D-BAUG
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Analysis I/II

Summer 2014

Exam problems

1. [10 Points] Consider the region

$$B = \left\{ z \in \mathbb{C} \setminus \{0\} \,\middle|\, \operatorname{Im}\left(\frac{z+2}{iz}\right) > 0 \right\}.$$

a) Sketch the region B in the complex plane.

b) The polynomial $z^3 + \frac{7}{2}z^2 + 7z + 6$ has one zero whose real part equals -1. Determine all the zeroes of this polynomial both in normal and in polar form.

c) Which of the above zeroes lie in B?

2. [10 Points] Determine the value of the following terms:

a) $\displaystyle\int_{-\pi}^{\pi} \frac{|\sin x|}{1+\cos^2 x}\,\mathrm{d}x$

b) $\displaystyle\int_{2}^{3} \frac{x-7}{(x+2)^2-9}\,\mathrm{d}x$

c) $\displaystyle\lim_{x\to 1} \frac{1-\cos(1-x^2)}{(1-x)^2}$

3. [10 Points] Determine the general real solution of the following differential equation

$$y^{(4)}(x) - y'(x) = 0,$$

where $y^{(4)}(x)$ denotes the fourth derivative of y w.r.t. x and determine from this all the solutions that satisfy the following conditions

$$\lim_{x\to\infty} y(x) = 2, \quad y(0) = 1, \quad y'(0) = 0.$$

Bitte wenden!

4. [**10 Points**] Remarks to the rating: Each statement is either true or false - please tick the appropiate box. If you want to remove a tick, please do this very clearly.

Every statement a) - j) gives +1 point, when your answer is correct, −1 when your answer is wrong and 0 when you do not answer the question. The total number of points of this question will, however, never be negative - we round up to 0.

a) The following

$$\int_1^3 \int_{-y}^0 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 \int_{-x}^3 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

yields a correct change of the order of integration.

true false
☐ ☐

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b) The region

$$D = \left\{(x,y,z) \in \mathbb{R}^3 \,|\, 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le \sqrt{1-x^2}\right\}$$

describes one eighth of a solid ball.

true false
☐ ☐

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c) Consider the vector field $\mathbf{F} = (2x, -y)$ and let C be the ellipse with semi-axes 2 and 3 and center $(3,2)$ oriented in an anti-clockwise manner, then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

true false
☐ ☐

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d) The following vector field $\mathbf{F} = (f(y), g(x))$ is conservative on $\mathbb{R}^2$.

true false
☐ ☐

Siehe nächstes Blatt!

The following three figures show each a three-dimensional vector field $\mathbf{F}$ in the x-y-plane. The vector fields looks identical to this in all other planes parallel to the x-y-plane i.e. $\mathbf{F}$ is independent of z and its z-component is constant and equal to 0.

The following two questions refer to the above three figures.

e) All three vector fields have zero divergence.

true	false
☐	☐

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f) For all three vector fields the rotation is given by the zero vector.

true	false
☐	☐

The following statement refers to the figure below which shows a two-dimensional vector field and two points P and Q:

g) Both $\operatorname{div}\mathbf{F}|_P > 0$ and $\operatorname{div}\mathbf{F}|_Q > 0$.

true	false
☐	☐

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h) Extend the function $f(x) = x^2 \sin x \cos x$ defined on the interval $[-\pi, \pi]$ periodically to all of $\mathbb{R}$ and let its Fourier series be given by

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then $a_0 \neq 0$ and $a_n = 0$, $n \geq 1$.

true	false
☐	☐

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i) Consider the 2π-periodic extension of the function $f(x) = x$ for $0 \leq x \leq 2\pi$. Then the Fourier series of the function converges at $x = 2\pi$ to π.

true false
☐ ☐

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j) Let $u_1(x, y)$ and $u_2(x, y)$ be two solutions of the partial differential equation

$$yu_{xx} + xu_{yy} = 0.$$

Then the function $u_1 - u_2$ is in general also a solution of this equation.

true false
☐ ☐

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5. **[10 Points]** Determine and classify the critical points of the following function

$$f(x,y) = -(x^2-1)^2 - (x^2-e^y)^2.$$

6. **[10 Points]** Consider the region

$$G = \left\{(x,y) \in \mathbb{R}^2 \,|\, r_0 \leq \sqrt{x^2+y^2} \leq 2,\ y \geq 0\right\}$$

with constant density $\varrho \equiv 1$.

a) Determine the center of mass of G when $r_0 = 1$.

b) Let $r_0 \in (0,2)$ the maximal value, for which the center of mass of G still lies in G. Determine a quadratic equation of the form $r_0^2 + pr_0 + q = 0$ which r_0 has to satisfy.

c) Determine r_0.

7. **[10 Points]** Determine the net outward flux of the vector field $\mathbf{F}(x,y,z) = (x^2, y, z)$ across the boundary of

$$B = \{(x,y,z) \in \mathbb{R}^3 \,|\, y^2 + z^2 \leq 1 + x^2 \leq 2\}.$$

Tip: Use the following cylindrical coordinates $x = x, y = r\cos\varphi, z = r\sin\varphi$.

8. **[10 Points]** Consider the function $f(x) = e^x$ defined on the interval $[0,\pi]$.

a) Sketch the odd and the even 2π-periodic extension of f on the interval $[-\pi, 3\pi]$.

b) Determine the Fourier series for the odd extension.

9. **[10 Points]** Determine a solution $u(x,t)$ of the following initial and boundary value problem using the method of separtating variables $u(x,t) = X(x) \cdot T(t)$.

$$\begin{cases} t^3 u_{xx} - u_t &= 0, & \text{f}0 < x < \frac{\pi}{2} \text{ and } t > 0 \\ u(0,t) &= 0, & \text{for } t > 0 \\ u(\pi/2,t) &= 0, & \text{for } t > 0 \\ u(x,0) &= 8\sin(6x), & \text{for } 0 < x < \frac{\pi}{2}. \end{cases}$$