ETH Zürich, Basisprüfung

Analysis I/II D-BAUG Sommer 2012

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Important points

- Duration of the exam / maximal number of points
  - Basisprüfung Analysis I/II: Exercises 1–10, 240 minutes, 63 points
  - Semesterkurs Analysis II: Exercises 6–10, 120 minutes, 32 points
- Permitted aids; 15 sheets DIN A4 (= 30 pages) self-authored summary (for the Analyis II exam only 10 sheets DIN A4); no calculator
- All answers must be justified and the appraoch of the solution must be clearly illustrated. Correct, but unjustified solutions will not give any points. If you make use of a theorem from the lecture, you have to specify precisely which theorem is used.
- The exam consists of 10 exercises, whereby the questions 4, 7 and 10 carry a slightly bigger weight. The maximal number of points for every (part of a) question is given in square brackets. You do not need to solve all questions to achieve the highest mark.
- Please start every question on a new sheet.
- Please write you name on **all** sheets and complete the header on the cover page (do not forget your ID number).
- Do not forget to order **all** sheets before you hand them in.

$* * *$ Lots of succes! $* * *$

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1. a) [**2 P**] Determine the real and imaginary part of the following complex number

$$z = \frac{4-8i}{3+4i} \, .$$

b) [**4 P**] Make a sketch in the complex plane of the domain determined by all complex numbers that satisfy the following two conditions:

i) $\arg\left((1+i)^2\right) \leq \arg(z^2) \leq \arg(-7)$,

ii) $\left|\frac{1+2\sqrt{2}i}{e^{i\pi}}\right| \leq \left|\frac{z}{(1+i)^2}\right| \leq |3+4i|$.

Bitte wenden!

2. a) [**2 P**] Compute the following limit

$$\lim_{x\to 1} \frac{\ln x}{\sin \pi x} \, .$$

b) [**4 P**] Determine the Taylor polynomial of degree four of the function $f(x) = \cos(\sin(x))$ at $x_0 = 0$.

3. [**6 P**] Consider the function

$$f(x) = \int_e^x \sqrt{\ln^2 t - 1}\, dt \qquad (1)$$

and determine the arc length of the function f over the interval $[e, e^3]$.

Hint: It is not possible to compute the integral (1) explicitly.

4. Compute the following integrals:

a) [**2 P**] $\int e^{-2x} \sin 6x \, dx$.

b) [**2 P**] $\int_2^4 \frac{1}{x\sqrt{x-1}} \, dx$.

c) [**3 P**] $\displaystyle\int \frac{3x^2 - 7x - 2}{x^3 - x^2 - 2x} \, dx$.

5. [**6 P**] Solve the following initial value problem

$$y'(x) \cdot (x+1) + y(x) = x^3, \; y(0) = \sqrt{5} \, .$$

6. Let S be the surface defined as the graph of $f(x, y) = 10 - x^2 - y^2$.

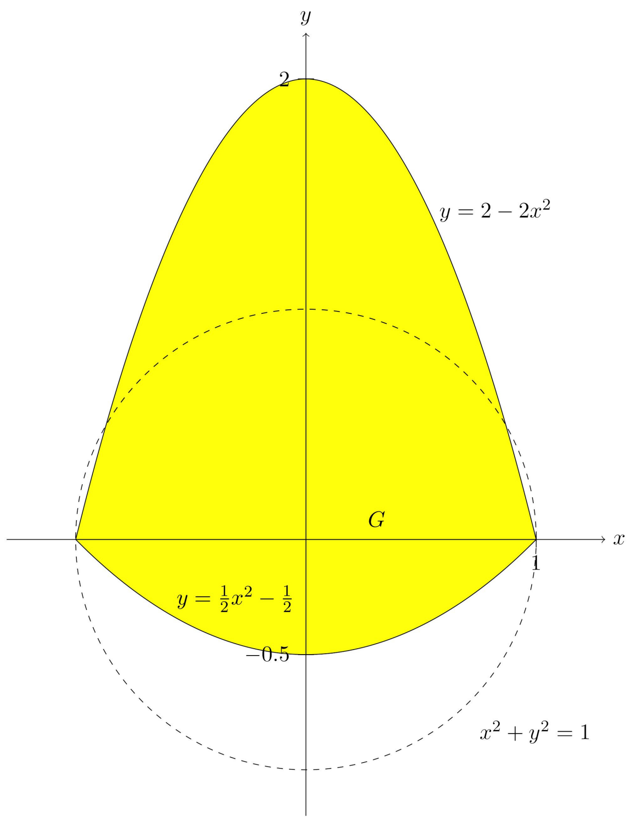
a) [**2 P**] Determine an equation for the tangent plane Σ to the surface S at the point $P = (0, 0, 10)$.

The temperature at (x, y, z) is given by the following function

$$T(x, y, z) = x^2 y + y^2 z + 4x + 14y + z \, .$$

b) [**4 P**] Among all possible directions tangential to the surface S at the point P, which direction will make the rate of change of temperature at P a maximum?

7. [**7 P**] Consider the function $f(x,y) = (x^2 + y^2 - 1)^2$. Determine the global extrema of f on the domain G (see figure).



8. [**6 P**] Compute the integral $\iint_S \vec{F} \cdot d\vec{S}$, where the vector field $\vec{F}$ is given by $F(x,y,z) = (2x, -y, -1)$ and the surface S is given by

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} = 1, z \geq 0 \right\} \quad \text{(upper half of an ellipsoid).}$$

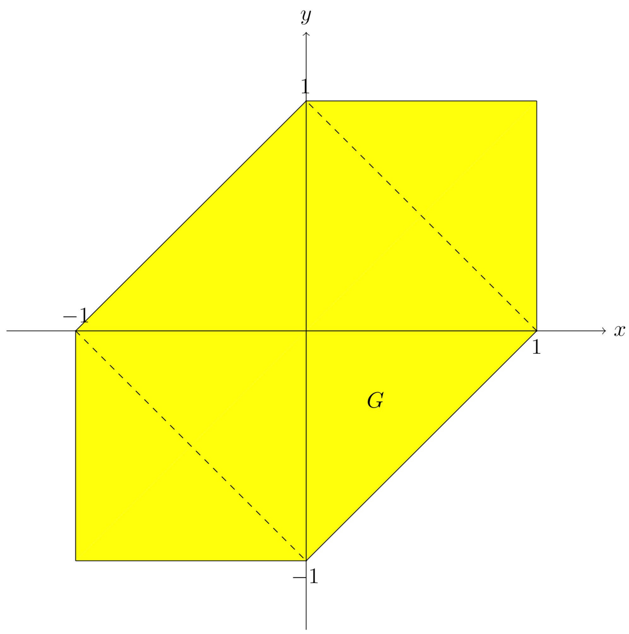
Here the normal vector points outwards.

Bitte wenden!

9. [**6 P**] An electric charge is distributed over the domain G (see figure) according to the following charge density

$$\sigma(x,y) = xy(x^2+y^2)$$

where the unit of σ is Coulomb per square meter: $\frac{C}{m^2}$.



Determine the total charge of G.

10. [**7 P**] Consider the function

$$f(x) = \begin{cases} (x-\pi)^2 & \text{for } x \in [0, 2\pi)\ , \\ f(x+2\pi) & \text{for all } x\ . \end{cases}$$

a) Determine the Fourier series of f on the interval $[-2\pi, 2\pi]$.
Hint: Start by making a sketch of the graph of f

b) Use the Fourier series (substitue $x = 0$) to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\ .$$