Tautological projection and the homomorphism property revisited

Prop: $\lambda_{g-r}\big|_{A_g^{\geq r+1}} = 0.$

Proof: Consider $\lambda_{g-r} \in CH^{*,op}_{\mathbb{Q}}(A_g^{\geq r+1})$.
Since an element of the operational Chow ring is defined by its evaluations on cycles, it suffices to show that if $V \xrightarrow{\varphi} A_g^{\geq r+1}$ is a map from a variety V, then $\varphi^*\lambda_{g-r} \cap [V] = 0$.
Let $\tilde{V} \xrightarrow{\eta} V$ be the normalization of V. Then

$$[V] = \eta_*[\tilde{V}] \Rightarrow \varphi^*\lambda_{g-r} \cap [V] = \varphi^*\lambda_{g-r} \cap \eta_*[\tilde{V}]$$
$$= \eta_*(\eta^*\varphi^*\lambda_{g-r} \cap [\tilde{V}]),$$

so we reduce to the case when V is normal. Let $G \xrightarrow{\pi} V$ be the semiabelian variety corresponding to φ. Then over the generic point of V, G has torus rank $\geq r+1$, so by Faltings-Chai, we have $G \cong G' \times T$ for T a torus of rank $\geq r+1$. From [SGA3] it follows that there exists $V' \xrightarrow{\psi} V$ finite étale s.t. $T|_{V'} \cong \mathbb{G}_m^{r+1} \times V'$. We conclude that $\psi^*\varphi^*\mathbb{E} \cong \mathcal{O}_{V'}^{r+1} \oplus \mathbb{E}_{g-r-1}$ and hence:

$$\varphi^*\lambda_{g-r} \cap [V] = \frac{1}{\deg\psi}\varphi^*\lambda_{g-r} \cap \psi_*[V']$$
$$= \frac{1}{\deg\psi}\psi^*\varphi^*\lambda_{g-r} \cap [V'] = 0. \quad \square$$

idea: The irreducible components of $M := \mathcal{H}om(T, \mathbb{G}_m)$ are finite and étale over V ⇝ take a finite number of these which generate M and consider V' the total space.

Repetition (tautological projection).

On $CH^*(\overline{A}_g)$ there is a morphism $taut: CH^*(\overline{A}_g) \to R^*(\overline{A}_g) \cong CH^*(LG_g)$ which sends α to the unique class $taut(\alpha) \in R^*(\overline{A}_g)$ s.t.:

$$\int_{\overline{A}_g} \alpha \cdot P(\lambda) = \int_{\overline{A}_g} taut(\alpha) P(\lambda)$$

for all $P(\lambda) \in R^*(\overline{A}_g)$.

polynomial in the λ_i's.

This is well-defined since $R^*(\overline{A}_g)$ is Gorenstein.

Extending these ideas to $CH^*(A_g)$ is possible but more tricky:

Def: If $\alpha \in CH_*(A_g)$, we write $\overline{\alpha} \in CH_*(\overline{A}_g)$ for <u>some</u> class with $\overline{\alpha}|_{A_g} = \alpha$.

Theorem (CMOP): $\lambda_g|_{\overline{A}_g \setminus A_g} = 0$.

The theorem implies that $\overline{\alpha} \cdot \lambda_g$ is well-defined, i.e., independent of the choice of $\overline{\alpha}$.

Def: The tautological projection $taut(\alpha) \in R^*(\overline{A}_g) \cdot \lambda_g = R^*(A_g)$ is defined by:

$taut(\alpha) := taut(\overline{\alpha} \cdot \lambda_g)$.

Geometric constructions:

Natural question: Is there a geometric construction of the tautological projection?

Answer: yes! In fact there are two closely related constructions.

Construction 1:

Consider
$$\begin{array}{c} \mathbb{E} \oplus \mathbb{E}^{\vee} =: \mathbb{H} \\ \downarrow \\ A_g \end{array}$$

There is a natural symplectic form ω on $\mathbb{H}$ induced by the dual pairing $\varphi: \mathbb{E} \oplus \mathbb{E}^{\vee} \to \mathcal{O}$. In matrix form:

$$\omega = \begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}, \text{ i.e., } \omega(x,y) = \begin{cases} 0 & \text{if } x,y \in \mathbb{E} \text{ or } x,y \in \mathbb{E}^{\vee} \\ y(x) & \text{if } x \in \mathbb{E} \text{ and } y \in \mathbb{E}^{\vee} \\ x(y) & \text{if } x \in \mathbb{E}^{\vee} \text{ and } y \in \mathbb{E}. \end{cases}$$

__Def__: Define $LG_{\mathbb{H}} \xrightarrow{\pi} \overline{A}_g$ to be the space of Lagrangian subbundles of $\mathbb{H}$, i.e., a section of $LG_{\mathbb{H}}$ is a subbundle $V \subseteq \mathbb{H}$ of rank g s.t. $V^{\perp} = V$.

The map π is an LG_g-bundle. We have:

__Thm__: $CH^*(LG_{\mathbb{H}}) \cong CH^*(\overline{A}_g) \otimes CH^*(LG_g)$ as a $CH^*(\overline{A}_g)$-algebra, where the algebra generators are $c_i(S)$, with $S \subseteq \pi^*(\mathbb{E} \oplus \mathbb{E}^{\vee})$ the universal subbundle.

Proof: This follows from $CH^*(LG_{\mathbb{H}}) \cong CH^*(\overline{A}_g)[c_1(S), \ldots, c_g(S)] \big/ (c(\pi^*\mathbb{H}) - c(S) \cdot c(S^{\vee}))$.

The subbundle $\mathbb{E} \subseteq \mathbb{H}$ defines a section $s: \overline{A}_g \to LG_g$.

Given a class $\alpha \in CH^*(\overline{A}_g)$, we get a class in $CH^*(LG_g)$ as follows:

$$(s_*[1] \cdot \pi^*\alpha) \in CH^*(\overline{A}_g) \otimes CH^*(LG_g) \cong CH^*(\overline{A}_g \times LG_g).$$

So we can define:

$$\beta := \pi_{2*}(s_*[1] \cdot \pi^*\alpha) \in CH^*(LG_g).$$

<u>Prop</u>: $\beta = \mathrm{taut}(\alpha) \in CH^*(LG_g)$

<u>Proof</u>: Write $x_i := c_i(S)$. We write M for the set of all monomials in $k[x_1, \ldots, x_g]$ which are linear in each x_i. For $m \in M$, we write $m^\vee$ for the dual element w.r.t. to pairing induced by the identification $\mathbb{Z} \cdot M \cong R^*(\overline{A}_g)$. Then:

$$s_*[1] = \sum_{m \in M} m(\lambda) \otimes m^\vee(X).$$ (it is the pullback of the diagonal under:

$$LG_{\mathbb{H}} \times_{\overline{A}_g} \overline{A}_g \xrightarrow{(id, s)} LG_{\mathbb{H}} \times_{\overline{A}_g} LG_{\mathbb{H}}).$$

Therefore:

$$\pi_{2*}(s_*[1] \cdot \pi^*(\alpha)) = \pi_{2*}\Big(\sum_m m(\lambda) \cdot \alpha \otimes m^\vee(X)\Big)$$

$$= \sum_m \Big(\int_{\overline{A}_g} m(\lambda)\alpha\Big) \cdot m^\vee(X) = \mathrm{taut}(\alpha).$$

$\square$

A posteriori, it is clear that this can be done in a simpler way:

Construction 2 (eliminating extra steps):

Consider:

$$\begin{tikzcd} & \overline{A}_g \times LG_g \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \overline{A}_g & & LG_g \end{tikzcd}.$$

Setting $X_i := c_i(\pi_2^* S)$, we define

$$\Delta := \sum_{m \in M} m(\lambda) \otimes m^\vee(X) \in CH^*(\overline{A}_g \times LG).$$

Then:

$$\mathrm{taut}(\alpha) = \pi_{2*}(\Delta \cdot \pi_1^*(\alpha)).$$

To define the tautological projection on A_g, one extends α to $\lambda_g \overline{\alpha}$ on $\overline{A}_g$ and uses the above constructions.

In positive characteristic, one can do slightly better (see later).

The homomorphism property revisited

Remember: $\alpha, \beta \in CH^*(A_g)$ satisfy the homomorphism property if $\mathrm{taut}(\alpha \cdot \beta) = \mathrm{taut}(\alpha)\,\mathrm{taut}(\beta)$.

I say that α satisfies the homomorphism property if (α, β) satisfies the homomorphism property for all β.

Expl: $\alpha \in R^*(A_g)$ satisfies the homomorphism property.

Lem: If $\bar{\alpha} \cdot \lambda_g \in R^*(\bar{A}_g)$, then α satisfies the homomorphism property.

Proof:
$$\int_{\bar{A}_g} \bar{\beta} \cdot \underbrace{\bar{\alpha} \cdot \lambda_g}_{= \mathrm{taut}(\alpha) \cdot \lambda_g} \cdot P(\lambda) = \int_{\bar{A}_g} \beta \cdot \mathrm{taut}(\alpha) \cdot \lambda_g \cdot P(\lambda)$$
$$= \int_{\bar{A}_g} \mathrm{taut}(\beta) \cdot \mathrm{taut}(\alpha) \cdot \lambda_g \cdot P(\lambda).$$

Cor: $[A_1 \times A_{g-1}]$ satisfies the homomorphism property.

Proof:
$$[\bar{A}_1 \times \bar{A}_{g-1}] \cdot \lambda_g = [* \times \bar{A}_{g-1} \cdot \lambda_{g-1}]$$
$$= [* \times \bar{A}_{g-1}] \cdot \lambda_{g-1}.$$

So it is enough to show that $[[\mathbb{G}_m] \times A_{g-1}]$ is tautological on torus rank 1.

Remember from Aitor's talk:

$$\begin{tikzcd} A_g^{\leq 1} & \mathcal{X}_{g-1} \arrow[l, "i"'] \arrow[d, shift right] \\ & A_{g-1} \arrow[u, bend right, "s"'] \end{tikzcd} \qquad \overset{=}{\delta}$$

$$N_{\delta | A_g^{\leq 1}}$$
$$\parallel$$
$$[[\mathbb{G}_m] \times A_{g-1}] \propto i_* s_* [1] = i_* \left(\frac{\Theta^{g-1}}{(g-1)!} \right).$$

$$= \frac{\delta^{g-1}}{(g-1)!} i_* [1]$$

$$\propto \delta^g \propto \lambda_g.$$

$$\Rightarrow [* \times \overline{A}_{g-1}] \cdot \lambda_{g-1} \propto \lambda_g \cdot \lambda_{g-1}. \qquad \square$$

Cor: The classes $NL_{g,d}$ for $d \geq 0$ satisfy the homomorphism property.

Proof: [IL24] $\Rightarrow$ there exist Hecke operators $T_n : CH^k(\overline{A}_g) \to CH^k(\overline{A}_g)$ satisfying:

1) $T_n(\alpha \cdot \beta) = \alpha \cdot T_n(\beta) \ \forall \alpha \in R^*(\overline{A}_g)$
2) $T_d([\overline{A}_1 \times \overline{A}_{g-1}]) = \overline{NL}_{g,d} + \gamma + \delta$ where $\gamma \in \langle \overline{NL}_{g,d'} | d' < d \rangle$, $\delta|_{A_g} = 0$.

Hence:

$$\overline{NL}_{g,d}\cdot\lambda_g = T_d(\lambda_g\cdot[\bar{A}_1\times\bar{A}_{g-1}]) + \underbrace{\gamma\lambda_g}_{\in R^*(\bar{A}_g) \text{ by induction}} + \cancel{\delta X_g}^{\,0}$$

$$= \lambda_g\cdot\lambda_{g-1}\underbrace{T_d([1])}_{vol(T_d)\cdot[1]} + \gamma\lambda_g \in R^*(\bar{A}_g). \quad \square$$

This explains most of the data we have so far.

By an extension of the above ideas, one can show the following more general result:

<u>Theorem</u>: The classes:

$$\left[\prod_{j=1}^{i_1-1}\lambda_j\cdot A_{i_1}\times\ldots\times\prod_{j=1}^{i_{k-1}-1}\lambda_j\cdot A_{i_{k-1}}\times A_{i_k}\right] =: \alpha$$

satisfy the homomorphism property for all i_j s.t. $i_1+\ldots+i_k=g$.

<u>Expl</u>: 1) $[\underbrace{A_1\times\ldots\times A_1}_{u \text{ times}}\times A_{g-u}]$

2) $[A_2\times A_{g-2}]\cdot\lambda_{g-2} = [\lambda_1 A_2\times\lambda_{g-3}A_{g-2}]$

<u>Proof</u>: We prove this by showing that $\bar{\alpha}\cdot\lambda_g\in R^*(\bar{A}_g)$. First consider the special case

$\bar{\alpha} = [\bar{A}_1\times\ldots\times\bar{A}_1\times\bar{A}_{g-u}]\cdot\lambda_g$

$= [[\mathbb{G}_m] \times \dots \times [\mathbb{G}_m] \times \lambda_{g-u} \bar{A}_{g-u}]$.

$\bar{\alpha} \cdot \lambda_g \hookrightarrow \bar{A}_g$ factors through $[[\mathbb{G}_m] \times \dots \times [\mathbb{G}_m] \times \bar{A}_{g-u+1}]$ and from the preceding proof, we know that $\bar{\alpha}$ lies in the class $[[\mathbb{G}_m] \times \dots \times [\mathbb{G}_m] \times \lambda_{g-u} \cdot \lambda_{g-u+1} \cdot \bar{A}_{g-u+1}]$. Hence by induction, $\bar{\alpha} \cdot \lambda_g \propto \lambda_g \cdot \lambda_{g-1} \cdot \dots \cdot \lambda_{g-u}$. In particular, $\lambda_1 \cdot \dots \cdot \lambda_g = [[\mathbb{G}_m] \times \dots \times [\mathbb{G}_m]]$ is pushed forward from the torus rank g locus.

We turn to the general case. Observe that

$$\bar{\alpha} \cdot \lambda_g = [* \times \dots \times * \times \lambda_{i_k} \bar{A}_{i_k}]$$

where $*$ denotes a tautological point class on $\bar{A}_{i_j}$. By the first part of the proof, each $*$ can be chosen to be of the form

$$[\underbrace{[\mathbb{G}_m] \times \dots \times [\mathbb{G}_m]}_{i_j \text{ times}}]$$

which reduces to the previous situation. $\square$

__Cor__: $\lambda_g \cdots \lambda_{g-r}$ vanishes on $A_g^{\leq r}$.

__Proof__: $\lambda_g \cdots \lambda_{g-r} = \lambda_g \cdot [\bar{A}_1 \times \dots \times \bar{A}_1 \times \bar{A}_{g-r}]$.

For many of the above results, we first guessed that they were true using arguments in positive characteristic. For example, the last result is obvious in positive characteristic, using that there is an explicit cycle $[V_r]$ (to be explained) representing λ_{g-r} on $\bar{A}_g$ which is supported on $A_g^{\leq r}$. I will use this as an excuse to introduce some concepts related to $\bar{A}_g$ in positive characteristic. Specifically, the Ekedahl-Oort (EO) stratification of A_g (extends to $\bar{A}_g$, similarly to $\mathbb{E}$).

In the following, fix a prime p.

<u>Def</u>: Consider the Frobenius morphism $f: A_g \to A_g$. We form the fiber product:

$$\begin{array}{ccc} \mathcal{X}_g^{(p)} & \xrightarrow{F} & \mathcal{X}_g \\ \downarrow & \square & \downarrow \\ A_g & \xrightarrow{f} & A_g \end{array}$$

$H^1(X) \to H^1(X^{(p)})$

$F^*\Omega^1_{\mathcal{X}_g/A_g} \xrightarrow{0} \Omega^1_{\mathcal{X}_g^{(p)}/A_g}$

The dual $V: \mathcal{X}_g \to \mathcal{X}_g^{(p)}$ of F is called the Verschiebung. F and V induce morphisms in cohomology:

$$V: H^{1(p)}_{dR} \to H^1_{dR}, \quad F: H^1_{dR} \to H^{1(p)}_{dR}$$

Since F is totally inseparable, $F|_{\mathbb{E}} = 0$. Also, FV and VF are the multiplication by p maps, so in cohomology $VF = FV = 0$. The final important property is that V and F are adjoints w.r.t. the symplectic forms on H^1_{dR} and $H^{1(p)}_{dR}$:

$$\langle V\cdot,\cdot\rangle = \langle F\cdot,\cdot\rangle.$$

$\langle\cdot,\cdot\rangle : \mathcal{H}^1_{dR} \times \mathcal{H}^1_{dR} \to \mathcal{O}_{A_g}$ given by identifying $\mathcal{H}^1_{dR,\mathcal{X}_g} \xrightarrow{\sim} \mathcal{H}^1_{dR,\mathcal{X}_g^\vee} \cong \mathcal{H}^{1\,\vee}_{dR,\mathcal{X}_g}$ (polarization)

The last isomorphism is given by sending $\varphi \in \mathcal{H}^{1\,\vee}_{dR,\mathcal{X}_g}$ to $\varphi\otimes id(\xi)$ where $\varphi\otimes id : \mathcal{H}^1_{dR,\mathcal{X}_g} \otimes \mathcal{H}^1_{dR,\mathcal{X}_g^\vee} \to \mathcal{H}^1_{dR,\mathcal{X}_g^\vee}$ and ξ is the $(1,1)$-component of $c_1(\mathcal{L}) \in \mathcal{H}^2_{dR,\mathcal{X}_g\times\mathcal{X}_g^\vee}$ in $\mathcal{H}^1_{dR,\mathcal{X}_g} \otimes \mathcal{H}^1_{dR,\mathcal{X}_g^\vee}$, where $\mathcal{L}$ is the polarization.

From this, one can conclude:

1) $\operatorname{rank}(F) = \operatorname{rank}(V) = g$
2) $\ker(F) = \operatorname{im}(V) = \mathbb{E}$.

Idea: Study degeneracy loci of F and V.

<u>Def</u>: An abelian variety A is said to have p-rank f, if $\dim(V^g\mathbb{E}_g^{(p)}) = f$. An abelian variety is said to have a-number a, if $\operatorname{rank}(V)|_{\mathbb{E}_g} = g-a$.

We write V_f for the locus of abelian varieties of p-rank $\le f$ and T_a for the locus of abelian varieties of a-number $\le a$.

<u>Rem</u>: • Alternatively, $f(A) = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mu_p, A))$ and $a(A) = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\alpha_p, A))$.

• It is known that the loci T_a and V_f have expected codimension.

<u>Cor</u>: The cycle classes $[T_a]$ and $[V_f]$ are tautological and can be computed explicitly (see [vdG99], based on work of Pragacz and Ratajski).

For example:

$$[V_f] = (p-1)(p^2-1)\cdots(p^{g-f}-1)\lambda_{g-f}.$$

Fact: $\overline{V_f} \subseteq A_g^{\leq f}$.

The Cartier dual of μ_p is $\mathbb{Z}/p$

<u>Proof</u>: We have $p^{f(A)} = |A[p]^\vee(\overline{k})|$ which is lower semicontinuous, since $\mathcal{X}_g[p]^\vee$ is finite and flat. So, on $\overline{V_f}$, we have $|\mathcal{X}_g|_{\overline{V_f}}[p]^\vee(\overline{k})| \leq p^f$. Since for a torus T of rank r it holds that $|T[p]^\vee(\overline{k})| = p^r$, we conclude $r \leq f$. $\square$

$T[p] = \mu_p^r$
$\rightsquigarrow (\mu_p^r)^\vee = (\mathbb{Z}/p)^r$

<u>Cor</u>: $\lambda_{g-r}|_{A_g^{\geq r+1}} = 0$.

<u>Proof</u>: $\lambda_{g-r} \propto [\overline{V_r}]$ which is supported on $A_g^{\leq r}$.

The above also provides a method for defining tautological projection in a cleaner way in positive characteristic (i.e., avoiding the $\lambda_g \cdot \tilde{\alpha}$ construction):

Consider:

$$\begin{tikzcd} & V_0 \arrow[dl, "i"'] \arrow[dr, hook, "j"] & \\ A_g & & \overline{A}_g \end{tikzcd}$$

Then for every $\alpha \in CH^*(A_g)$, we have

$$\mathrm{taut}(\alpha) \cdot [V_0] = \mathrm{taut}(j_* i^* \alpha).$$

[vdG] and [E vdG] exploit the above ideas further. They construct a stratification of A_g by 2^g strata defined by studying the behaviour of V relative to a filtration of $\mathbb{E}$

<u>Def</u>: Let $U \to A_g$ be the locus in $Fl_{\mathbb{E}}$ consisting of the flags $\{\mathbb{E}_i\}_{i=1,\dots,g}$ which are V-stable,

i.e.: $\qquad V(\mathbb{E}_i) \subseteq \mathbb{E}_i^{(p)} \quad \forall i.$

Here, $\mathbb{E}_i^{(p)} = f^*(\mathbb{E}_i) \subseteq f^*(\mathcal{H}^1_{dR}) = \mathcal{H}^{1\,(p)}_{dR}$

Formation of relative de Rham commutes with base change.

(one can see U as the moduli space of abelian varieties A and a V-stable symplectic filtration of $A[p]$)

<u>Observation</u>: [vdG99]: $U \xrightarrow{\pi} A_g$ is finite.

On U, we now have many more options for defining interesting loci on A_g:

<u>Def</u>: Write $\nu_A(i) := \dim(V(\mathbb{E}_i|_{[A]}))$

<u>Important facts</u>: 1) $\nu_A(i) \leq g \quad \forall i$

2) $\nu_A(i) \leq \nu_A(i+1) \leq \nu_A(i)+1.$

(1), (2) $\Rightarrow$ there are 2^g possible ν's. These are called <u>final types</u>.

<u>Def</u>: Fix a final type ν. We let $Z_\nu \subseteq A_g$ be the locus of abelian varieties of final type ν.

Thm [EvdG]: 1) Z_ν is non-empty and locally closed in A_g.

2) There is an explicit combinatorial method for computing $\operatorname{codim}_{A_g}(Z_\nu)$.

3) The Z_ν define a stratification of A_g.

4) The classes $[\overline{Z}_\nu] \in CH^*(A_g)$ are tautological (Fulton).