# Tropical abelian varieties and their moduli

## §0. Motivation

- Tropical geometry (fans, cone stacks) records <u>combinatorial information</u> about degenerations of smooth varieties
- Conversely, this allows us to define <u>toroidal compactifications</u> of moduli of these smooth varieties.

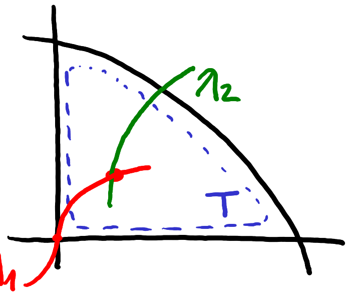
<u>Examples</u>

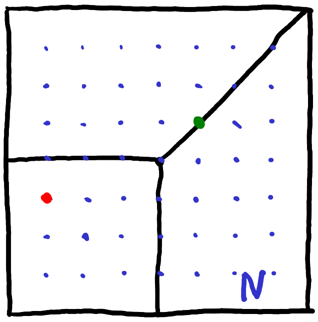
- Toric varieties

  $T \cong (\mathbb{C}^*)^n$ torus

  $N = \mathrm{Hom}(\mathbb{G}_m, N) \cong \mathbb{Z}^n$

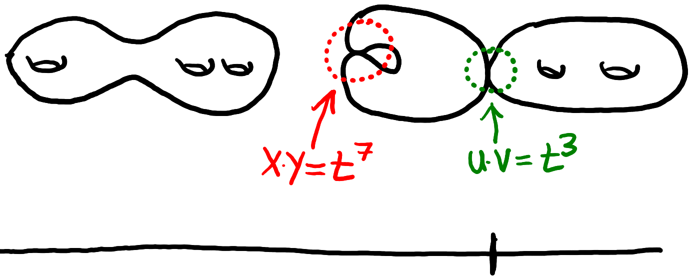
  ↑ 1-param. subgroups




$N \otimes \mathbb{R}$

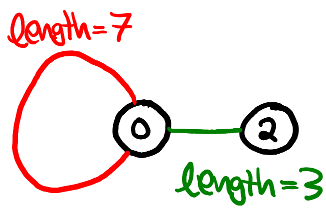
- Moduli of curves




Spec R ↗ DVR

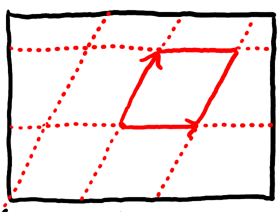
$t=0$ ↗

⇝




tropical curve

<u>Main goal</u> moduli of abelian varieties



$(X = V/\Lambda_t, \phi_t)$

??

Spec R

$t=0$ ↗

⇝

this talk!

$\boxed{??}$

tropical abelian variety

## §1. Principally polarized abelian varieties

**Def** A <u>principally polarized tropical abelian variety (pptav)</u> of dimension $g$ is given by the data $X=(V/\Lambda, Q(\cdot,\cdot))$ of

- a real vector space $V$ of dimension $g$ containing a lattice $\Lambda$ of rank $g$ (st. $V=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$)
- a positive semi-definite symmetric bilinear form

$$Q(\cdot,\cdot) : V\times V \longrightarrow \mathbb{R}$$

whose null-space

Null(Q) is <u>rational</u>

$$\mathrm{Null}(Q)=\{v\in V: Q(v,v)=0\}=\{v\in V: Q(v,w)=0\ \forall w\in V\}$$

has a basis (as $\mathbb{R}$-vector space) of elements in $\Lambda\subseteq V$.

$$(V/\Lambda, Q(\cdot,\cdot)) \xrightarrow[\text{isom.}]{\phi} (V'/\Lambda', Q'(\cdot,\cdot))$$

$\Leftrightarrow$ $V \xrightarrow[\sim]{\phi} V'$ st. $\phi(\Lambda)=\Lambda'$ and $Q'(\phi(v),\phi(w))=Q(v,w)\ \forall v,w\in V$

**Exa** $V=\mathbb{R}^2\supseteq\Lambda=\langle\binom{1}{0},\binom{1}{1}\rangle$, $Q\left(\binom{x}{y},\binom{w}{z}\right)=3\cdot x\cdot w$

$\leadsto$ $\mathrm{Null}(Q)=\mathrm{Span}\binom{0}{1}\rightarrow$ rat'l

**Note** $\Lambda \xrightarrow[\sim]{\phi_{\mathbb{Z}}} \mathbb{Z}^g \leadsto V \xrightarrow{\phi} \mathbb{R}^g$

$\Rightarrow$ any $X$ is of the form

$$X\cong(\mathbb{R}^g/\mathbb{Z}^g, Q(v,w)=v^T\cdot Q\cdot w)$$

positive semi-def. matrix $Q\in \mathrm{Mat}_{g\times g}(\mathbb{R})$. w/ rat'l kernel

**Idea** $\mathcal{X}$ abelian var. dim $g$ $\leadsto$ $\mathcal{X}\cong\mathbb{C}^g/\mathbb{Z}^{2g}$, $\phi:\mathcal{X}\longrightarrow\mathcal{X}^\vee$ Polarization

$\wr$

$X$ tropical ab. var, dim $g$ $\leadsto$ $X\cong\mathbb{R}^g/\mathbb{Z}^g$, $Q$ trop. polariz.

## Interlude: Game plan

$$\left\{\begin{array}{l}\text{matrices } Q\geq 0\\ \text{w/ } \ker(Q) \text{ rational}\end{array}\right\} \xrightarrow[q]{Q\mapsto(\mathbb{R}^g/\mathbb{Z}^g,\, Q(\cdot,\cdot))} \left\{\begin{array}{l}\text{p.p. tropical}\\ \text{abelian var. } X\end{array}\right\}/\text{isom}$$

$\Omega_g^{rt}$ $\qquad\qquad\qquad\qquad$ $\mathcal{A}_g^{trop}$

### Fibers of $q$

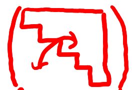
$$q(Q_1)\cong q(Q_2) \iff \exists\, S: \mathbb{Z}^g \xrightarrow{\sim} \mathbb{Z}^g :$$
$$(Sv)^T\cdot Q_2\cdot (Sw) = v^T\cdot Q_1\cdot w \quad \forall v,w\in\mathbb{R}^g$$

$$\iff \exists\, S\in GL_g(\mathbb{Z}): \quad \underbrace{S^T\cdot Q_2\cdot S}_{\text{defines action } GL_g(\mathbb{Z})\curvearrowright\Omega_g^{rt}} = Q_1$$

<u>Plan</u> Construct $\mathcal{A}_g^{trop}$ as $\mathcal{A}_g^{trop} = \Omega_g^{rt}/GL_g(\mathbb{Z})$.

## §2. Spaces of positive semi-definite matrices

<u>Def</u> For $g\geq 0$ we define the spaces

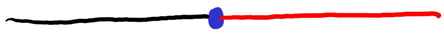



$$\begin{aligned}
\mathrm{Sym}_g &= \{A\in \mathrm{Mat}_{g\times g}(\mathbb{R}) : A \text{ symmetric}\} \cong \mathbb{R}^{\binom{g+1}{2}}\\
&\ \ \cup\!\!\!\mid\\
\Omega_g^{rt} &= \{Q\in\mathrm{Sym}_g : Q\geq 0,\ \ker(Q) \text{ rat'l}\}\\
&\ \ \cup\!\!\!\mid\\
\Omega_g &= \{Q\in\mathrm{Sym}_g : Q>0\}
\end{aligned}$$

rat'l closure of $\Omega_g$.

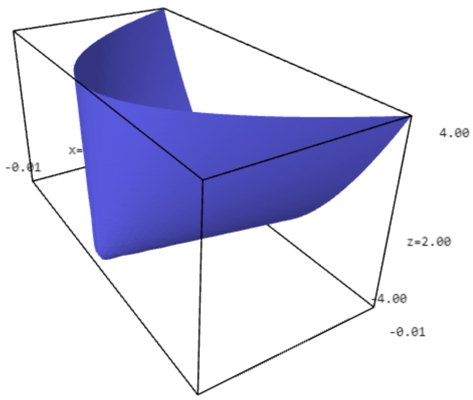
<u>Pro</u> $\Omega_g^{rt}$ and $\Omega_g$ are convex cones in $\mathrm{Sym}_g$. Moreover, we have $\Omega_g\subseteq\mathrm{Sym}_g$ is open and dense as a subset of $\Omega_g^{rt}$.

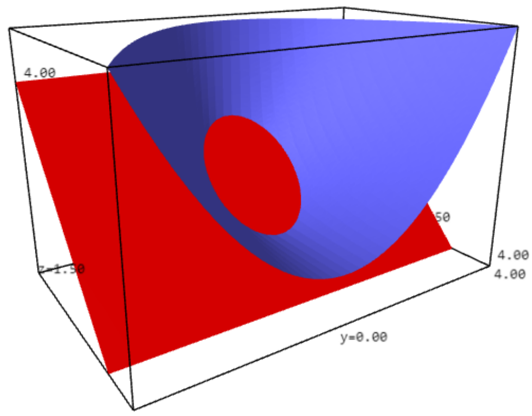
**Example** $(g=1)$ $\quad \mathrm{Sym}_1 \cong \mathbb{R} \supseteq \Omega_1^{rt} = \mathbb{R}_{\geq 0} \supseteq \Omega_1 = \mathbb{R}_{>0}$



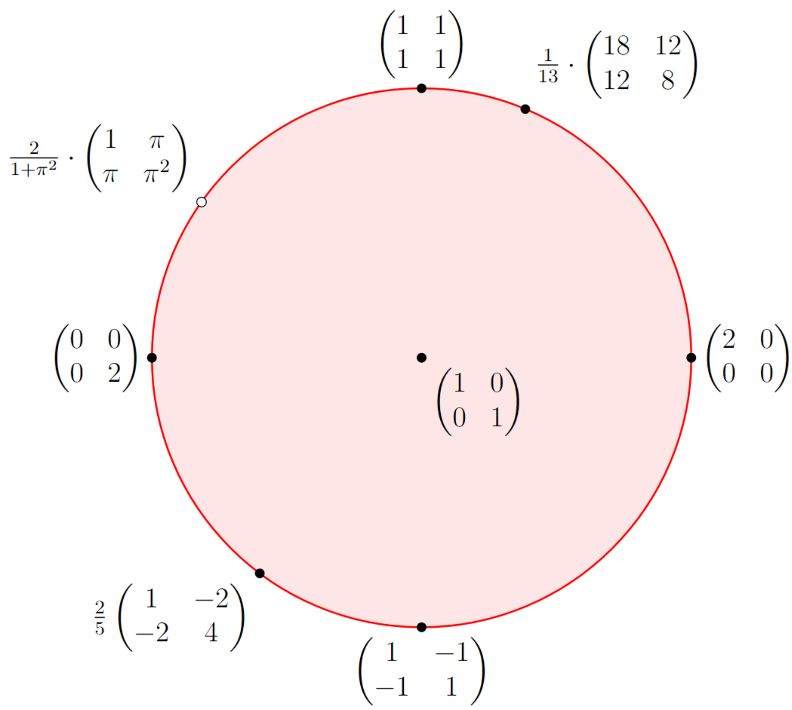
**Example** $(g=2)$ $\quad \mathrm{Sym}_2 = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : (a,b,c) \in \mathbb{R}^3 \right\}$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0 \iff a \geq 0 \;,\; c \geq 0 \;,\; ac - b^2 \geq 0$$




$\overline{\Omega_g}$ = filled quadratic cone $\rightsquigarrow$ cutting with plane $H = \{a + c = 2\}$.




external rays: $\mathrm{Cone}(v \cdot v^T)$ for $v \in \mathbb{Z}^2$ primitive

$\leftrightsquigarrow$ Pythagorean triples $x^2 + y^2 = z^2$, $x, y, z \in \mathbb{Z}$

## Facts

- extremal rays of $\Omega_g^{rt}$ : $\mathrm{Cone}(v \cdot v^T)$ for $v \in \mathbb{Z}^g$ primitive
  ↳ $\Omega_g^{rt}$ = cone spanned by these rays.
- $GL_g(\mathbb{Z}) \curvearrowright \Omega_g^{rt}$ acts transitively on the rays
- More generally: $Q \in \Omega_g^{rt}$ of rank $r$

$$\Rightarrow \exists\, S \in GL_g(\mathbb{Z}) : \quad S \cdot Q \cdot S^T = \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}, \; \underset{\text{pos. def.}}{Q' \in \Omega_r}$$

## Problems with defining $\mathcal{A}_g^{trop} = \Omega_g^{rt} / GL_g(\mathbb{Z})$

A) $\Omega_g^{rt}$ is not a rational polyhedral cone!
↳ need to subdivide into RPCs

B) Quotient $\mathcal{A}_g^{trop}$ no longer has cone structure!
↳ need to subdivide before taking quotient ⇝ cone stack

C) Quotient topology on $\mathcal{A}_g^{trop}$ is not Hausdorff!

$$X_n = \begin{pmatrix} 1 & 1/n \\ 1/n & 1/n^2 \end{pmatrix} \xrightarrow{n} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_n = \begin{pmatrix} 1/n^2 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{n} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\not\simeq$

$\simeq$

↳ need to subdivide, take quotient on finite list of orbit represent.

D) Action has infinite stabilizers!

$$\mathrm{Stab}_{GL_2(\mathbb{Z})}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \{\pm 1\}, b \in \mathbb{Z} \right\}$$

↳ need new definition of p.p. trop. abelian varieties!

# §3. Admissible decompositions

**Def** An <u>admissible decomposition</u> of $\Omega_g^{rt}$ is a collection $\Sigma$ of RPCs in $\Omega_g^{rt}$ such that

a) $\forall \sigma \in \Sigma$ and $\tau \preceq$ face $\Rightarrow \tau \in \Sigma$

b) $\forall \sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2$ is face of $\sigma_1, \sigma_2$

c) $\Omega_g^{rt} = \bigcup_{\sigma \in \Sigma} \sigma$

$\Sigma$ is (infinite) fan w/ $|\Sigma| = \Omega_g^{rt}$

e) $\forall \sigma \in \Sigma$, $S \in GL_g(\mathbb{Z})$: $\underbrace{S \cdot \sigma}_{\{S \cdot M \cdot S^T \,:\, M \in \sigma\}} \in \Sigma$

f) $\Sigma$ decomposes into finitely many $GL_g(\mathbb{Z})$-orbits

$GL_g(\mathbb{Z})$-invar. & finiteness.

## §3.1. Perfect cone decomposition

**Def** Given $Q \in \Omega_g$ pos. definite, let

$$\mu(Q) = \min_{v \in \mathbb{Z}^g \setminus \{0\}} Q(v,v) \quad \text{and} \quad M(Q) = \{v \in \mathbb{Z}^g \setminus \{0\} : Q(v,v) = \mu(Q)\}$$

minimum          minimal vectors

Let $\sigma[Q] = \mathrm{Cone}(v \cdot v^T : v \in M(Q))$

**Thm [Vor]** $\Sigma^{PC} = \{\sigma[Q] : Q \in \Omega_g\}$ is an adm. decomp. of $\Omega_g^{rt}$

perfect cone / first voronoi decompos.

**Example 2.20** ($g = 2$). Let's calculate $\sigma[Q]$ for some positive definite matrices $Q \in \Omega_2$. Consider the list

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Q_3 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}. \tag{20}$$

A short calculation shows that $\mu(Q_i) = 1$ for $i = 1, 2, 3$, and the minimal vectors are given by

$$M(Q_1) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, M(Q_2) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, M(Q_3) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}. \tag{21}$$

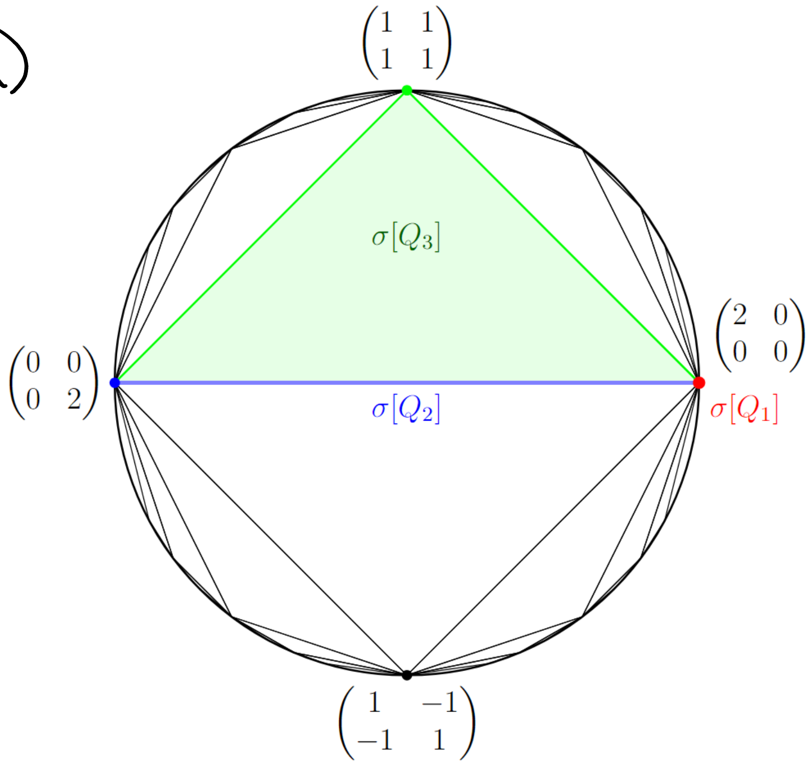
Consequently, we have that the cones $\sigma[Q_i]$ of the perfect cone decomposition are calculated as

$$\sigma[Q_1] = \mathrm{Cone}\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$\sigma[Q_2] = \mathrm{Cone}\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

$$\sigma[Q_2] = \mathrm{Cone}\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right).$$

Exa ($g=2$)




Facts

- Consider $Q_0, Q_1 \in \Omega_g$ given by

$$Q_0 = \begin{pmatrix} 2 & & 1 \\ & \ddots & \\ 1 & & 2 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} \begin{smallmatrix}2&0\\0&2\end{smallmatrix} & & & 1 \\ & 2 & & \\ & & \ddots & \\ 1 & & & 2 \end{pmatrix}$$

$\sigma[Q_0]$ = principal cone (simplicial)

$\sigma[Q_1] \in \Sigma_g^{pc}$ not simplicial for $g \geq 4$

$\overline{\mathcal{A}}_g^{pc}$ singular in codim 10.

- Cones can be enumerated (Voronoi's algorithm) $\rightsquigarrow$ complexity grows fast!

| g | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\#\Sigma_g^{pc}(\max)/\mathrm{GL}_g(\mathbb{Z})$ | 1 | 1 | 2 | 3 | 7 | 33 | 10916 | ?? (> 500000) |

Table 1: Number of $\mathrm{GL}_g(\mathbb{Z})$-orbits of maximal cones in the perfect cone decomposition

## §3.2. Second Voronoi decomposition

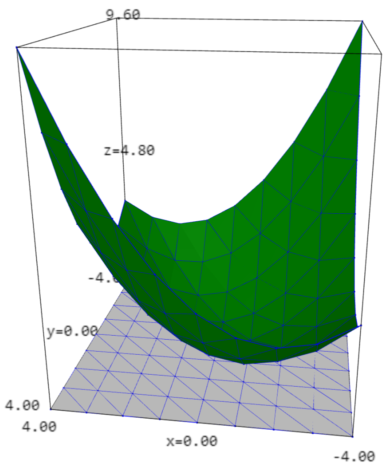
**Def** For $Q \in \Omega_g^{rt}$ consider the convex hull

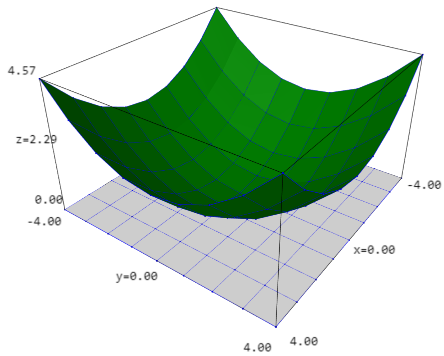
$$\mathcal{C}_Q = \mathrm{Conv}\big((v, Q(v,v)) : v \in \mathbb{Z}^g\big) \subseteq \mathbb{R}^{g+1}$$

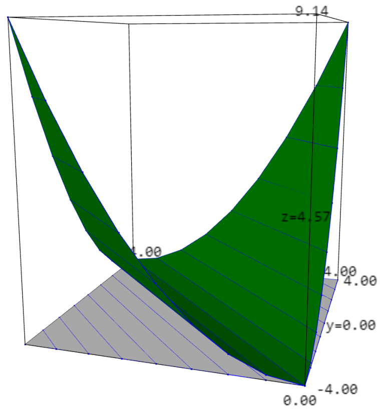
$$\text{lower facets of } \mathcal{C}_Q \xmapsto{(x_1,\dots,x_g,y)\mapsto(x_1,\dots,x_g)} \mathrm{Del}(Q)$$ Delaunay subdiv.

Polyhedral subdiv. of $\mathbb{R}^g$; $\mathbb{Z}^g$-invariant

**Exa**




$Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\qquad$ $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad$ $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

**Thm** [Vor.] Given $D$ a Delaunay subdivision, then

$$\sigma_D = \overline{\{Q \in \Omega_g^{rt} : \mathrm{Del}(Q) = D\}} \quad \text{is a RPC}$$

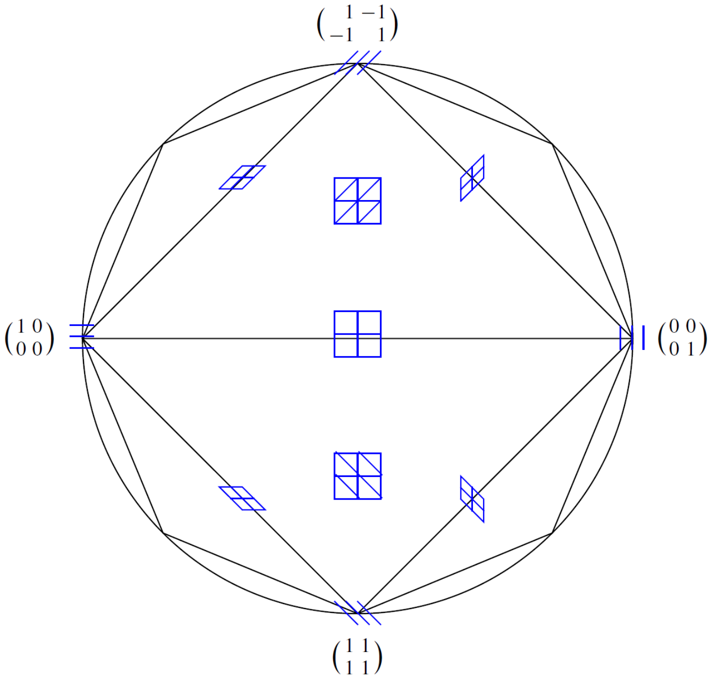
and $\Sigma^{vor} = \{\sigma_D : D \text{ Del. subdiv.}\}$ is an adm. decomposition.

2nd Voronoi decomposition

## Exa (g=2)

Turns out:

$$\Sigma_2^{vor} = \Sigma_2^{PC}$$




[Chan - Combinatorics of the tropical Torelli map, Fig.5]

## Facts

- Number of orbits of maximal cones:

| g | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\#\Sigma_g^{\mathrm{Vor}}(\max)/\mathrm{GL}_g(\mathbb{Z})$ | 1 | 1 | 1 | 3 | 222 | ?? (> 250000) |

- For $g \geq 4$, there are rays $\rho \in \Sigma_g^{vor}(1)$ not spanned by $v \cdot v^T$
  ↳ boundary of $\bar{\mathcal{A}}_g^{vor}$ no longer irred. divisor

- For $g \geq 5$, there are non-simplicial cones in $\Sigma_g^{vor}(3)$
  ↳ $\bar{\mathcal{A}}_g^{vor}$ singular in codimension 3

## Comparison $\Sigma_g^{PC}$ vs. $\Sigma_g^{vor}$

| g | $g \leq 3$ | $g = 4, 5$ | $g \geq 6$ |
|---|---|---|---|
| PC vs. Vor | $\Sigma_g^{PC} = \Sigma_g^{vor}$ | $\Sigma_g^{vor} \longrightarrow \Sigma_g^{PC}$ strict refinem. | neither refines the other |

$\Sigma_g^{PC} \cap \Sigma_g^{vor} = \Sigma_g^{mat}$ matroidal locus ← simplicial, see [Melo-Viviani 12]

# §4. Cone stacks

$\Sigma$ admissible decomp. of $\Omega_g^{rt} \overset{??}{\rightsquigarrow}$ cone stack $cs(\Sigma)$

<u>Def</u> (wrong)

- <u>Objects</u> cones $\sigma \in \Sigma$
- <u>Morphisms</u>

$$\mathrm{Mor}(\sigma_1, \sigma_2) = \{ S \in GL_g(\mathbb{Z}) : S \cdot \sigma_1 \text{ face of } \sigma_2 \}$$

<u>Problem</u> $\mathcal{S} = \mathrm{Cone}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Sigma_2^{PC} \rightsquigarrow \mathrm{Aut}(\mathcal{S}) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \begin{matrix} a, c = \pm 1 \\ b \in \mathbb{Z} \end{matrix} \right\}$

infinite!

<u>Idea</u> Take union $\Omega_0^{rt} \sqcup \Omega_1^{rt} \sqcup \cdots \sqcup \Omega_g^{rt}$

$\Sigma \rightsquigarrow$ subdivis. $\Sigma_0 \quad \Sigma_1 \quad \cdots \quad \Sigma_g = \Sigma$ via maps

$$i_A : \Omega_h^{rt} \longrightarrow \Omega_g^{rt}, \quad Q \mapsto A^T Q A, \text{ for } A : \mathbb{Z}^g \twoheadrightarrow \mathbb{Z}^h$$

<u>Def</u> (right)

- <u>Objects</u> Cones $\sigma \in \Sigma_h$ $(h = 0, \dots, g)$ w/ $\sigma \cap \Omega_h \neq \emptyset$
- <u>Morphisms</u>

$$\mathrm{Mor}(\underset{\text{in } \Omega_{h_1}^{rt}}{\sigma_1}, \underset{\text{in } \Omega_{h_2}^{rt}}{\sigma_2}) = \{ A : \mathbb{Z}^{h_1} \twoheadrightarrow \mathbb{Z}^{h_2} : i_A(\sigma_1) \text{ face of } \sigma_2 \}$$

<u>Exa</u> ($\Sigma_2^{PC}$)

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|---|---|---|---|---|
| $\sigma_0$ | $()$ | $()$ | $()$ | $()$ |
| $\sigma_1$ | | $\pm(1)$ | $\pm(1\ \ 0), \pm(0\ \ 1)$ | $\pm(1\ \ 0), \pm(0\ \ 1), \pm(1\ \ -1)$ |
| $\sigma_2$ | | | $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}$ | $\pm\begin{pmatrix} 1 & 0 \\ 0 & \pm1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ \pm1 & 0 \end{pmatrix}, \pm\begin{pmatrix} \pm1 & 1 \\ 0 & -1 \end{pmatrix}$ $\pm\begin{pmatrix} 1 & \pm1 \\ -1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ \pm1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ 1 & \pm1 \end{pmatrix}$ |
| $\sigma_3$ | | | | $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\pm\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ |

Table 3: Matrices $A$ defining morphisms $\sigma_i \to \sigma_j$ in $\Sigma$ is displayed in row $i$, column $j$; here $(), (), (), ()$ denote the unique matrices of dimensions $0 \times 0, 0 \times 1, 0 \times 2$ and $0 \times 3$, respectively. Also, all signs $\pm$ appearing in the formulas can be chosen independently of one another, so a formula containing two instances of $\pm$ stands for four different matrices.

Thanks for your attention!