Degenerations of abelian varieties & construction of the moduli spaces

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0. Semiabelian schemes and semistable reduction

Def: A <u>semiabelian variety</u> G is a commutative, reductive connected group scheme $/k$

G semiabelian

$\Updownarrow$

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

$T \simeq \mathbb{G}_m^r$ h-dimensional abelian variety

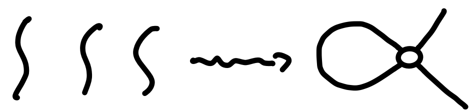
r = torus rank

$g = \dim(G)$

Def: A <u>semiabelian scheme</u> is a group scheme G/S whose fibers G_s are semi abelian varieties

- $s \longmapsto r(G_s)$ is NOT constant

  just lower-semicont.

  eg: 

  ell. curves torus

- If $s \longmapsto r(G_s)$ is constant then

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

globally on S, where A/S is an abelian scheme.

- Semiabelian schemes satisfy semistable reduction !!

· <u>Example</u>

· C prestable curve (at worst nodal) $\rightsquigarrow$ $\mathrm{Pic}^{0}(C)$ is an ab. variety

$$0 \longrightarrow T \longrightarrow \mathrm{Pic}^{0}(C) \longrightarrow \mathrm{Pic}^{0}(\tilde{C}) \longrightarrow 0$$

torus of rank $b_1(\Gamma)$ ↳ stable graph

normalization of C

· Split semiabelian schemes

$$0 \longrightarrow T \longrightarrow G \longrightarrow X \longrightarrow 0$$

are classified by

$$\mathrm{Hom}(M, X)$$

$$M = \mathrm{Hom}(T, \mathbb{G}_m)$$

$$N = \mathrm{Hom}(\mathbb{G}_m, T) = M^{\vee}$$

1. Degeneration data

Degeneration setup (✱)

$S = \operatorname{Spec}$ (normal domain R)

$\eta = \operatorname{Spec}(K) = \operatorname{Spec}(\operatorname{Frac}(R))$

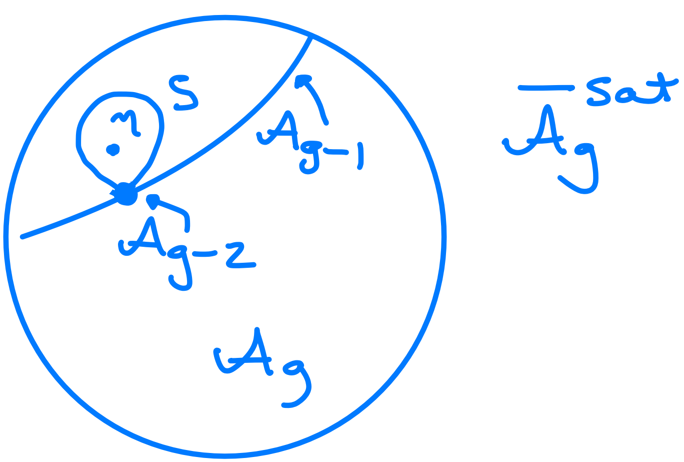
$S_0 = \operatorname{Spec}(R/I)$

G/S = semiabelian scheme

Assume G_η is an ab. variety,

$\lambda : G_\eta \longrightarrow G_\eta^\vee$ polarization

G_0 has constant torus rank r



Want to produce simpler data that has some moduli

1.1. Raynaud extensions

Over $\hat{S}$ (the I-adic completion), one can extend

$$0 \longrightarrow T_0 \longrightarrow G_0 \longrightarrow X_0 \longrightarrow 0$$

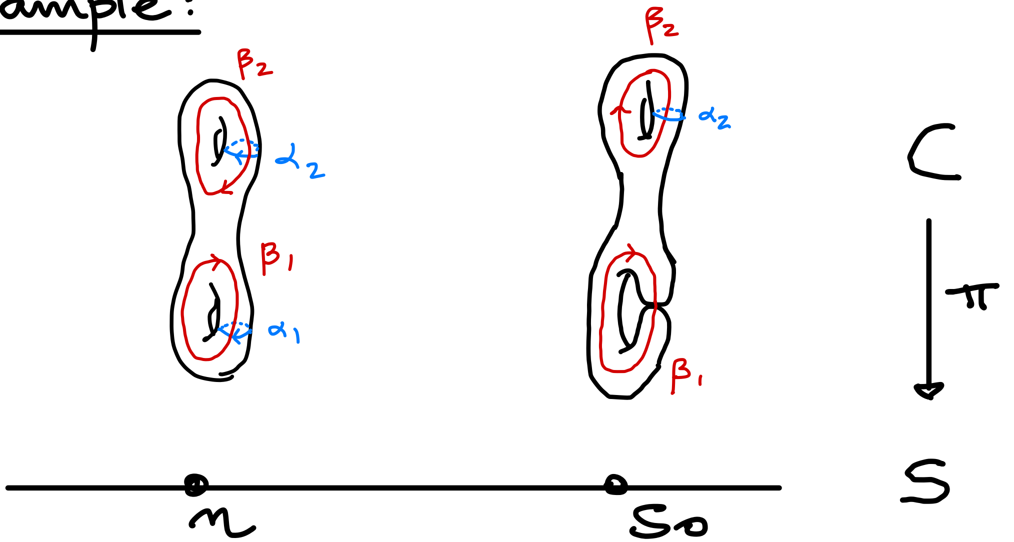
to a split semiabelian scheme:

$$0 \longrightarrow T \longrightarrow \tilde{G} \longrightarrow X \longrightarrow 0$$

$\tilde{G}$ is the Raynaud extension of G.

- Raynaud extensions are functorial and do not depend on choices
- Raynaud extensions preserve duality.

Example:



$$\mathrm{Jac}(C_\eta) = \frac{H^0(C_\eta, \omega_{C_\eta})^\vee}{\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \rangle} = G_\eta$$

$$\begin{tikzcd}
0 \arrow[r] & T_0 \arrow[r] & \mathrm{Jac}(C_0) \arrow[r] \arrow[d, equal] & \mathrm{Jac}(\tilde{C}_0) \arrow[r] \arrow[d, equal] & 0 \\
& & \dfrac{H^0(C_0, \omega_{C_0})^\vee}{\langle \beta_1, \alpha_2, \beta_2 \rangle} & \dfrac{H_0(\tilde{C}_0, \omega_{\tilde{C}_0})^\vee}{\langle \alpha_2, \beta_2 \rangle} &
\end{tikzcd}$$

If $S =$ disk, one can find

an open neighbourhood of 0 such that $\alpha_1, \beta_1, \alpha_2, \beta_2$ are sections of the local system

$$s \longmapsto H_1(C_s, \mathbb{Z})$$

Then the Raynaud extension is

$$s \longmapsto \frac{H^0(C_s, \omega_{C_s})^\vee}{\langle \beta_1(s), \alpha_2(s), \beta_2(s) \rangle}$$

In the degeneration setup $(\star)$,

we obtain the following over $\hat{S}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \tilde{G} & \longrightarrow & X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \lambda & & \uparrow \lambda_X & & \\
0 & \longrightarrow & T^\vee & \longrightarrow & \tilde{G}^\vee & \longrightarrow & X^\vee & \longrightarrow & 0
\end{array}$$

Why? λ_η extends to $\lambda_G : G \longrightarrow G^\vee$
and

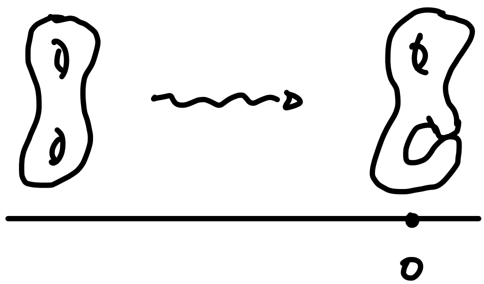
Which is classified by Hom's

$$\begin{tikzcd} \underline{M}_s \arrow[r, "c"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & X^\vee \arrow[d, "\lambda_X"] \\ \underline{N}_s \arrow[r, "c^\vee"] & X \end{tikzcd}$$

where ϕ is injective.

1.2. The period subgroup

Motivation: In



disk $\doteq S$

α_1 did not appear in the Raynaud extension

$$s \longmapsto \frac{H^0(C_s, \omega_{C_s})^\vee}{\langle \beta_1(s), \alpha_2(s), \beta_2(s) \rangle} = \tilde{G}_s$$

but over $S \setminus \{0\}$, we have

$$\mathbb{Z}\alpha_1(s) \subset \frac{H^0(C_s, \omega_s)^\vee}{\langle \beta_1(s), \alpha_2(s), \beta_2(s) \rangle}$$

this is called a <u>PERIOD</u>

Theorem (Chai-Faltings) I

In the degeneration setup $(\star)$, there is an homomorphism

$$i : N \longrightarrow \tilde{G}_\eta$$

making the diagram

$$\begin{tikzcd} & \tilde{G}_\eta \arrow[dr] & \\ N = N_\eta \arrow[ur, "i"] \arrow[rr, "c^\vee"] & & X_\eta \end{tikzcd}$$

commute

We want to unpack this

$$\tilde{G} = \underline{\mathrm{Spec}}_X \bigoplus_{m \in M} c(m)$$

So $i : N \longrightarrow \tilde{G}_\eta$ over $\underline{N} \xrightarrow{c^\vee} X$

is the same as an homomorphism

$$(c^\vee)^* \bigoplus_{m \in M} c(m)_\eta \longrightarrow \mathcal{O}_{N,\eta}$$

which is the same as sections

$$\tau(n, m) \in H^0(\eta, \underbrace{c^\vee(n)^* c(m)^{-1}_\eta}_{(c^\vee(n) \times c(m))^* P_X^{-1}})$$

satisfying

$$\tau(n_1 + n_2, m) = \tau(n_1, m) + \tau(n_2, m)$$

$$\tau(n, m_1 + m_2) = \tau(n, m_1) + \tau(n, m_2)$$

under canonical isomorphisms.

Theorem (Chai-Faltings) II

τ satisfies two extra conditions:

- $\tau(n_1, \phi(n_2)) = \tau(n_2, \phi(n_1))$
- $n \longmapsto \tau(n, \phi(n))$ extends to a section of $c^\vee(n) \times c(\phi(n)) P_X^{-1}$ for all n, and this section vanishes at $\widehat{S}_0$ if $n \neq 0$

1.3. The tropicalization

If we choose trivializations

$$\vartheta_\eta = K \xrightarrow{\sim} \left(c^\vee(n) \times c(m)\right)^* P_{X,\eta}^{-1}$$

then τ defines a function

$$\tau : N \times M \longrightarrow K$$

such that

- $\tau(n_1, \phi(n_2)) = \tau(n_2, \phi(n_1))$
- $\tau(n, \phi(n)) \in R \ \forall n$ and
  $\tau(n, \phi(n)) \in I \ \forall n \neq 0.$

If R is a DVR with valuation v, the <u>polarized tropical abelian variety</u> associated with this degeneration data is the data of (N, ϕ, Q):

- N is a lattice of rank $r \leq g$
- $\phi : N \longrightarrow N^\vee$ is injective
- $Q : N \times N^\vee \longrightarrow \mathbb{Z}$ is symmetric:

  $Q(n_1, \phi(n_2)) = Q(n_2, \phi(n_1))$

  and positive definite:

  $Q(n, \phi(n)) > 0$ if $n \neq 0$

Summary:

$G_\eta \longrightarrow \eta$ abelian scheme,

$\lambda_\eta : G_\eta \longrightarrow G_\eta^\vee$

semistable reduction

G/S semiabelian

$\lambda_G : G \longrightarrow G^\vee$

Raynaud ext.

$X/\hat{S}$ abelian scheme

$c, c^\vee : M, N \longrightarrow X^\vee, X$

$\phi : N \longrightarrow M$

periods

$\tau . \vartheta_{N\times M} \longrightarrow (c\times c^\vee)^* \mathcal{P}^{-1}_{X,\hat\eta}$

"symmetric", "positive" trivializ

<u>Claim:</u> $(X, \lambda_X, \phi, c, c^\vee, \tau)$

determine G/S

<u>will come back later</u>

2 Local picture of $\overline{\mathcal{A}_g}$

If we assume that G/S is principally polarized then ϕ, λ_X are isomorphisms, c determines $c^\vee$ so the only data is:

$(X, \lambda_X) \in \mathcal{A}_h$

$$[c : M \longrightarrow X^\vee] \in \overbrace{\mathcal{X}_h \times_{\mathcal{A}_h} \cdots \times_{\mathcal{A}_h} \mathcal{X}_h}^{r \text{ times}}$$

A section of $(c \times \lambda_X^{-1} \circ c)^* P_X$ that "degenerates".

a) Identify the univ. bundle

$$
\begin{tikzcd}
H^0_{sym}\left((c \times \lambda_X^{-1} c)^* \mathcal{P}_X^{-1} \right) \arrow[d] & \in & \mathcal{V} \arrow[d] \\
{[c : M \longrightarrow X^\vee, \lambda_X]} & \in & \mathcal{X}_h^\vee \times \dots \times \mathcal{X}_h^\vee
\end{tikzcd}
$$

$$M \cong \mathbb{Z} x_1 + \dots + \mathbb{Z} x_r \quad \star$$

$$H^0_{sym}\left((c \times \lambda_X^{-1} \circ c) \mathcal{P}_X^{-1} \right) =$$

$$= \bigoplus_{i \leq j} \mathcal{P}_X \big|_{c(x_i) \times \lambda_X^{-1} c(x_j)} \simeq$$

$$\simeq \bigoplus_{i \leq j} p_{ij}^* \mathcal{P}_X ,$$

$$\begin{tikzcd}
& \mathcal{X}_h^\vee \times \cdots \times \mathcal{X}_h^\vee \arrow[dl, "\lambda_X \circ \pi_i"'] \arrow[d, "p_{ij}"] \arrow[dr, "\pi_j"] & \\
\mathcal{X}_h & \mathcal{X}_h \times \mathcal{X}_h^\vee \arrow[l] \arrow[r] & \mathcal{X}_h^\vee
\end{tikzcd}$$

After identifying $\mathcal{X}_h \simeq \mathcal{X}_h^\vee$,

$\mathcal{V} = \mathbb{G}_m^{\binom{r+1}{2}}$- bundle given by

$$p_{ij}^* P_X \ (i<j), \quad p_i^* \Theta^{-2}$$

universal symmetric divisor, trivialized.

<u>b) Compactify the univ. bundle</u>

$\rightsquigarrow$

allow sections to degenerate

$\begin{bmatrix}\text{cocharacter group} \\ \text{of } \mathcal{D}\end{bmatrix} \equiv \mathrm{Sym}^2(M^\vee)$

$\Rightarrow$ We want

$$\sigma \subset \mathrm{Sym}^2(M^\vee)\otimes\mathbb{R} \equiv \left\{\begin{array}{l}\text{symmetric} \\ \text{bilinear forms} \\ B: M_{\mathbb{R}}\times M_{\mathbb{R}}\to\mathbb{R}\end{array}\right\}$$

but only forms that are positive-definite, so $\sigma \subset \Omega_r^{rc}$, $\mathring{\sigma} \subset \Omega_r^{rc}$

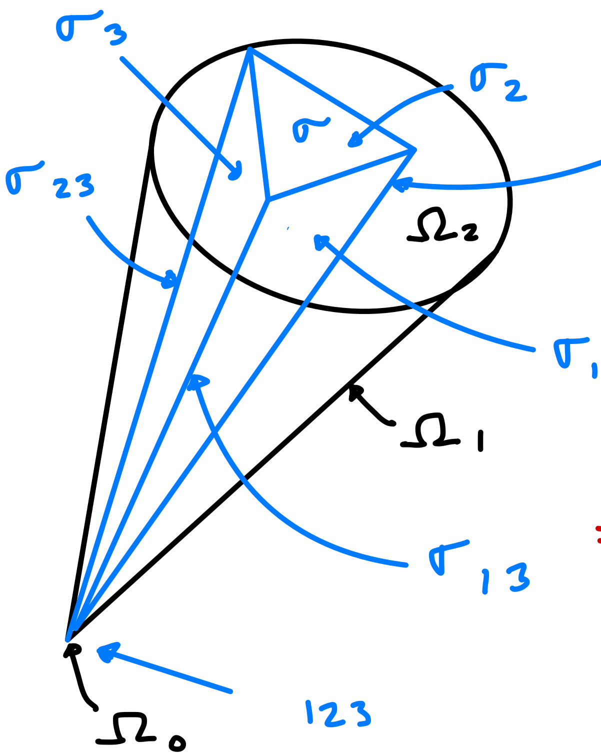
this is exacty a cone that appears in an admissible decomposition of $\mathcal{A}_g^{trop}$!!

$$\overline{\mathcal{D}}^\sigma = \text{compactification of } \mathcal{D} \text{ along } \sigma$$

$\hookrightarrow$ Allow sections whose degeneration is in σ.

$Z^\circ(\sigma)$ = locally closed stratum corresponding to σ

$$Z^{partial}(\sigma) = \bigsqcup_{\substack{\sigma' \lneq \sigma \\ int(\sigma') \subseteq \Omega_r}} Z^\circ(\sigma')$$



Here,

$$Z^{partial}(\sigma) = \bigsqcup_{\sigma' \in \left\{ \begin{matrix} \sigma, \sigma_1, \sigma_2, \sigma_3, \\ \sigma_{12}, \sigma_{13} \end{matrix} \right\}} Z^\circ(\sigma')$$

Choice of ☆ is not canonical

$\rightsquigarrow$

This identifies $\overline{\mathcal{V}}^{\sigma} \equiv \overline{\mathcal{V}}^{\sigma'}$ if σ, σ' are $GL_r(\mathbb{Z})$-translates.

Given compatible decompositions

$$\begin{array}{cccc} \Sigma_0 & \Sigma_1 & & \Sigma_g \\ \| & \| & & \| \\ \{\sigma_\alpha^0\} & \{\sigma_\alpha^1\} & \cdots & \{\sigma_\alpha^g\} \\ \cap & \cap & & \cap \\ \Omega_0^{rc} & \Omega_1^{rc} & & \Omega_g^{rc} \end{array}$$

such that $\mathrm{int}(\sigma_\alpha^r) \subseteq \Omega_r$

(like in Johannes' talk)

Choose finitely many maximal cones $\sigma_{\alpha_1}^{r_1}, \dots, \sigma_{\alpha_k}^{r_k}$ that represent the $GL_g(\mathbb{Z})$-orbits.

$$\overline{A_g}^{\Sigma} = \bigsqcup_{i=1}^{k} \left[Z^{\text{partial}}(\sigma_{\alpha_i}^{r_i}) \big/ \Gamma(\sigma_{\alpha_i}^{r_i}) \right]$$

stabilizer of the cone

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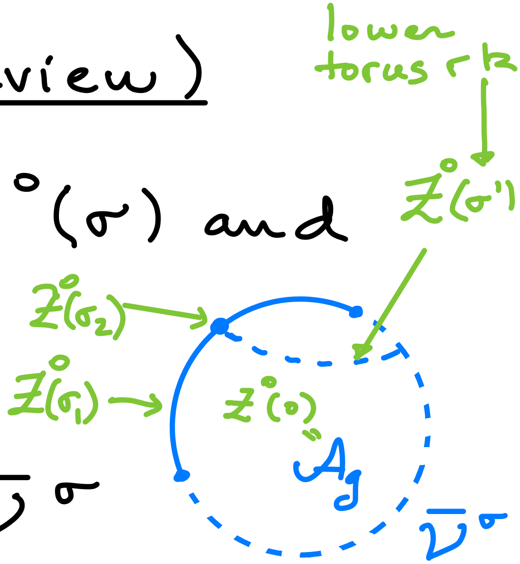
Johannes' autom. groups

3. Gluing (overview)

How to glue $\mathcal{Z}^\circ(\sigma)$ and $\mathcal{Z}^\circ(0) = \mathcal{A}_g$?



$$\mathcal{Z}^\circ(\sigma) \subseteq \overline{\mathcal{D}}^\sigma$$

has the same dim as $\mathcal{A}_g$!!

3.1. Mumford's construction

Let S_σ be the formal completion of $\overline{\mathcal{D}}^\sigma$ along $\mathcal{Z}^\circ(\sigma)$.

On S_σ we have the degeneration data. $(X, \lambda_X, c: M \longrightarrow X^\vee, \tau)$

Mumford constructs a semiabelian scheme

$$G_\sigma^\heartsuit / S_\sigma, \quad \lambda : G_\sigma^\heartsuit \longrightarrow G_\sigma^{\heartsuit\vee}$$

such that

- $G_\sigma^\heartsuit \Big|_{S_\sigma \setminus \widehat{Z^\circ(\sigma)}}$ is an ab. scheme
- $G_\sigma^\heartsuit \Big|_{\widehat{Z^\circ(\sigma)}} =$ extension of X given by c

Mumford's construction is <u>HARD</u> but is the origin for constructing compactifications of $G/\overline{A_g}$

Locally in the étale topology, $G_\sigma^\heartsuit$ <u>lifts</u> to a semiabelian scheme on $\overline{D_\sigma}$

3.2. How to glue

Pick étale neighbourhoods of all the $Z^{\text{partial}}(\sigma_{\alpha_i}^{r_i})$ such that the Mumford family lifts. Let $\mathcal{U}$ be their union.

Because the $G^{\heartsuit}_{\sigma,\eta}$ are abelian, one has a relation $R' \subset \mathcal{U} \times \mathcal{U}$.

$$R = \text{normalization}(\overline{R'})$$

$$\overline{A_g} = \mathcal{U} /\!/ R$$

4. Example

$$\begin{array}{c} \Omega \\ \cup \\ \bullet \ \Sigma^{\le 1} = \{ \{0\}, \ \mathbb{R}_{\ge 0} v^t v : v \in \mathbb{Z}^g \} \end{array}$$

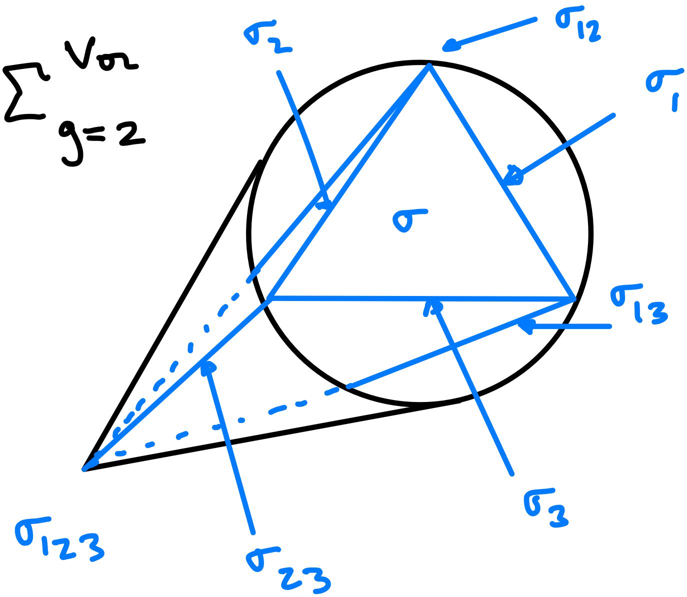
gives a partial compactification, which is canonical because the rays $\mathbb{R}_{\ge 0} v^t v$ are extremal

$$\begin{tikzcd}
G^{\le 1} \arrow[dd] & = & \mathcal{X}_g \arrow[dd] & \sqcup & P_X^{-2} \arrow[d, "\mathbb{G}_m\text{-bundle}"] \\
 & & & & \mathcal{X}_{g-1} \times [\mathcal{X}^\vee_{g-1}/\pm 1] \arrow[d, "(g-1)\text{-dim}"] \\
\overline{\mathcal{A}}_g^{\le 1} & = & \mathcal{A}_g & \sqcup & [\mathcal{X}^\vee_{g-1}/\pm 1]
\end{tikzcd}$$

normal bundle is Θ^{-2}

Automorphisms of the cone: $v \longmapsto -v$

$\bullet \; \sum_{g=2}^{\mathrm{Vor}}$



$$\sigma_1 \equiv \sigma_2 \equiv \sigma_3 \mod GL_2$$

$$\sigma_{12} \equiv \sigma_{23} \equiv \sigma_{13} \mod GL_2$$

$$\overline{\mathcal{A}_2} = \underset{Z^0(\sigma)}{\mathcal{A}_2} \sqcup \underbrace{\left[\mathcal{X}_1 / \pm 1\right]}_{Z^0(\sigma_{12})} \sqcup \underbrace{\left[\frac{\mathbb{C}^*}{\mathbb{Z}_2 \times \mathbb{Z}_2}\right] \sqcup \left[\frac{pt}{S_3 \times \mathbb{Z}_2}\right]}_{Z^{\mathrm{partial}}(\sigma)}$$

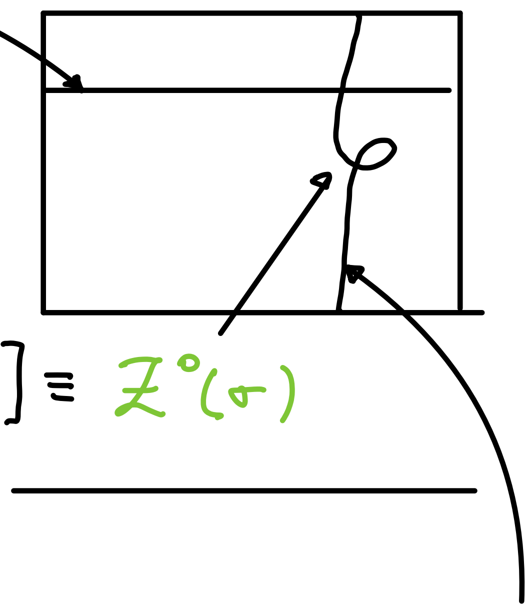
Compare with $\overline{\mathcal{M}_2} = \overline{\mathcal{A}_2}$

$$\overline{\mathcal{M}_2} = \underbrace{\mathcal{M}_2 \sqcup \delta_1}_{\mathcal{A}_2} \sqcup \overline{\delta_0}$$

δ_0 parametrized by

<u>zero section</u> $\equiv [\quad 1\,] \equiv \delta_1 \cap \delta_0$

not stratum



$$\begin{array}{ccc} \overline{\mathcal{M}_{1,2}} & = & \overline{\mathcal{X}_1} \\ \big\downarrow & & \big\downarrow \\ \overline{\mathcal{M}_{1,1}} & = & \overline{\mathcal{A}_1} \end{array}$$

$[\quad] \equiv Z^{\circ}(\sigma)$

$[\quad] \equiv$ nodal fiber $\equiv \delta_0^2 \equiv Z^{\text{partial}}(\sigma)$

Note: δ_1 and $\delta_0 \cap \delta_1$ are not strata of $\overline{\mathcal{A}_2}$!!