[KKN] (Kajiwara, Kato, Nakayama)

Degenerating elliptic curves:

Consider (R, m), R m-adically complete, $m = (q)$. The Tate curve E_q is a formal neighbourhood of the nodal curve in $\mathcal{M}_{1,1}$, analytically defined as $\mathbb{G}_m / q^{\mathbb{Z}}$. Algebraically, it is cut out by:

$$y^2 + xy = x^3 + a_4 x + a_6$$

$$-a_4 = 5 \sum_n \frac{n^3 q^n}{1-q^n}$$

$$-a_6 = \sum_n \frac{7n^5 + 5n^3}{12} \cdot \frac{q^n}{1-q^n}$$

(use a branch of the logarithm to transform to $\mathbb{C} / \langle 1, \tau \rangle$ and then consider the Weierstrass $\wp$-function).

<u>Idea</u>: Want to remember an infinitesimal smoothing of the curve $\rightsquigarrow$ log geometry.

Equip R with the log structure generated by q.

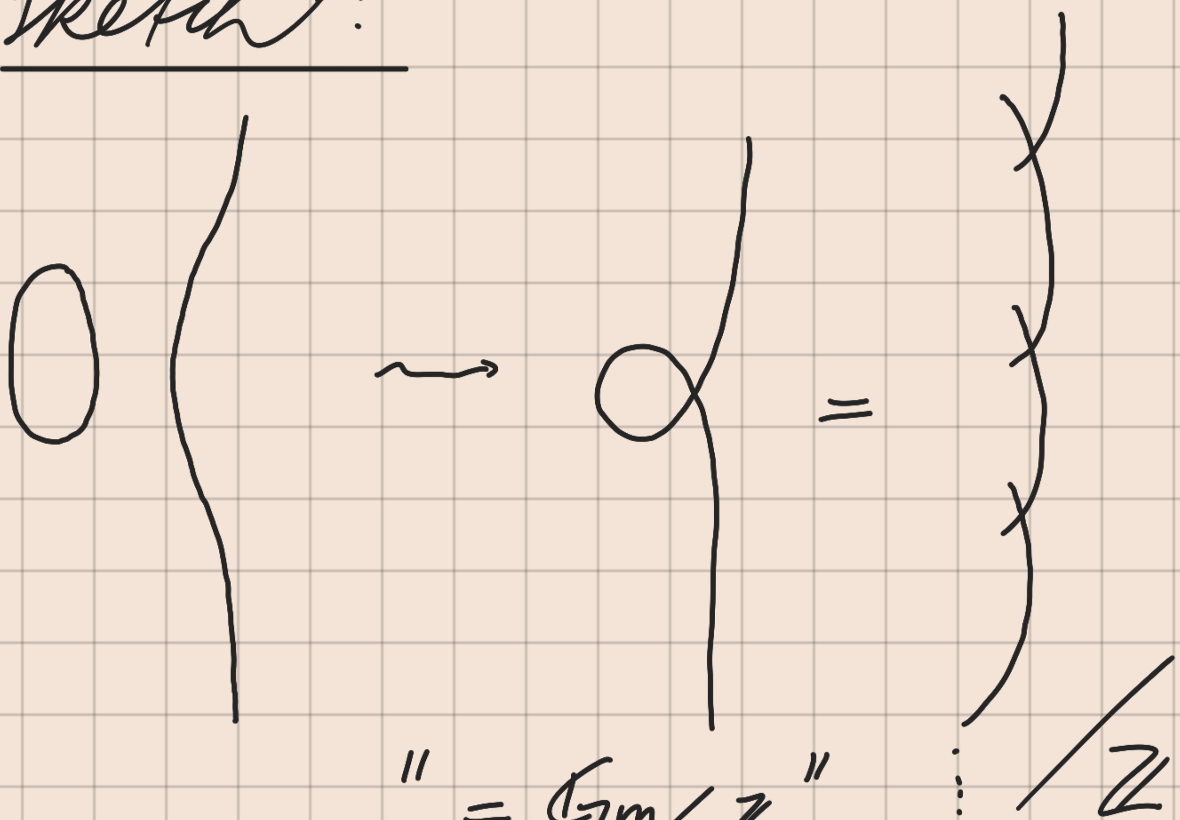
$\mathbb{G}_{m,\log}(U) := M_U^{gp}(U)$

We will define a proper group object $\mathcal{E}_q$ s.t. $\mathcal{E}_q|_\eta = E_q$

("$\mathcal{E}_q = \mathbb{G}_{m,\log}/q^{\mathbb{Z}}$").

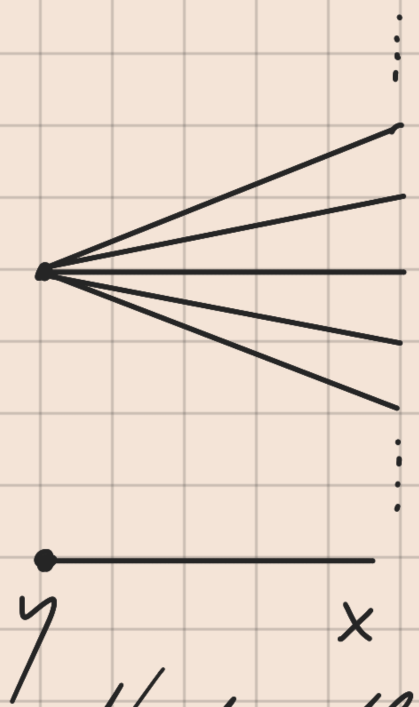
The construction is essentially the construction of a Raynaud extension + log geometry.

<u>Sketch</u>:



"$= \mathbb{G}_m/q^{\mathbb{Z}}$"

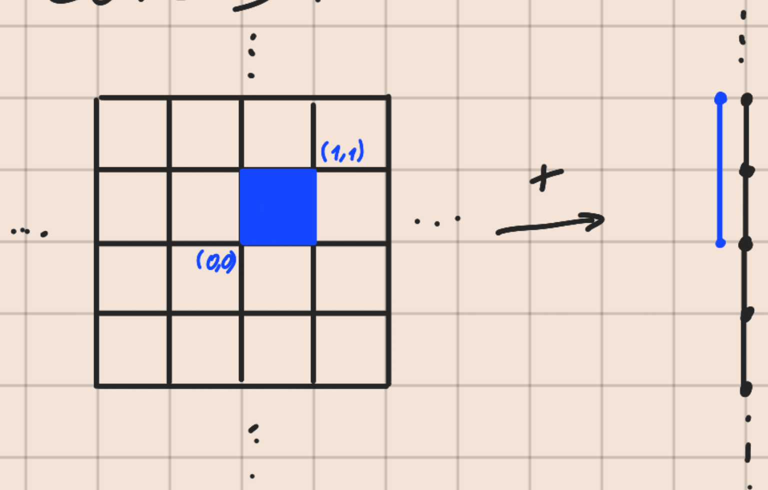
Tropically:



this face looks like:

Δ:

Note that the addition map $\Delta \times \Delta \to \Delta$ doesn't map cones into cones:



Solution: replace $\mathbb{Z}$ with $2\mathbb{Z}$ and repeat.

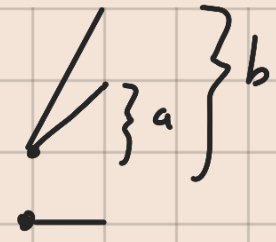
Def: $V_{a,b,n}(U) := \{ \varphi \in M_U^{gp}(U) \mid q^a \mid \varphi \mid q^b \}$

on $\operatorname{Spec}(R/m^n)$ $\quad (x \mid y \Leftrightarrow x^{-1} y \in M_U(U))$

$(a,b \in \mathbb{Z},\ a<b,\ n \in \mathbb{N})$

$$V_{a,b,n} \simeq \operatorname{Spec}\left(R/_{m^n}\right) \times_{\mathbb{A}^1} X_\sigma$$

σ:



σ:

$by - x \geq 0$

$x - ay \geq 0$.

$V_{a,b,n} \cap V_{b,c,n}$ is a dense open subset (look at the cones). So we can glue:

$$\mathbb{G}_{m,\log,n}^{(q,\mathbb{Z})} := \bigcup_{k \in \mathbb{Z}} V_{k,k+1,n}.$$

We have $q V_{k,k+1} = V_{k+1,k+2}$, so we can write $\mathbb{G}_{m,\log,n}^{(q,\mathbb{Z})} / q^{\mathbb{Z}}$

$\mathcal{E}_q^{(\mathbb{Z})}$:= Algebrization of the formal scheme $\varinjlim_n \mathbb{G}_{m,\log,n}^{(q,\mathbb{Z})} / q^{\mathbb{Z}}$.

Observe now that:

$\varphi \in V_{a,b}(U)$, $\varphi' \in V_{a',b'}(U)$

$\Rightarrow \varphi \cdot \varphi' \in V_{a+a',b+b'}(U)$.

So we have well-defined multiplication:

$$V_{k,k+\ell} \times V_{k',k'+\ell'} \to V_{k+k',k+k'+(\ell+\ell')}.$$

This induces a multiplication:

$$\mathcal{E}_q^{(\ell\mathbb{Z})} \times \mathcal{E}_q^{(\ell'\mathbb{Z})} \to \mathcal{E}_q^{((\ell+\ell')\mathbb{Z})},$$

motivating the following:

<u>Def</u>: $\mathcal{E}_q := \varinjlim_{\ell} \mathcal{E}_q^{(\ell\mathbb{Z})}$

<u>Theorem</u>: $\mathcal{E}_q$ is a flat proper group object in the category of log stacks, s.t. $\mathcal{E}_q|_\eta = E_q$. Also, for any n, we have $\mathcal{E}_q|_{\mathrm{Spec}(R/m^n)} \cong \mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$.

<u>Def</u>: Let $q \in M_B^{gp}(B)$. We set:

$$\mathbb{G}_{m,\log}^{(q)}(U) := \{ f \in M_U^{gp}(U) \mid \exists i,j \in \mathbb{Z} : q^i | f | q^j \text{ locally on } U \}$$

$(x|y \Leftrightarrow x^{-1}y \in M_U(U))$.

$\mathbb{G}_{m,\log}^{(q)}$:

There is a log étale monomorphism $G \to E_q$ with G a semiabelian variety with special fiber $\mathbb{G}_m$, fitting into a SES:

$$0 \to G \to E_q \to \mathbb{G}_{m,\log}^{(q)} / q^{\mathbb{Z}} \cdot \mathbb{G}_m \to 0$$

(G is the algebraization of $\mathbb{G}_{m,n} / q^{\mathbb{Z}}$).

Degenerating abelian varieties, take II

Fundamental problem:

$$\begin{array}{ccc} A & \subseteq & X \\ \downarrow & & \downarrow \\ \eta & \subseteq & \mathrm{Spec}(R) \end{array}$$

Ab. var

X proper + group scheme $\not\Rightarrow$ ab. var. ↯ (e.g. $\mathcal{M}_{1,1}$ is not proper)

not true for Stacks/LogSch

Idea: package all the "degeneration data" from Aitor's talk as a group object in Stacks/LogSch.

Repetition of degeneration data

Setup:

- $S = \mathrm{Spec}\, R$, R complete w.r.t. I int. dom.
- η generic pt. of S
- $G \to S$ semiab. variety
- $S_0 = \mathrm{Spec}(R/I)$ closed subset

2 constraints: 1) G_η is ab. var.

2) $G_0 := G \times_S S_0$ has constant torus rank r

(1) $\Rightarrow$ can (and will) assume $\exists$ polarization $\lambda : G_\eta \to G_\eta^\vee$

(2) $\Rightarrow$ G is split over $\hat{S}_0$:

$$0 \to \hat{T} \to \hat{G} \to \hat{X} \to 0.$$

torus of rk. r; ab. var.

Raynaud extension

$\Rightarrow \exists \; 0 \to T \to \tilde{G} \to X \to 0,$ ($= G$ modulo periods)

(can extend this splitting over S)

$G \mapsto (\tilde{G}, T, X)$ is a functor and

$G^\vee \mapsto (\tilde{G}^\vee, T^\vee, X^\vee)$.

So get:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, dotted, "\lambda_T"'] \arrow[drr, red, "=0"] & \tilde{G} \arrow[r] \arrow[d, "\lambda"] & X \arrow[r] \arrow[d, dotted, "\lambda_X"] & 0 \\
0 \arrow[r] & T^\vee \arrow[r] & \tilde{G}^\vee \arrow[r] & X^\vee \arrow[r] & 0
\end{tikzcd}$$

$\lambda_T \leftrightarrow \varphi \in \mathrm{Hom}(N, M)$, where

$N := \mathcal{H}om(\mathbb{G}_m, T)$,

$M := N^\vee$

$\tilde{G} \xrightarrow{\lambda} \tilde{G}^\vee$ is dominant

$\Rightarrow \varphi$ inj. $\Leftrightarrow$ im φ has finite index

$\tilde{G}$ is a T-torsor over X

$$\mathrm{Ext}^1(X,T) = \mathrm{Ext}^1(X, \mathcal{H}om(M,\mathbb{G}_m)) \overset{M \text{ loc. free}}{=} \mathcal{H}om(M, \underbrace{\mathcal{E}xt^1(X,\mathbb{G}_m)}_{=X^\vee})$$

$\leftrightarrow c \in \mathcal{H}om(M, X^\vee)$

($\tilde{G} = \underline{\mathrm{Spec}}_X(\bigoplus_v \mathcal{L}_{c(v)})$ locally)

λ gives:

$$\begin{tikzcd} N \arrow[r, "c"] \arrow[d, "\varphi"'] & X \arrow[d, "\lambda_X"] \\ M \arrow[r, "c^\vee"] & X^\vee \end{tikzcd}$$

- λ_X is a polarization
- φ is injective with f. cokernel

Conclusion: $\tilde{G}$ is uniquely determined by $(X, N, \varphi, c, c^\vee, \lambda_X)$

What is the missing data to get from G to G?

Answer (Chai–Faltings):

N records the "loops in G whose diameter goes to ∞" but doesn't remember the relative speeds.

Theorem (Chai–Faltings):

There exists a lift:

$$\begin{tikzcd}
N \arrow[r, "u"] \arrow[dr, "c"'] & \tilde{G} \arrow[d] \\
 & X
\end{tikzcd}$$

of c such that "$\tilde{G}/N \cong G$".

Conclusion: The full set of degeneration data uniquely characterizing G is: $(X, \underset{(\cong \tau)}{u}, c^\vee, \varphi, \lambda)$

The goal is now to do the same procedure as for E_q in higher dimensions. In this case, the "u" discussed at the beginning of the talk becomes essential:

<u>Def</u>: A log 1-motif (over S) is:

1) An extension G of an abelian scheme X by a torus T.
2) an étale sheaf of abelian groups N which étale locally on S is a constant sheaf of finitely generated free abelian groups.
3) A homomorphism $u: N \to G_{log}$.

Here G_{log} is the pushout:

$$\begin{array}{ccccccc}
0 \to T & \longrightarrow & G & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to T_{log} & \longrightarrow & G_{log} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

$T_{\log} := \mathcal{H}om(M, \mathbb{G}_{m,\log})$

Where $M := \mathcal{H}om(T, \mathbb{G}_m)$.

In the degeneration data:

$$(X, u, c^\vee, \varphi, \lambda)$$

log 1-motives give us (X, u).

From this we can reconstruct "τ" as:

$$\langle \cdot , \cdot \rangle : M \times N \to \mathbb{G}_{m,\log} / \mathbb{G}_m .$$

$$M \times N \xrightarrow{(id, u)} M \times G_{\log} \to M \times G_{\log} / G$$

$$\cong M \times T_{\log} / T \cong M \times \mathcal{H}om(M, \mathbb{G}_{m,\log} / \mathbb{G}_m)$$

$$\to \mathbb{G}_{m,\log} / \mathbb{G}_m =: \mathbb{G}_{m,\mathrm{trop}}$$

Also, we can reconstruct "$\tilde{G}$" as follows:

First, we set:

$$\mathcal{H}om(M, \mathbb{G}_{m,trop})^{(N)}(U) :=$$

$$\left\{ \varphi \in \mathcal{H}om(M, \mathbb{G}_{m,trop})(U) \;\middle|\; \begin{array}{l} \forall \bar{u} \to U,\ \exists\, n_1, n_2 \in N_{\bar{u}} \text{ s.t.} \\ \langle m, n_1 \rangle \mid \varphi_{\bar{u}}(m) \mid \langle m, n_2 \rangle \\ \forall m \in M \end{array} \right\}$$

$$\left(a \mid b \Leftrightarrow a^{-1}b \in \overline{M}_{U,\bar{u}} \subseteq \overline{M}^{gp}_{U,\bar{u}} \right)$$

This is a subgroup sheaf and $N \to \mathcal{H}om(M, \mathbb{G}_{m,trop})$ factors through it.

$n \longmapsto \langle \cdot, n \rangle$

In the E_q example, this corresponds to $\mathbb{G}^{(q)}_{m,log} / \mathbb{G}_m$.

There is a natural map

$$G_{log} \to G_{log}/G \cong \mathcal{H}om(M, \mathbb{G}_{m,trop})$$

so we can set $G_{log}^{(N)}$ to be the preimage of $\mathcal{H}om(M, \mathbb{G}_{m,trop})^{(N)}$ under this morphism.

<u>Def</u>: A log abelian variety with constant degeneration is a sheaf of abelian groups isomorphic to $G_{log}^{(N)}/N$ for some choice of pointwise polarizable log 1-motif $[u: N \to G_{log}]$.

<u>Theorem</u>: A log abelian variety is a log abelian variety of constant degeneration iff the rank of the torus part of G is constant.

Expl: E_q is not a log abelian variety with constant degeneration due to condition (4). It does become one on $\mathrm{Spec}(R/(q^n))$ though.

Theorem: $[u: N \to G_{log}] \mapsto G_{log}^{(N)}/N$ is an equivalence of categories between the category of pointwise polarizable log 1-motifs and the category of log abelian varieties of constant degeneration.

The missing items in the degeneration data require the notion of a dual log 1-motif

Def: Let $[u : N \to G_{log}] =: L$ be a log 1-motif.
We define $G^\vee_{log}$ to be the sheaf of abelian varieties on $(\mathcal{L}Sch/S)_{ét}$ associated to the presheaf:

$$U \longmapsto \left\{ (F, h) \;\middle|\; \begin{array}{l} F \in Ext^1(X, \mathbb{G}_m), \\ h : N \to F_{log} \text{ s.t.} \\ \begin{tikzcd} N \arrow[r] \arrow[d] & G_{log} \arrow[d] \\ F_{log} \arrow[r] & X \end{tikzcd} \\ \text{commutes.} \end{array} \right\}$$

This fits into an exact sequence:

$$0 \to T^\vee_{log} \to G^\vee_{log} \to \underset{\substack{\| \\ Ext^1(X, \mathbb{G}_m)}}{X^\vee} \to 0$$

where :

1) $G^\vee_{log} \to X^\vee, (F,h) \mapsto F$

2) $T^\vee_{log} \to G^\vee$

$$\begin{tikzcd} & N \arrow[d, "(\varphi, \mathrm{proj})"] \\ \varphi \in \mathcal{H}om(N, \mathbb{G}_{m,log}) \mapsto & 0 \to \mathbb{G}_{m,log} \to \mathbb{G}_{m,log} \oplus X \to X \end{tikzcd}$$

$$\mathbb{G}_{m,log} \oplus X = (\mathbb{G}_m \oplus X)_{log}$$

Note that $m \in M \leftrightarrow T \to \mathbb{G}_m$, so we can define $M \to G^\vee_{log}$:

$m \mapsto F$

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & X \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{G}_m \arrow[r] & F \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

$(\mathrm{Ext}^1(M, X) \xrightarrow{m} \mathrm{Ext}^1(\mathbb{G}_m, X))$.

The map $h : N \to F_{log}$ is obtained as $N \to G_{log} \to F_{log}$.

We thus obtain a dual log 1-motif:

$$[u^*: M \to G^\vee_{log}] =: L^\vee$$

<u>Def</u>: A polarization of a log 1-motif is a morphism $(\varphi, \lambda): L \to L^\vee$ of log motifs such that:

1) λ induces a polarization on X,
2) φ is injective with finite cokernel,
3) The morphism $T_{log} \to T^\vee_{log}$ obtained by restricting λ, coincides with the one induced by φ.
4) $\langle \varphi(y), y \rangle \in \overline{M}_S \setminus \{0\} \subseteq \overline{M}^{gp}_S$ $\forall y \in N \setminus \{0\}$ locally on S.
"$\langle \cdot, \cdot \rangle$ is positive definite"

We say that a log 1-motif is pointwise polarizable if there exists a polarization at each $\bar{s} \to S$.

From this we get the full degeneration data, so we can reconstruct a log abelian variety from it:

<u>Def</u>: A log abelian variety is a sheaf of abelian groups A on $(LSch/S)_{ét}$, such that étale locally on S:

1) There exist locally constant sheaves M, N of finitely generated free abelian groups and a pairing $\langle \cdot, \cdot \rangle : M \times N \to \overline{\mathbb{G}}_{m,log}$

2) A fits into an SES:

$$0 \to G \to A \to \underline{Hom}(M, \mathbb{G}_{m,trop})^{(N)} / \overline{N} \to 0$$

($\overline{N}$ is the image of N in $\underline{Hom}(M, \mathbb{G}_{m,\log})$)

3) For every $\bar{s} \in S$, there is a homomorphism $\varphi : \overline{N}_{\bar{s}} \to \overline{M}_{\bar{s}}$ such that:

i) φ is inj. of f. cokernel

ii) $\langle \varphi(n), n' \rangle = \langle \varphi(n'), n \rangle \ \forall n, n' \in \overline{N}_{\bar{s}}$

iii) $\langle \varphi(n), n \rangle \in \overline{M}_{S,\bar{s}} \quad \forall n \in \overline{N}_{\bar{s}}$

4) The diagonal is represented by finite morphisms.

<u>Rem</u>: In (3), we have to descend $\langle \cdot, \cdot \rangle$ to $\overline{M} \times \overline{N}$. This is well-defined, since if $\langle m, \cdot \rangle = \langle m', \cdot \rangle$, then $\langle m, n \rangle = \langle m', n \rangle \ \forall n \in \overline{N}$.

<u>Expl</u>: Log abelian varieties of constant degeneration are log abelian varieties (see Aitor's talk for the properties of φ).

Expl : $\mathcal{E}_q$ is a log abelian variety :

- $N = q^{\mathbb{Z}}$, $M = \mathcal{H}om(N, \mathbb{Z})$
- $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{G}_{m,\mathrm{trop}}$
  $(m, n) \mapsto q^{m(n)}$.

$\boxed{\begin{array}{l}\mathcal{H}om(\mathcal{H}om(N,\mathbb{Z}), \overline{\mathbb{G}}_{m,\log}) \\ \parallel \\ \overline{\mathbb{G}}_{m,\log} \otimes N\end{array}}$

$$0 \to G \to \mathcal{E}_q \to \mathbb{G}_{m,\log}^{(q)} / \mathbb{G}_m \cdot q^{\mathbb{Z}} \to 0$$

$$\mathbb{G}_{m,\log}^{(q)} = \mathcal{H}om(q^{\mathbb{Z}}, \mathbb{G}_{m,\mathrm{trop}})^{(q)} / (q^{\mathbb{Z}})^\vee$$

- At the generic point η :
  - $G = E_q = G_{\log} = \mathcal{E}_q$

    $N \to G_{\log}$ is the 0-map.

- At the closed point :
  - $G = \mathbb{G}_m$, $G_{\log}^{(q)} = \mathbb{G}_{m,\log}^{(q)}$
  - $\mathcal{E}_q = \mathbb{G}_{m,\log}^{(q)} / q^{\mathbb{Z}}$.
  - $N \to G_{\log}$, $q \mapsto q$.

Note that $\overline{N} \cong \overline{M} \cong i_* \mathbb{Z}$, $i : \{x\} \to S$, the closed point

Application: Néron models of Jacobians of curves [Holmes-Molcho-Orecchia-Poiret]

Setup: $C \to S$ log smooth family of curves over a log smooth base S. We write S° for the locus where $C \to S$ is smooth; $C^\circ := C \times_S S^\circ$.

Def: $X \to S$ is a Néron model for $\mathrm{Pic}^0(C^\circ) \to S^\circ$ if it is smooth and:

1) $X \times_S S^\circ = \mathrm{Pic}^0(C^\circ)$

2) For any smooth $T \to S$, the restriction:

$$X(T) \to \mathrm{Pic}^0(T \times_S S^\circ) \text{ is a bijection.}$$

(i.e., every line bundle on S° extends uniquely to an element of X).

The existence of a Néron model as a separated q.c. algebraic space is tricky and does not always hold. [Holmes]

On the other hand, in the category of log abelian varieties, a Néron model always exists and has a modular interpretation:

<u>Thm [HMOP]</u>:

$\mathrm{LogPic}^{0}_{C/S}$ satisfies the Néron mapping property.

<u>Def</u>: A log line bundle on $C \to S$ is a $\mathbb{G}_{m,\log}$-torsor L on C s.t. for any strict $\bar{s} \to S$, $L_{\bar{s}}$ has

bounded monodromy.

Bounded monodromy means:
Let $C \to (\operatorname{Spec} \mathbb{C}, M)$ be a log smooth curve and $\mathcal{C}$ the associated tropical graph. The nodes of C are of the form $xy - q$ for $q \in M \setminus \mathbb{C}^\times$. We thus get a map $\ell : H \to \overline{M}$, sending e to q.
The torsor $L_\xi \rightsquigarrow \xi \in H^1(C, \overline{M}_C^{gp})$ $= \operatorname{Hom}(H_1(\mathcal{C}), \overline{M}^{gp})$ is said to be of bounded monodromy if

$$(\gamma \cdot \gamma)^{-n} \mid \xi(\gamma) \mid (\gamma \cdot \gamma)^n \text{ for some } n, \forall \gamma.$$

(here, $e \cdot f = \delta_{ef} \cdot \ell(e)$).

__Def__:
__LogPic__$_{C/S}(T) := \{$Log line bundles on $C \times_S T\}$.

Using the language we developped, we can rephrase this as follows: From Leyka's talk, recall that the matrix Q corresponding to the image of C in $\bar{A}_g$ is given by:

$$Q = \sum_{e \in \Gamma(\mathcal{C})} \ell(e)\, e^* \otimes e^*$$

Therefore, we may write:

$$(\gamma \cdot \gamma)^{-n} = \langle \varphi(\gamma), \gamma \rangle$$

where we identify $N \cong H_1(\Gamma)$, $\varphi : N \to M,\ \gamma \mapsto Q(\gamma, \cdot)$

$$(\gamma \cdot \gamma)^{-n} \mid \xi(\gamma) \mid (\gamma \cdot \gamma)^{n}$$

$$\iff \xi \in Hom(M, \mathbb{G}_{m,trop,S})^{(N)}$$

$$\mathcal{H}om(M, \mathbb{G}_{m,trop,S})^{(N)} / N := \mathrm{TroPic}_{C/S}.$$

Thm [MW]: There is a SES:

$$0 \to \mathrm{Pic}^{[0]}_{C/S} \to \mathrm{LogPic}_{C/S} \to \mathrm{TroPic}_{C/S} \to 0$$

semiabelian scheme.

$$\overline{M} := R^1\pi_* \overline{\mathbb{G}}_{m,trop}, \ \overline{N} = \overline{M}^\vee$$

$\langle \cdot, \cdot \rangle$ is the dual pairing.

At any point $\bar{s} \to S$, φ is given by the matrix above.

<u>Cor</u>: The Torelli map $\mathcal{M}_g \to A_g$ extends uniquely to a map $\overline{\mathcal{M}}_g \to A_g^{log}$.

In particular, $\mathcal{M}_g \to A_g^\Sigma$ extends iff the corresponding map $\Sigma_{g,n} \to \Sigma$ is a map of cone stacks.