NC pair (log smooth space)? [Abramovich–Chen–Marcus–Ulirsch–Wise, '14]







Rem:

$$\Sigma_X = \operatorname{colim}_{U \to X} \Sigma_U$$

Where $U \to X$ runs over all "étale subsets with a global chart"

To summarize the results so far, every toroidal compactification of a smooth DM stack gives a cone stack. We will now study how this cone stack changes under modifications of this compactification.

2 Log alterations and cone stacks

Some generalities on log schemes:

Def: A log scheme is a triple

(X, M, α) where X is a scheme, M is a sheaf of monoids on $X^{\text{ét}}$ and $\alpha : M \to (\mathcal{O}_X, \cdot)$ is a morphism of sheaves such that $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$.

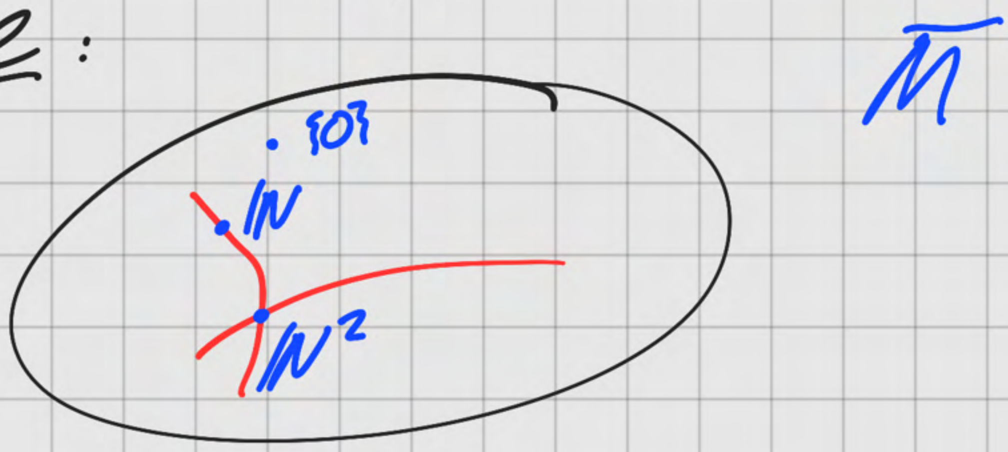
<u>Expl</u>: (The only example you should think about):

If (X, D) is a an NC pair, then we can set $M_X(U) := \{ f \in \mathcal{O}_X(U) \mid f|_{U \setminus D} \in \mathcal{O}_X^\times \}$.

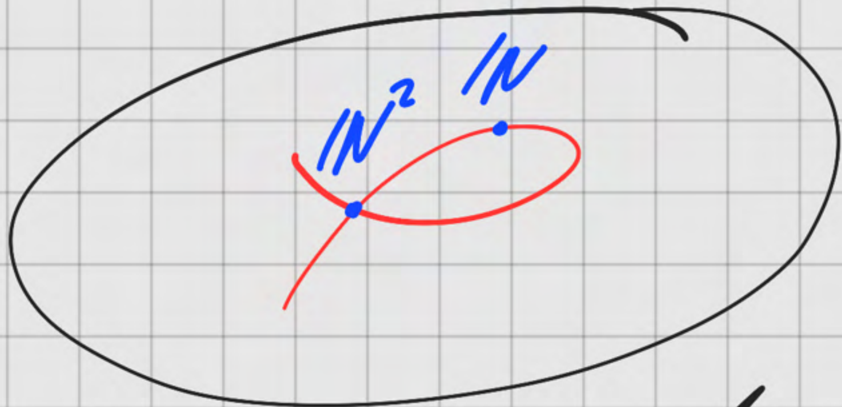
$\rightsquigarrow$ M_X is the "sheaf of functions vanishing at infinity".

Most of the information of M_X is contained in $\overline{M}_X := M_X / \mathcal{O}_X^\times$.

<u>Expl</u>:



Expl (étale topology):



Expl: A toric variety is a log scheme, by taking:

$$M := \mathcal{O}_X^* \cdot \langle x^u \mid u \in \sigma^\vee \rangle$$

on an affine.

The log. structure tracks the strata of the compactification, i.e. the rank of $\overline{M}$ at a point is the number of boundary divisors containing that point (étale-locally).

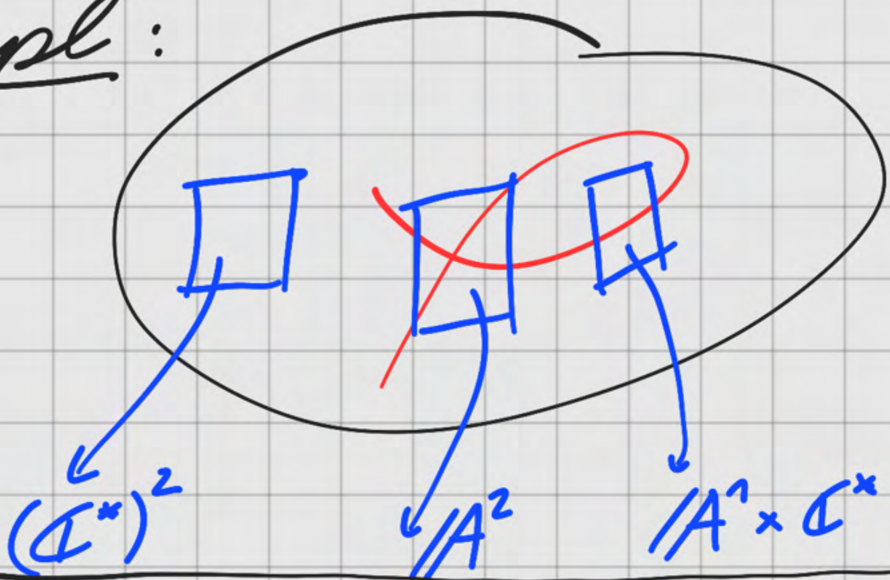
Log geometry has a name for the open subsets we used to construct the cone complex:

Def: A chart of a log scheme is map $X \to X_\sigma$ to some toric variety, such that M_X is isomorphic to the pullback of the log structure on X_σ (intuitively: M_X is generated by global sections).

All our log structures will have charts locally.

– charts

Expl:



Def: A log scheme is log smooth if it has charts étale locally s.t. $X \to X_\sigma$ is étale.

A map $X \to Y$ of log schemes is log smooth if étale locally it is isomorphic to $Y \times_{X_\Delta} X_{\Delta'} \to Y$ for some toric varieties $X_\Delta, X_{\Delta'}$.

Locally: log smooth $\Leftrightarrow$ has at worst toric singularities.

Def: The sheaf of logarithmic differentials $\Omega^{\log}_{X|Y}$ is defined by some universal property which I don't care about. I will define it for a map of NC pairs $(X,D) \to (Y,E)$:

$$\Omega^{\log}_{X|Y} := \Omega_{X|Y}\left\langle \frac{df}{f} \,\middle|\, f \in M_X \right\rangle \Big/ \left\langle \frac{de}{e} \,\middle|\, e \in M_Y \right\rangle$$

I.e., the sheaf of differentials regular away from D with at most logarithmic singularities.

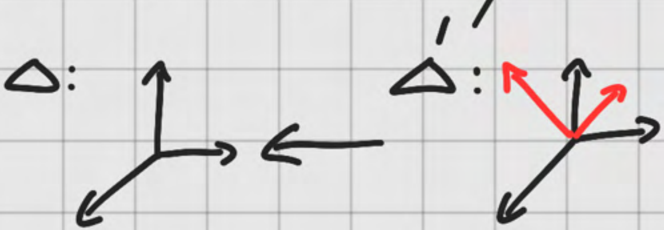
Proposition: If f is log smooth, the sheaf of log differentials is locally free.

Pf: It suffices to show $\Omega^{\log}_{X_\Delta} = \mathcal{O}_X^d$ for any toric variety. But this follows from the fact that

$$M \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Delta} \to \Omega^{\log}_{X_\Delta} \quad v \otimes g \mapsto g \cdot \frac{dx^v}{x^v}$$

is an isomorphism. Here M is the character lattice of X_Δ (logarithmically, would take $M^{gp}_{X_\Delta}(X_\Delta)$)

__Expl__: Toric morphisms are log smooth (If $X_\Delta \to X_{\Delta'}$ is a toric map, $X_\Delta \cong X_\Delta \times_{X_{\Delta'}} X_{\Delta'}$). In particular the proper, log étale maps correspond to subdivisions of Δ:

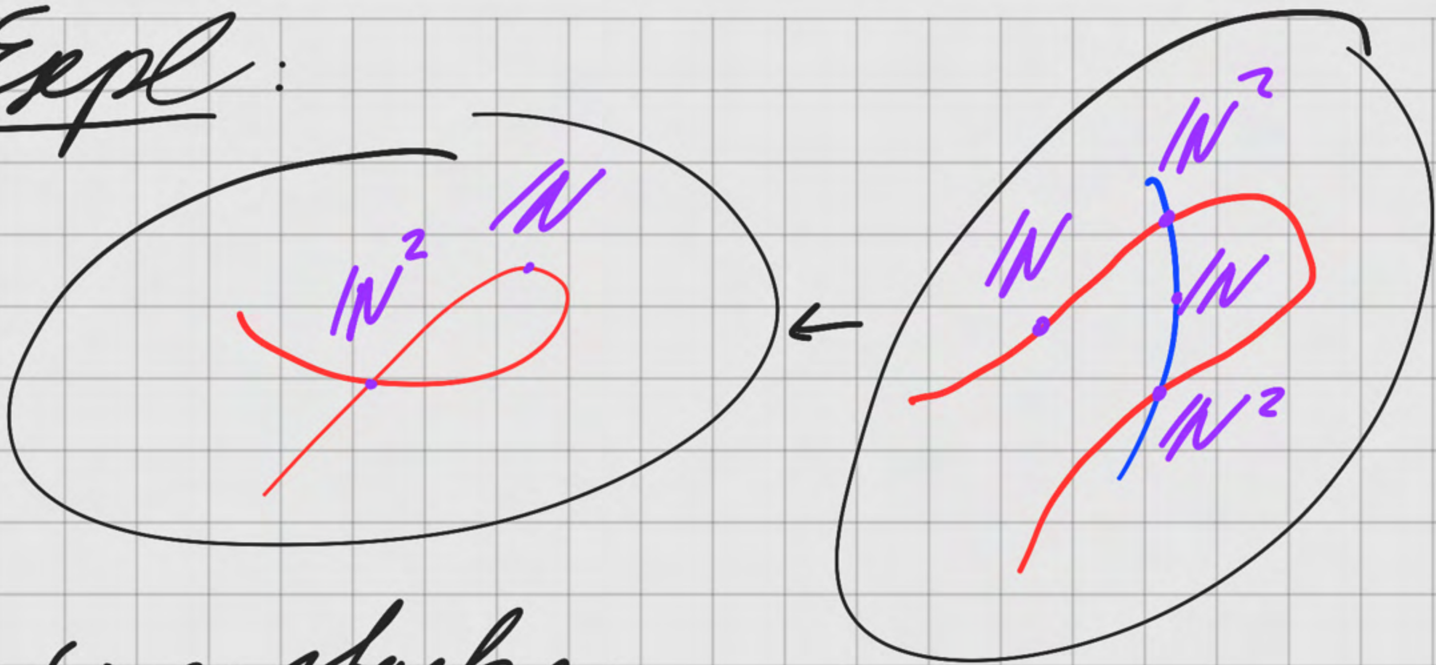


These are blowups iff they can be written as an iterated subdivision along hyperplanes.

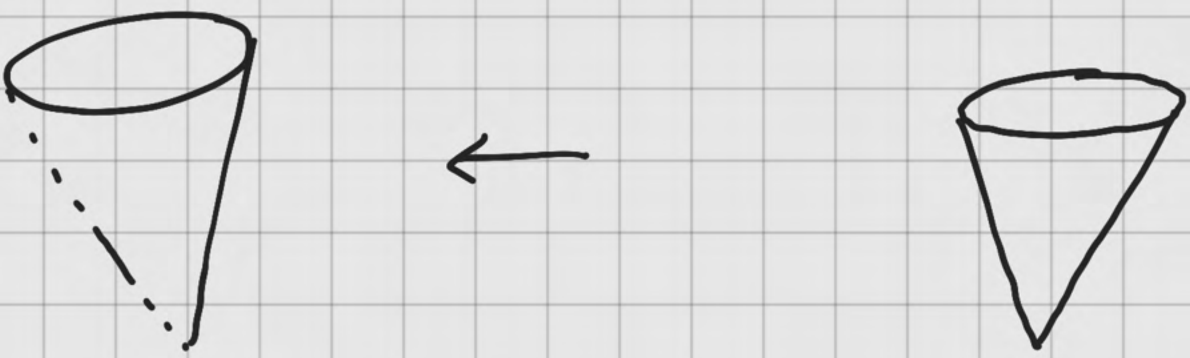
__Def__: Let X be a log scheme. A morphism $X' \to X$ is a log blowup (modification) if it is étale-locally on X it is pulled back from a toric blowup (subdivision).

__Cor__: Log modifications are log étale, proper monomorphisms.

Expl:



cone stacks:



i.e:

<u>Thm</u>: Let (X, D) be an NC pair. Then the log modifications of (X, D) correspond to subdivisions of the cone stack.

The key part of the proof is showing that a morphim which is locally a subdivision is a global subdivision. This works because cones don't have any non-trivial covers.

Brief digression on root stacks

For stacks there is a second log étale construction, namely taking root stacks of the boundary divisor components:

[Abramovich]

A (Cartier) divisor D on X is equivalent to a map $X \to [\mathbb{A}^1/\mathbb{G}_m]$.

On $\mathbb{A}^1$, we can take an n-th root of $\{0\}$ by considering $[\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{\cdot n} [\mathbb{A}^1/\mathbb{G}_m]$ which takes x to x^n.

We then write:

$$\begin{tikzcd}
\sqrt[n]{D/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\cdot n"] \\
X \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}$$

On the dense point of $[\mathbb{A}^1/\mathbb{G}_m]$, $\cdot^n$ is an isomorphism whereas it is a μ_n-gerbe at the closed point. I.e., $\sqrt[n]{D/X}$ is obtained from X by adding μ_n-stabilizers along D.

You may now expect that I will tell you how to think about root stacks in the category of cone stacks. Instead, I will give a warning that this is not as straightforward as one may think: For example, the map $[\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{\cdot^n} [\mathbb{A}^1/\mathbb{G}_m]$ corresponds to the map of cones:

$\mathbb{R}_{\geq 0} \xrightarrow{\cdot n} \mathbb{R}_{\geq 0}$ which suggests that the tropicalization of $\sqrt[n]{\{0\}/\mathbb{A}^1}$ is $\mathbb{R}_{\geq 0}$, i.e. the same as the tropicalization of $\mathbb{A}^1$. This problem arises, since the construction of the cone stack of a space is not functorial.

__Def__: An __alteration__ of log schemes is a finite composition of root stacks and subdivisions.

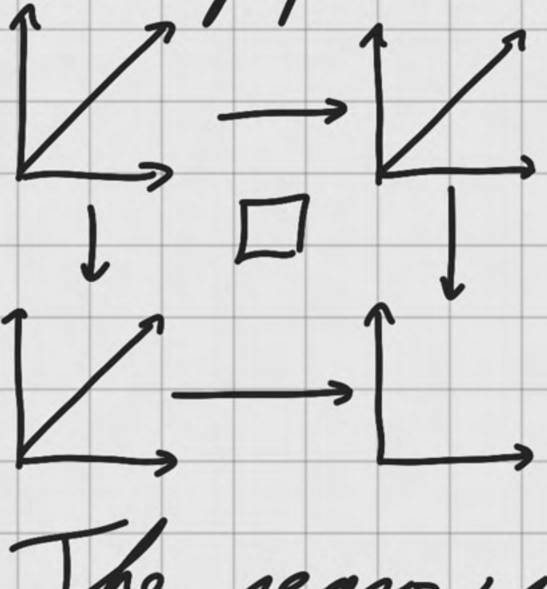
3 Application: weak semi-stable reduction for log smooth maps

<u>Def</u>: A map $X \to S$ is called weakly semistable if it is flat with reduced fibers and S is smooth.

<u>Question</u>: Given a flat family $X \to S$, can one alter S s.t. the pullback becomes semistable?

This seems definitely wrong (take $X \to S$ a blowup). But what about the following!

Blowup of $\mathbb{A}^2$ at 0:



The reason why this seems to work on the level of fans but not on the algebraic side is because fiber products don't commute with taking cone complexes. Instead, the procedure for taking fiber products of fans $\Delta_1 \to \Delta \leftarrow \Delta_2$ corresponds to:

1) Normalizing $X_{\Delta_1} \times_{X_\Delta} X_{\Delta_2}$,
2) Taking the component which contains the identity (e,e) of the torus.

<u>Warning</u>: Fiber products of log schemes do not commute with the forgetful map to schemes!

Idea of weak SS reduction:

1) Resolve singularities of the morphism to make it log smooth (char 0)
2) Use combinatorics to compute a weak SS reduction in the log smooth case. (arbitrary char)

We will focus on (2):

<u>Thm</u> [Molcho, <u>Adiprasito - Liu - Temkin</u>, Abramovich]

If $X \to B$ is a proper, surjective log smooth morphism between log smooth schemes (i.e. toroidal embeddings), then there exists an alteration $B' \to B$ and a modification $X' \to X \times_B B'$ (fiber product in the category of fs schemes!) of the main component of $X \times_B B'$ such that $X' \to X$ is weakly semistable.

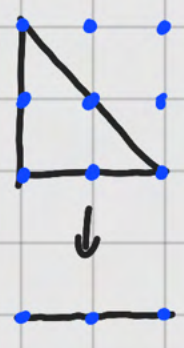
This can be done

1) functorially
2) universally (allowing log algebraic stacks)

<u>Pf</u>: Associate to $X \to B$ a map of cone stacks $\Sigma_X \to \Sigma_B$.

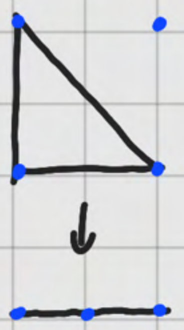
On the level of cone stacks, weakly semistable is equivalent to:

1) Each cone $\sigma \in \Sigma_X$ is mapped surjectively onto a cone $\tau \in \Sigma_B$.
2) If (σ, N) is mapped to (τ, N'), then $N \to N'$ is surjective.

<u>Expl</u>:  is weakly semistable

$\leftrightarrow \mathbb{A}^2 \to \mathbb{A}^1$.

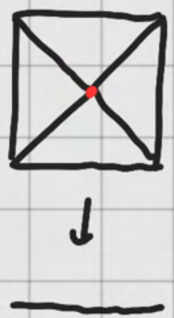
But taking a sublattice:



corresponds to:

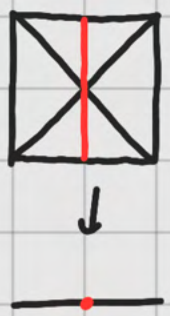
$$\mathbb{A}^2 \xrightarrow[(x^2,y^2)]{} \mathbb{A}^2_2 \to \mathbb{A}^1$$

which has non-reduced fibers. This can be resolved by taking a sublattice on the base.



is not weakly ss.

This can be resolved by subdivision:



Idea [Molcho, Kapranov-Sturmfels-Zelevinsky]:

The cones in the subdivision of Σ_B are the loci where the fiber polytope is constant.

E.g.:

Need to make sure that:

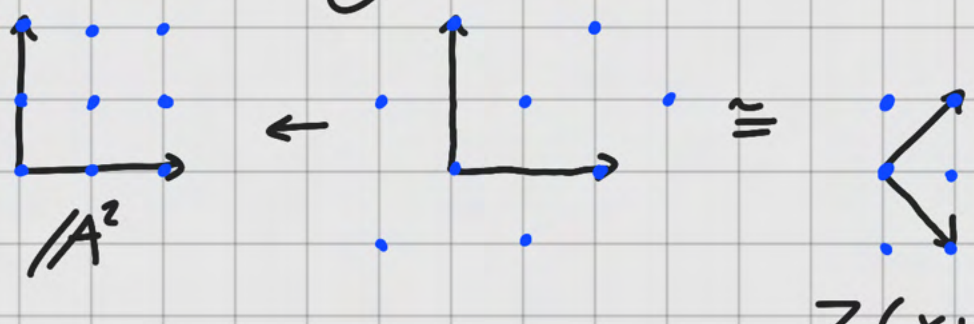
can't happen (not proper since dominant but not surjective).

Using this, the fibers are now constant on each cone, so every cone maps surjectively onto a cone.

Remaining problem: lattices may not surject.

The problem is that simply changing the base lattice is not an alteration.

F.e., it may make B non-smooth:



$Z(xy-z^2)$.

Idea: take root stacks along torus divisors. This "decreases the length of the rays" and turns out to resolve the non-reduced fibers.

$\square$

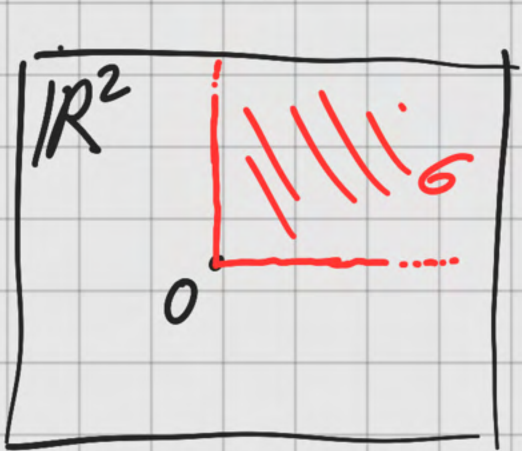
Appendise

Toric varieties and fans

Toric varieties are varieties with a nice combinatorial description:

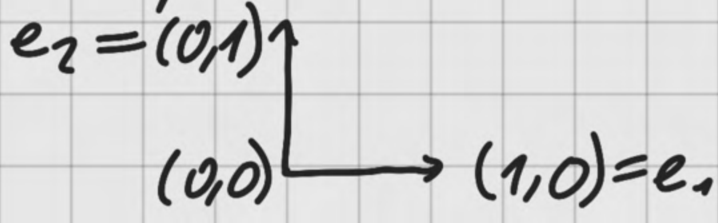
Def: A (strictly convex) rational polyhedral cone (RPC) in $\mathbb{R}^n$ is the convex hull of finitely many rays through the origin, with rational slope, (all lying in some open half space).

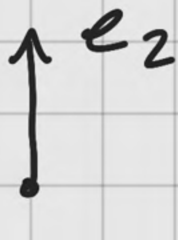
Exple:



or:

Draw simpler:

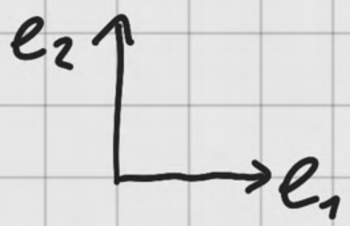




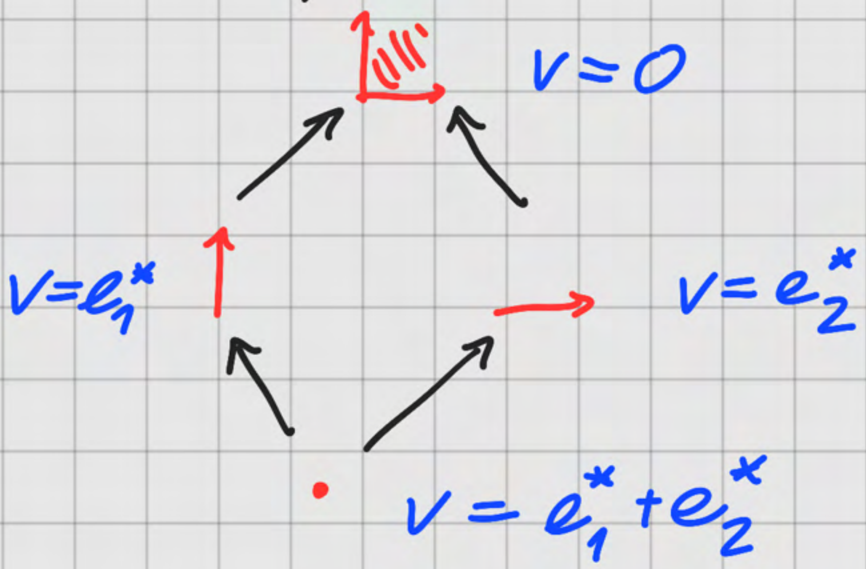
Note: the rays are not part of the data, i.e. the cone spanned by e_1, e_2 is the same as the one spanned by e_1, e_2, e_1+e_2.

<u>Def</u>: A face of a cone σ is a cone which is obtained as $\sigma \cap \{v=0\}$ for some $v \in (\mathbb{R}^n)^*$ satisfying $v(\sigma) \subseteq \mathbb{R}_{\geq 0}$.

<u>Ex</u>:



has 4 faces:

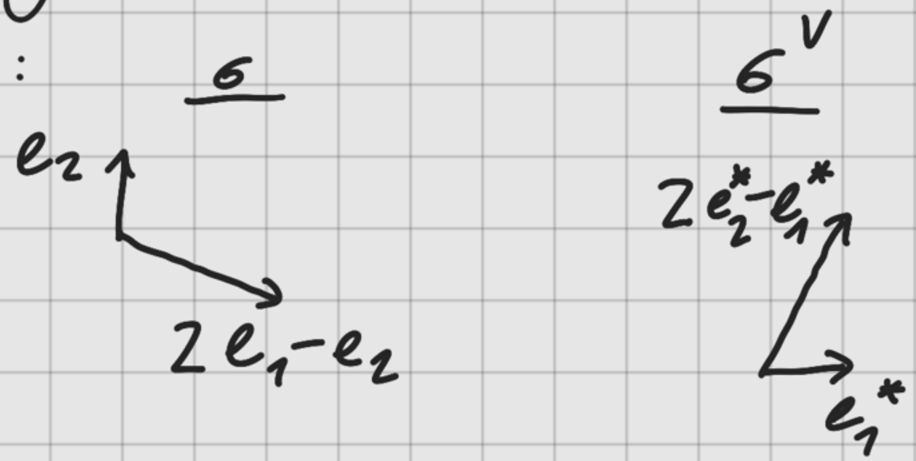


<u>Def</u>: The dual of a cone σ is:

$$\sigma^\vee := \{v \in (\mathbb{R}^n)^* \mid v(\sigma) \subseteq \mathbb{R}_{\geq 0}\}.$$

It is again an RPC:

Expl:



Lem: If σ is an RPC, $\sigma \cap \mathbb{N}^n$ is a finitely generated monoid.

$\rightsquigarrow$ we can define a variety:

Def: $X_\sigma := \operatorname{Spec}(k[\sigma^\vee])$

where:

$$k[\sigma^\vee] = k[x^v \mid v \in \sigma^\vee \cap \mathbb{N}^n] \Big/ \langle x^u x^v - x^{u+v} \rangle.$$

X_σ has a dense open subset with coordinate ring:

$$k[(\sigma^\vee)^{gp}] = k[\sigma^\vee][x^{-v} \mid v \in \sigma^\vee \cap \mathbb{N}^n]$$

This is an algebraic torus $T \cong (\mathbb{C}^*)^n$

<u>Lem</u>: 1) The action of T on itself by multiplication extends to an action $T \curvearrowright X_\sigma$.

2) There is an inclusion - reversing bijection between T-orbits and faces of σ

3) This construction defines a bijection between the set of affine toric varieties with torus T and cones σ.

4) An affine toric variety is smooth iff the rays of its cone are a $\mathbb{Z}$-basis of $\mathbb{N}^n \cap \text{span}(\sigma)$.

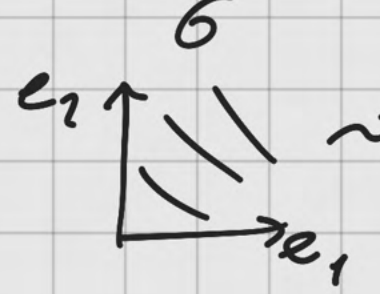
To get non-affine toric varieties, glue cones along common faces

$\rightsquigarrow$ <u>Def</u>: A fan in $\mathbb{R}^n$ is a collection Δ of RPCs such that:

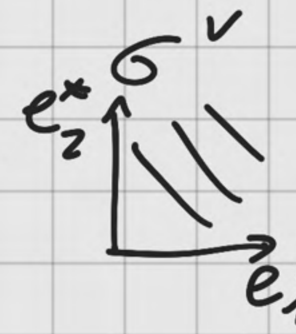
1) $\sigma \in \Delta \Rightarrow$ every face of σ is in Δ

2) $\sigma, \tau \in \Delta \Rightarrow \sigma \cap \tau \in \Delta$ and $\sigma \cap \tau$ is a face of σ and τ.

<u>Expl</u>:



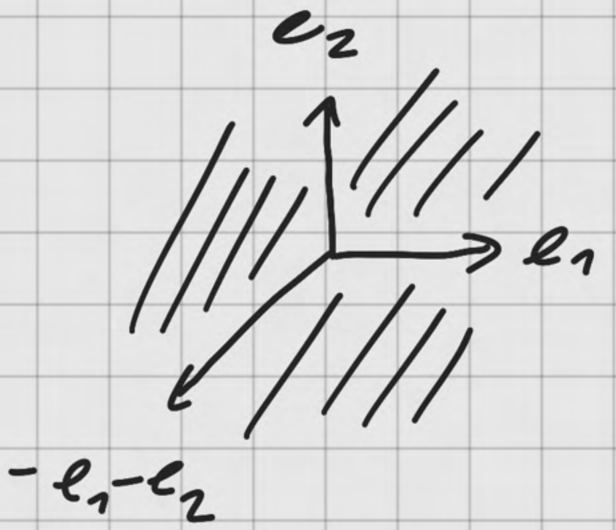
$\rightsquigarrow$



$\rightsquigarrow$ $k[\sigma^\vee]$

$$k[x^{e_1^*}, x^{e_2^*}] \cong k[x,y]$$

$$\rightsquigarrow X_\sigma = \mathbb{A}^2 \circlearrowleft (\mathbb{C}^*)^2$$



is a fan. Each affine part is isomorphic to $\mathbb{A}^2$. Tracing the gluing maps, we get:

$$X_\Delta \cong \mathbb{P}^2.$$

Important points for us:

- Toric varieties $\longleftrightarrow$ fans
- A toric variety has a distinguished divisor D given by the vanishing of monomials x^u.

- Many properties of toric varieties (morphisms, Chow groups, smoothness) can be read from the fan.