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Immersed Lagrangian quantum cohomology

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Abstract

We develop a version of immersed Lagrangian quantum cohomology for closed, graded, exact, generic Lagrangian immersions satisfying a positivity assumption, which slightly generalizes and completes the work done in [AB19], and endow it with a canonical $\mathscr{A}_\infty$-structure. In particular, this shows that such Lagrangian immersions may be admittet as objects in the Fukaya category of the ambient symplectic manifold.

derived (?)

Contents

1 Preliminaries **5**
1.1 A very quick introduction to symplectic geometry and almost complex structures . . . 5
1.2 Introduction to pseudoholomorphic curves and Floer theory 6
1.3 A sketch of the construction of embedded Lagrangian quantum cohomology 10

2 Immersed Lagrangian quantum cohomology: vector space structure **17**
2.1 Gradings 17
2.2 Moduli spaces of pearly trajectories 19
2.3 Transversality of pearls 26
2.4 Compactness of moduli spaces of pearls in dimension 0 35
2.5 Compactifications of moduli space of pearls in dimension 1 38
2.6 The pearl complex 41
2.7 Invariance from parameters 43
2.8 Lagrangian quantum cohomology is Floer cohomology 47

3 Immersed Lagrangian quantum cohomology: A_∞ structure **53**
3.1 The pearly product 53
3.2 Unit 62
3.3 Associativity and higher operations 64
3.4 An example of computation 64

Bibliography **69**

Introduction

The present work is a report of the author's journey into Floer theory for Lagrangian immersions as part of his Master thesis at ETH during the fall semester of 2021, and it is in some way the continuation of previous work on Floer theory initiated during the autumn semester of 2020 [Amb20]. In the cited work, the author started the theory of Morse functions and Mikahil Gromov's revolutionary work on pseudoholomorphic curves in symplectic manifolds to then develop the machinery of Hamiltonian and Lagrangian Floer cohomology, to in the end move onto Biran and Cornea's Lagrangian quantum cohomology.

We give a brief overview of the situation. Hamiltonian Floer cohomology is a generalization of Morse cohomology to an infinite dimensional setting in the presence of a symplectic form. Andreas Floer's work was motivated by a conjecture by Arnold introduced during the sixties, ~~stating~~ asserting that the number of fixed points of a symplectic diffeomorphism arising from a Hamiltonian function is bounded below by the sum of the Betti numbers of the manifold. One constructs the cohomology by trying to do Morse theory with an "action functional" on the space of free loops of the symplectic manifold. By changing a bit our point of view, we may see Hamiltonian Floer cohomology as generated by the intesection points of the diagonal submanifold of our symplectic manifold with the graph of a Hamiltonian diffeomorphism, two compact Lagrangian submanifolds of our symplectic manifold. Indeed, the idea behind Lagrangian Floer theory is to consider the intersection of two compact Lagrangian submanifolds meeting transversally and build a~~n~~ cohomology starting from the intersection points. Using Hamiltonian perturbation, one can then define Lagrangian Floer cohomology for any pair of Lagrangian submanifold, in particular for a single Lagrangian submanifold, of our symplectic manifold. It is well-known, that one can endow Lagrangian Floer homology with an associative product, the so-called Donladson product. It is also well-known that such structure gives us far more informations at the chain level, where the product is not associative, but is subject to A_∞-relations, giving rise to the so-called Fukaya category. Another chain-isomorphic ?? approach to Lagrangian Floer theory is known: Lagrangian quantum cohomology, which has been studied intensively by Biran and Cornea at the end of the 00's. Lagrangian quantum cohomology is a deformation of the Morse cohomology of a compact Lagrangian (subject to a serie of assumptions), in which we let "blow up" finitely many points on Morse trajectories to pseudoholomorphic disks with boundary on such a Lagrangian. By considering similar but three-ended trajectories, one endows Lagrangian quantum cohomology with a product which is associative in cohomology. One can see the so-called "pearl" complex generating this cohomology as the limit of the standard Floer complex of a single Lagrangian (hence with

series? certain?

perhaps chain-level ?

Hamiltonian perturbation), when the Hamiltonian tends to the identity. This was the ending point of the previous work of the author and will be the starting point of this thesis.

In this thesis, we first introduce the basics of symplectic and almost complex geometry and give a quick look at the structure of the proof of Gromov's non-squeezing theorem, for which one already needs some arguments about transversality and compactness of certain moduli spaces of spheres. Then, we go over some results about the decomposition of pseudoholomorphic curves into simple (i.e. with a dense set of injective points) pieces: it is very well-known that in the closed case there is a dichotomy between simple and multiply-covered curves (i.e. up to finitely many points, all the points of a curve have the same covering multiplicity), while the case with boundary is a bit more hard, and has been worked out in the early 00's by Lazzarini in the embedded case and recently by Perrier in the generic immersed one. We then outline rather quickly some steps of the proof of the theorems of Lazzarini and Perrier. After giving a quick remainder of Lagrangian quantum cohomology in the embedded case we move to the main topic of this thesis: the immersed case. This is the content of Chapter 1 of this thesis.

Floer theory for immersed Lagrangians has become of interest in recent times, and is an important part of Biran and Cornea's program in symplectic topology, where for instance immersed Lagrangian cobordism is taken into consideration when building a new way of looking at Fukaya categories via an equivalence relation modeled on a cobordism which is always immersed by definition (and surpisingly, at least to the author, this construction does not use holomorphic curves). First, we construct the graded vector space structure of Lagrangian quantum homology for certain exact Lagrangian immersions with only transverse double points and no other singularities, mainly following previous work of Alston and Bao, while refining it and relaxing the starting assumptions. As we are in the exact case, our configurations will include no smooth pseudoholomorphic disks, and there will be no bubbling of smooth disks and spheres, but may include disks with a singular point (a so-called "teardrop") eventually connected via Morse flowlines. Aside from that, the main difference from the embedded case is that our "pearl" complex is also generated by ordered double points, which implies that we have to take care of configurations starting or/and ending at double points. We show that our cohomology is well-defined, independent of the parameters we use (a Morse-Smale pair together with an autonomous almost complex structure) and chain isomorphic to the standard definition of Lagrangian Floer cohomology via Hamiltonian perturbations. This is the content of Chapter 2 of this thesis.

We then move to defining the ring and A_∞ structures of immersed Lagrangian quantum cohomology and providing an example of computation. We first define a ring structure on the chain level, counting eight different types of configurations for each combination of critical point/double point allowed in configurations with three ends and show that it is well defined and independent ~~from~~ of the parameters. in cohomology Then, we show that this product admits a unit which is canonical in cohomology. After that, we show that this product is associative in cohomology and sketch the definition of the higher structures endowing the pearl complex with an A_∞-structure. In the end, we compute the A_∞-structure of the pearl complex for a class of immersed Lagrangians of smoothings of A_N-surfaces (for which Alston earlier computed the ~~vector space~~ additive structure of the quantum cohomology). Unfortunately, this example does not give us a proof that the product we defined is non-commutative in general, but we expect to be so because of the rigidity of the definition of the "core" of configurations contributing to the product. This is the content of Chapter 3 of this thesis and is, as far as the author knows, new, altough very similar to (and inspired by) previous work of Biran and Cornea and of Fukaya (in the Morse case).

This thesis was written during the summer of 2021 and is the result of a work lasted ~~seven~~ months, which was mainly focused on Lagrangian quantum homology for immersions but also included some weeks of thinking about other topics such as: decomposition of pseudoholomorphic disks with boundary on Lagrangian immersions (following Lazzarini and Perrier), Lagrangian cobordism, the shadow metric and the cobordism category (following more recent work of Biran and Cornea) and persistence homology applied to symplectic topology (following a new book by Leonid Polterovich, Daniel Rosen, Karina Samvelyan and Jun Zhang), mainly in view of the PhD the author will start at ETH under the supervision of Prof. Paul Biran.

NO!

Rewrite the entire intro!

Acknowledgments

I want to thank Prof. Paul Biran firstly for accepting to work with me for my Master thesis and for suggesting me a cool main topic which perfectly fits my previous work, as well as other topics which gave me a broader view onto the world of symplectic topology and its major trends at the moment. Moreover, I want to thank him for his initial guidance trought the world of immersed Floer theory and for the freedom he gave me after to work on my ideas.

1 — Preliminaries

1.1 A very quick introduction to symplectic geometry and almost complex structures

In this section we will introduce the very basics of symplectic and almost complex geometry and fix some notation.

Let M be a smooth manifold. A symplectic form on M is a non-degenerate and closed differential 2-form $\omega \in \Omega^2(M) := \Gamma(T^*M \wedge T^*M)$. We call a tuple (M, ω), with M and ω as above, a symplectic manifold. It's quite easy to see that a symplectic manifold has to be even dimensional and orientable. It is well known that symplectic manifolds have no local invariants other than the dimension of the manifold itself, so that symplectic geometry is quite different from Riemannian geometry: indeed, we could interpret closedness of the sympelctic form as some kind of flatness. If L is a smooth manifold of half the dimension of M such that there is an immersion $\imath : L \to M$ such that $i^*\omega = 0$ we say that $\imath$ is a Lagrangian immersion of L in M; if moreover $\imath$ has transverse double points and no triple points we say that it is generic. We define the set of ordered self-intersections of a generic Lagrangian immersion $\imath$ as $R := \{(p,q) \in L \times L : \imath(p) = \imath(q),\ p \neq q\}$. A diffeomorphism between symplectic manifolds is said to be a symplectomorphism if it pullbacks the symplectic form of the target manifold to the symplectic form of the domain. Moreover, a vector field $X \in \Gamma(TM)$ is said to be symplectic if $i_X\omega \in \Omega^1(M)$ is a closed 1-form; note that flows of symplectic vector fields are symplectomorphisms. We mimic the construction of the Riemannian gradient. Consider a smooth function $H \in C^\infty(M)$ with compact support, which, in this context, symplectic geometers like to call Hamiltonian, then we can define the Hamiltonian vector field $X^H \in \Gamma(TM)$ associated to H trough

$$i_{X^H}\omega = dH$$

i.e. they put $-dH$ on the RHS).

(notice that a lot of people use~~s~~ the minus sign convention here). ~~This operation is of course legal~~ as the form ω is assumed to be non-degenerate. We generally denote by φ_t^H the flow of the vector field X^H; note that such flows are made of symplectomorphisms as Hamiltonian vector fields are symplectic by construction. It is quite easy to see that trajectories of Hamiltonian vector fields are contained in level sets of the associated Hamiltonian functions.

The def of X^H is good

We define almost complex structures on smooth manifolds. An almost complex structure on a smooth manifold M is a smooth $(1,1)$-tensor field $J \in \Gamma(TM \otimes T^*M)$ such that $J^2 = -id$ when

perhaps a few words on non-autonom. Hamilt. functions and the flows they generate.

and J induces a canonical orient.

STYLE: It is

seen as an isomorphism of the tangent bundle TM. We call a tuple (M, J), with M and J as above, an almost complex manifold. It's quite easy to see that an almost complex manifold has to be even dimensional and orientable. A smooth map $f : (M, J) \to (M', J')$ between almost complex manifolds is said to be (J, J')-holomorphic if

$$Df \circ J = J' \circ Df$$

identify $\mathbb{R}^{2n} = \mathbb{C}^n$, via $(x_1, y_1, \dots, x_n, y_n) \leftrightarrow (x_1 + iy_1, \dots)$

on TM. On $\mathbb{R}^{2n}$ there is a standard almost complex structure: multiplication with the imaginary unit i; we call such an almost complex structure the standard almost complex structure and denote it by J_0. An almost complex structure J on M is integrable if there is an atlas of M with (J, J_0)-holomorphic charts. It's not hard to see that integrable almost complex manifolds are complex manifolds (in the sense of having an atlas with biholomorphic transition functions) and viceversa (see for instance [Can08]).
One can show that oriented surfaces admit almost complex structures, and that almost complex structures on surfaces are always integrable. We define a Riemann surface to be a surface with an (integrable) almost complex structure.

Consider a symplectic manifold (M, ω). An almost complex structure J on M is said to be tamed by ω if for any $x \in M$ and $v \in T_xM$, $\omega_x(v, Jv) > 0$; we denote the space of almost complex structure tamed by ω by $\mathcal{J}_t(M, \omega)$. An almost complex structure $J \in \mathcal{J}_t(M, \omega)$ tamed by ω is compatible with ω if for any $x \in M$ and $v, u \in T_xM$, $\omega_x(Jv, Ju) = \omega_x(v, u)$. We define the Riemannian metric induced by ω and $J \in \mathcal{J}_c(M, \omega)$ by $(g_{J,\omega})_x(v, w) := \omega(v, Jw)$, for $x \in M$ and $v, u \in T_xM$; it is not hard to see that this indeed defines a Riemannian metric on M. Reasoning at the level of symplectic linear algebra, one can show that for a symplectic manifold (M, ω), the spaces $\mathcal{J}_t(M, \omega)$ and $\mathcal{J}_c(M, \omega)$ are always non-empty and contractible.

For the details of these constructions, see [Can08; MS17; MS12]).

1.2 Introduction to pseudoholomorphic curves and Floer theory

After Gromov's revolutionary paper [Gro85], the study of pseudoholomorphic curves became a ~~major~~ central ~~trend~~ topic in symplectic topology. Given an almost complex manifold (M, J) and a Riemann surface (Σ, j), a pseudoholomorphic or J-holomorphic curve is a (j, J)-holomorphic map $\Sigma \to M$. One of the foundational results Gromov proved using pseudoholomorphic curves is the following *non-squeezing* theorem.

Theorem 1.1. *Let $\varphi : B^{2n}(R) \to B^2(r) \times \mathbb{R}^{2n-2}$ be a symplectic embedding. Then $R \leq r$.*

Gromov's idea for the proof was basically as follows. First, we have to compactify $B^2(r) \times \mathbb{R}^{2n-2}$ in a nice way in order for the new embedding to be symplectic again. We endow the image of the embedding, a ball in the new space, with an almost complex structure which is basically the standard one on the standard ball (but pushed forward by the embedding), and then show that there always exists a pseudoholomorphic curve in a nice homology class passing trough the center of such ball.

Perhaps instead of the non-squeezing put here Gromov's Thm. on non-exist. of exact (closed) Lagr. submanif. of $\mathbb{R}^{2n}$?

The problem is that, as we don't know much about the embedding and about the way which the aforementioned almost complex structure gets extended to the whole manifold, there is absolutely *no way* to construct such a pseudoholomorphic curve explicitly. To show existence, we will deform the almost complex structure by a path in the space of compatible ones, to a nice looking one, for which we easily get any information about its pseudoholomorphic curves, and then introduce a strong compactness theorem, involving a cobordism between the two moduli spaces, that allows us to deduce some information about pseudoholomorphic curves of the first complex structure by looking at the second one: this wild step is achieved by viewing families of pseudoholomorphic curves as zero sections of infinite dimensional vector bundles. In this process, one major point is that pseudoholomorphic curves in our fixed homology class generically come in a family which is finite dimensional smooth manifold. From this point on, we use some standard machinery from algebraic geometry called Lelong inequality (see [GH11]) to conclude.

~~Hence, the proof goes trought two major steps: the study of *regularity* (which is usually called *transversality*) and *compactness* of moduli spaces of pseudoholomorphic curves.~~ In Morse theory (see [AD14]), one goes trought similar steps to define Morse cohomology, where instead of pseudoholomorphic curves one looks at negative gradient flowlines of some nice function (in this case, transversality reduces to triviality, as one can see moduli spaces as transverse intersections of balls). It was Floer's idea (see [Flo87; Flo88; Flo89]) to in fact combine the theory of pseudoholomorphic curves developed by Gromov with the well known ideas coming from Morse theory to build an infinite dimensional version of the latter in order to solve a famous conjecture proposed by Arnold on the number of periodic orbits of a Hamiltonian diffeomorphism. In this section we briefly review the basics of Floer cohomology for Lagrangian intersections in the embedded monotone case, first following the original idea by Floer, i.e. using Hamiltonian perturbations (for that, we will follow [AD14; Poz94]), and then following the method developed more recently by Biran and Cornea (see [BC07; BC08; BC09]) using so-called *pearly* trajectories.

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Pseudoholomorphic curves

Let (M, ω) be a compact symplectic manifold and fix a compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, let (Σ, j) be a compact connected Riemann surface. For smooth maps $u : \Sigma \to M$, we define the operator $\overline{\partial_J} u := \frac{1}{2}(du + J \circ du \circ j)$. Of course, u is J-holomorphic if and only if $\overline{\partial_J} u = 0$; picking complex coordinates (s, t) on Σ, u is J-holomorphic if and only if $\partial_s u + J_u \partial_t u = 0$. J-holomorphic curves present many rigidity properties which are analogous to those of holomorphic curves (altough proofs are generally much harder), one can find these results in [MS12, Chapter 2].

To deal with transversality of moduli space of curves, we need the following two definitions.

Definition 1.2. *Let $u : \Sigma \to M$ be J-holomorphic. u is said to be multiply covered if there is another compact Riemann surface Σ', a J-holomorphic curve $u' : \Sigma' \to M$ and a holomorphic branched covering $\sigma : \Sigma \to \Sigma'$ of degree greater than one such that $u = u' \circ \sigma$. u is said to be somewhere* *injective if there is* $z \in \Sigma$ *such that*

$$Du(z) \neq 0 \text{ and } u^{-1}(u(z)) = \{z\}$$

Such points are called injective points of u*. A curve with a dense set of injective points is called simple.*

Standard theory (see [MS12, Chapter 3]) basically tells us that for a generic choice of the compatible almost complex structure, moduli spaces of pseudoholomorphic curves $u : \Sigma \to M$ which have a dense set of injective points are finite dimensional smooth manifolds. In practice, when dealing with a problem requiring some kind of transversality, one has to find suitable decomposition criteria for interesting pseudoholomorphic curves and then use some additional assumptions to conclude that all the interesting curves are regular. The decomposition results depends on the kind of Riemann surface and boundary conditions we are working with. The following results take care of this in the closed, compact and compact immersed case respetively.

Proposition 1.3 ([MS12])**.** *Assume that* Σ *is closed and consider a* J*-holomorphic curve* $u : \Sigma \to M$*. Then* u *is simple if and only if it is not multiply covered. Moreover, any* J*-holomorphic sphere* $u : \mathbb{C}P^1 \to M$ *is covered by a simple* J*-holomorphic sphere.*

This means that in the closed case there is a simple-multiply covered dichotomy. This is not the case when there are boundary conditions. Indeed, this is easily illustrated by the "Lantern" example in [Laz00]: the map $u(z) = z^3$ from the disk, seen as one-point compactification of the complex upper halfplane, to the sphere, seen as one-point compactification of the complex plane, amounts to "wrapping plastic wrap around a watermelon" for an angle of 3π, so that it is clearly non simple. However, it may be decomposed into simple pieces (wrapping of π each) which cover the whole image.

Proposition 1.4 ([Laz00; Laz11])**.** *Let* L *be a Lagrangian submanifold of* M*. Assume that* Σ *is compact with boundary and consider a non-constant* J*-holomorphic curve* $u : (\Sigma, \partial\Sigma) \to (M, L)$ *with finite energy. Then there are finitely many simple* J*-holomorphic curves* $v_1, ..., v_k : (\Sigma_i, \partial\Sigma_i) \to (M, L)$*, and finitely many integers* $m_1, ..., m_k \in \mathbb{Z}$ *such that in* $H_2(M, L)$ *we have*

?
Σ is compact...

$$[u] = \sum_{i=1}^{k} m_i [v_i]$$

Moreover, if $\Sigma = D$*, any* Σ_i *is isomorphic to* D *and we have*

$$\bigcup_{i=1}^{k} v_i(D) = u(D)$$

Before stating the next proposition, we need to define pseudoholomorphic curves with corners.

Definition 1.5. *Consider an immersion* $\imath : P \to N$ *of a manifold* P *into a manifold* M*. Let* $\gamma : (-\epsilon, \epsilon) \to \imath(P)$ *be a path such that* $\gamma(0)$ *is a double point of* $\imath$*, and consider a lift* $\tilde{\gamma} : (-\epsilon, \epsilon) - \{0\} \to P$ *of* γ *not defined on* 0*. Let* $p = \lim_{t\to 0^-} \tilde{\gamma}(t)$ *and* $q := \lim_{t\to 0^+} \tilde{\gamma}(t)$*. If* $p \neq q$ *we say that* γ *has a branch jump of type* (p, q) *at* $t = 0$*.*

Definition 1.6. *Consider a generic Lagrangian immersion $\imath : L \to M$ with set of ordered double points R. A J-holomorphic α-marked disk with corners and boundary on $\imath$ is a tuple $\overline{u} := (u, \Delta, \alpha, l)$ where:*

1. *u is a continuous map $(D, \partial D) \to (M, \imath(L))$ which is smooth and J-holomorphic on $\mathrm{int}(D)$;*
2. *$\Delta := \{z_1, ..., z_k\} \subset \partial D$ is a finite ordered subset of the circle with $z_i \neq z_j$ for $i \neq j$, coming with a decomposition $\Delta := \Delta^+ \sqcup \Delta^-$ of Δ into so-called outgoing and incoming points;*
3. *α is a map $\{1, ..., k\} \to R$;*
4. *l is a continuous lift $\partial D - \Delta \to L$ of u, i.e. we have $u = \imath \circ l$ on $\partial D - \Delta$, such that if $z_i \in \Delta^+$, then*
$$\left(\lim_{\theta \to 0^-} l(e^{i\theta} z_i), \lim_{\theta \to 0^+} l(e^{i\theta} z_i) \right) = \alpha(i)$$
and if $z_i \in \Delta^-$, then
$$\left(\lim_{\theta \to 0^+} l(e^{i\theta} z_i), \lim_{\theta \to 0^-} l(e^{i\theta} z_i) \right) = \alpha(i)$$
i.e. l has a branch jump of type $\alpha(i)$ at $z_i \in \Delta^+$ when moving counterclockwise, and a branch jump of type $\alpha(i)$ at $z_i \in \Delta^-$ when moving clockwise;
5. *the energy $E(u) := \int_{D^2 - \Delta} u^* \omega$ is finite.*

We will call the points of Δ corners of branch jumps of $\overline{u}$. We denote the clockwise limit on ∂D as limits from left and counterclockwise ones as limits from right.

The map l is the big new structure which is not present when one concentrates on embedded Lagrangians: it keeps track of the type of branch jumps a specific disk has and it ensures that we have only a finite number of jumps.

Proposition 1.7 ([Per19])**.** *Let $\imath : L \to M$ be a generic Lagrangian immersion. Let $u : (D, \partial D) \to (M, \imath(L))$ be a J-holomorphic disk with corners and boundary on $\imath$ with finite energy. Then, there are finitely many simple J-holomorphic disks $v_1, ..., v_k : (D, \partial D) \to (M, \imath(L))$ with corners and boundary on $\imath$ with finite energy and finitely many positive integers $m_1, ..., m_k \in \mathbb{Z}_{>0}$ such that in $H_2(M, \imath(L))$ we have*
$$[u] = \sum_{i=1}^{k} m_i [v_i]$$
and moreover
$$\bigcup_{i=1}^{k} v_i(D) = u(D)$$

These results are proved analyzing the set of accumulation points of the set of multiple points of a pseudoholomorphic curve. In the closed case the proof is not that hard, as this set defines an equivalence relation, while the case with boundary is much more difficult. The idea in [Laz00; Laz11]

what is the statement
in the closed case?

is to analyze the so called *frame* of a pseudoholomorphic curve with Lagrangian boundary condition. Consider a pseudoholomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$ and let $C(u) := u^{-1}(u(\text{Crit(u)}))$; for $z, z' \in \Sigma - C(u)$ define the relation $z\mathcal{R}_u z'$ if and only if for any neighbourhoods $V, V' \subset \Sigma$ of z and z' respectively there are neighbourhoods $U \subset V$ and $U' \subset V'$ such that $u(U) = u(U')$. Let $\overline{\mathcal{R}_u}$ be the closure of $\mathcal{R}$ in Σ, which may not be an equivalence relation if Σ has indeed boundary. However, the frame $\mathcal{W}(u) := \overline{\mathcal{R}_u}(\partial\Sigma)$ has very interesting properties, and is in fact a graph. This is proved using a relative version of Carleman similarity principle (see[MS17, Chapter 3] and [Laz00, Section 3])~~,~~ and the fact that locally the projection $\overline{\mathbb{R}_u}$ is open. The point is then associating a Riemann surface to any connected component of $\Sigma - \mathcal{W}(u)$ such that it embeds nicely ? in Σ; then, one gets a pseudoholomorphic curve v on this "smaller" Riemann surface, which is easily seen to be simple. In the case of disks, showing that the domain is biholomorphic to a disk is easy if the frame is connected but requires some more work (which is the purpose of [Laz11]) in the case where it is not connected. Basically, the work in [Per19] directly extends Lazzarini ideas to immersions.

1.3 A sketch of the construction of embedded Lagrangian quantum cohomology

In this section we will briefly described Lagrangian quantum homology in the embedded case, skipping a lot of details, and provide an intuition for its construction.

Originally, Floer introduced what is nowadays called Hamiltonian Floer homology (see [Flo87] for the original paper, and [AD14] for a complete and detailed overview of the theory in the symplectically aspherical manifolds), also called Floer homology for Hamiltonian diffeomorphisms, in order to attempt a solution for a celebrated conjecture of Arnold, relating the topology of a symplectic manifold to the number of periodic orbits of a generic Hamiltonian function. Given a closed symplectic manifold (M, ω) satisfying some additional assumptions, a generic time-dependent Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ and a generic compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, we consider the complex freely generated by periodic Hamiltonian orbits of H and the differential idefined by counting cylinders which connect two Hamiltonian orbits and satisfy a perturbed Cauchy-Riemann equation, the so-called Floer equation (q.v. Section 2.8). The perturbation comes from the gradient of the chosen Hamiltonian H. Starting from the ideas of Gromov on pseudoholomorphic curves [Gro85], Floer showed that in this way one can define a well-defined homology ~~and~~ as well as a canonical ring structure on it, via the so called *pair of pants product* (see for instance [Sei08]). This idea and its Lagrangian counterpart (see [Poz94] for a nice and brief overview) revolutionized the world of symplectic topology ~~during~~ in the beginning of the nineties. The downsides of this approach are that, other than being quite difficult to compute in practice, it appeals to a quite heavy machinery, especially when dealing with regularity of spaces of interesting configurations.

It is well-known that Hamiltonian Floer homology is isomorphic to the Morse homology of M as vector space, while it is in general not as a ring. Starting from this observation, the idea is to find another way to compute Hamiltonian Floer homology by deforming singular homology (which in this setting is best represented by its Morse counterpart). In particular, the heuristic idea is to look at the Floer equation not as a perturbed Cauchy-Riemann equation but as a J-perturbed gradient flow equation, that is to take into account Morse flowlines of an autonomous Morse functions on M which may be interrupted by pseudoholomorphic spheres (and no disks, as there is no boundary condition to discuss). The result is the so called quantum homology of M: the quantum complex is generated by critical points of a Morse function on M, the quantum differential counts Morse flowlines (as sphere contributions cancel out) between critical points and the quantum product counts Y-shaped configurations interrupted by a pseudoholomorphic sphere in the middle. In this case proving regularity is very easy due to Proposition 1.3 combined with a dimension count. Showing that the complex is well-defined is quite easy, as bubbling of pseudoholomorphic spheres is a codimension 2 phenomenon. It has been shown [PSS96] that the Hamiltonian quantum homology and the quantum homology of M are in fact isomorphic as rings (via a similar isomorphism to the one we will see in Section 2.8). One may also view quantum homology as Hamiltonian Floer homology for an Hamiltonian tending to the identity.

They don't cancel out! The do not appear in moduli spaces of dim 0!

clearly & VISIBLY define $H_2^D(M,L)$, otherwise the reader might miss it!

We will now outline the idea behind the generalization of quantum homology in the Lagrangian setting as developed in [BC07; BC08] starting from a direct application of the idea above to the Lagrangian setting. Assume that (M,ω) is a closed symplectic manifold and assume that $L \subset M$ is a monotone embedded Lagrangian, meaning that there is positive proportionality on $H_2^D(M,L)$ (the image of the Hurewitch map on $\pi_2(M,L)$) between the Maslov map μ on L (which may be roughly identified as a winding number) and integration by ω (see [Oh93]) and that the minimal Maslov number of L is at least 2.

Say $2n := \dim M$. We will now explain which sort of problems one encounters when trying to build a quantum cohomology theory for Lagrangians. We want to again deform Morse cohomology by counting trajectories on a Lagrangian with a possible bubble on them, which in this case will be a disk bubble, as we have boundary conditions: however, as we will see, bubbling of pseudoholomorphic disks is a codimension one phenomenon, and that is a big obstruction to the definition of a proper differential. We will try first to define a cohomology and then introduce the actual Lagrangian quantum homology.

More formally, pick a Morse function $f : L \to \mathbb{R}$ on L, a pseudogradient vector field $X \cancel{\in \Gamma(TL)}$ on L adapted to f, a compatible almost complex structure $J \in \mathcal{J}(M,\omega)$, a class $A \in H_2^D(M,L)$ and two critical points $x, y \in \mathrm{Crit}(f)$ of f. Consider the following moduli space:

? should by a small-er group here.

$$\mathcal{L}'(A,x,y,J) = \{u : (D^2, S^1) \to (M,L) :\ u(-1) \in W^u(x),\ u(1) \in W^s(y),\ [u] = A\}/\mathrm{Aut}(D^2)$$

which is the moduli space of unparametrized configurations of the following form:

something missing here?

We will assume that $\mathcal{L}'(A,x,y,J)$ is, for a generic choice of compatible almost complex structure, a smooth manifold (this may in fact be proved via Proposition 1.4 as we're in a monotone setting). It's easy to compute the dimension of such moduli spaces using the following evaluation maps. Define

$$ev_- : u \in \mathcal{L}_2(A,J) \longmapsto u(-1) \in L, \quad ev_+ : u \in \mathcal{L}_2(A,J) \longmapsto u(1) \in L$$

what is $\mathcal{L}_2(A,J)$?

I'm not sure how to prove this, unless we assume

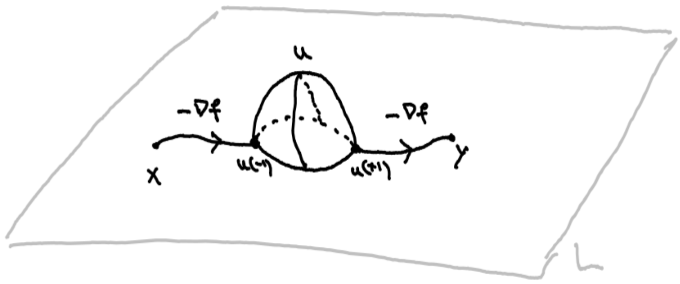

Figure 1.1: A quantum trajectory of the Morse function f.

Then

$$\mathcal{L}'(A,x,y,J) = (ev_- \times ev_+)^{-1}(W^u(x) \times W^s(y))$$

has dimension

$$\begin{aligned}\dim \mathcal{L}'(A,x,y,J) &= \dim \mathcal{L}_2(A,J) - \operatorname{codim}(W^u(x)) - \operatorname{codim}(W^s(y)) = \\ &= (n+\mu(A)-3+2) - (n-|x|) - |y| = \mu(A) + |x| - |y| - 1\end{aligned}$$

Consider the complex C^* made of critical points of f graded by Morse index. We define a map $d : C^* \to C^*[-1]$ by

$$dy := \sum_{x\in \mathrm{Crit}(f),\ A \in H_2^D(M,L):\ |x|-|y|+\mu(A)-1 = 0\ ?} |\mathcal{L}'(A,x,y,J)|_2 y$$

We investigate if d is a differential for the complex C^*. As usual in Floer theory, and as we will do tons of times in the remaining of this thesis we have to look at the compactness properties of our moduli space in dimension 0 and 1. One proves that the moduli spaces $\mathcal{L}'$ are compact in dimension 0. We look in dimension 1: our dream is that to compactify the moduli spaces L''s we have to add configurations ? differering only from the fact of containing broken Morse flowlines, as a computation of d^2 shows. However, this is not the case in general. Indeed, by Gromov compactness for embedded disks (see [Fra08]), we have the following four classes of configurations in the compactification $\overline{\mathcal{L}'}(A,x,y,J)$: ???

1. breaking of Morse trajectories;
2. bubbling of a pseudoholomorphic sphere ~~in~~ at an interior point of the disk; can happen also at a boundary put. (?)
3. bubbling of a pseudoholomorphic disk with boundary on L ~~in~~ at a boundary point of the original disk which is not an incidence point of the Morse trajectories;
4. bubbling of a pseudoholomorphic disk with boundary on L ~~in~~ at an incidence point of the Morse trajectories.

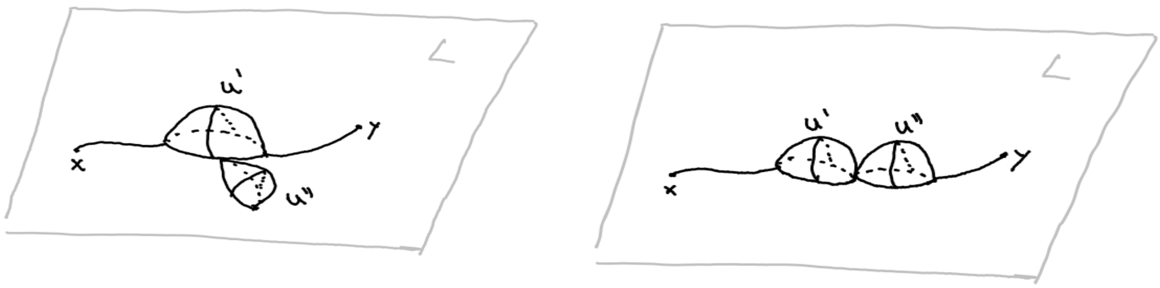

Figure 1.2: Cases 3 and 4.

We will denote by $\overline{\mathcal{L}'}$ the compactification of a moduli space $\mathcal{L}$. Let $x, y \in \mathrm{Crit}(f)$ and $A \in H_2^D(M, L)$ such that $\mu(A) + |x| - |y| = 2$. Consider a sequence $u_n \in \mathcal{L}'(A, x, y, J)$ which does not have a convergent subsequence. Case 1 is what we want to happen. We analyze the possibilities of the other three configuration to happen. Cases 2 and 3 can't happen, and altough the argument is very similar, we will treat them separately.

Case 2. A pseudoholomorphic sphere $v : S^2 \to M$ in the class $B \in H_2^D(M)$ may bubble off from a subsequence of u_n in an interior point. We look at the image, always denote by $B \in H_2^D(M, L)$, of B with respect to the homomorphism $\pi_2(M) \to \pi_2(M, L)$ (appearing in the long exact sequence in homotopy of the couple (M, L)). Then, in the limit $(u, v) \in \overline{\mathcal{L}'}(A, x, y, J)$, the disk u from which v bubbles off lies in the class $A' = A - B$. As L is monotone, $\mu(B) \geq 2$, so that $\mu(A') \leq \mu(A) - 2$. Then the moduli space $\mathcal{L}'(A', x, y, J)$ has dimension

$$\dim \mathcal{L}'(A', x, y, J) \leq \dim \mathcal{L}'(A, x, y, J) - 2 = -1$$

Hence (as we assumed every moduli space is smooth), configurations in case 2 can't exist.

Case 3. A pseudoholomorphic disk $u' : (D^2, S^1) \to (M, L)$ in the class $B \in H_2^D(M, L)$ may bubble of from a subsequence of u_n ~~in~~ at a boundary point which is not $1 \in D^2$ or $-1 \in D^2$. Then, in the limit $(u, u') \in \overline{\mathcal{L}'}(A, x, y, J)$, the disk u from which v bubbles off lies in the class $A' = A - B$. As L is monotone, $\mu(B) \geq 2$, so that $\mu(A') \leq \mu(A) - 2$. Then the moduli space $\mathcal{L}'(A', x, y, J)$ has dimension

$$\dim \mathcal{L}'(A', x, y, J) \leq \dim \mathcal{L}'(A, x, y, J) - 2 = -1$$

Hence (as we assumed every moduli space is smooth), configurations in case 3 can't exist.

Case 4. A pseudoholomorphic disk $u' : (D^2, S^1) \to (M, L)$ in the class $B \in H_2^D(M, L)$ may bubble of from a subsequence of u_n ~~in~~ at either the point -1 or the point $(+1)$. Then, in the limit $(u, u') \in$

$\overline{\mathcal{L}'}(A, x, y, J)$, the disk u from which v bubbles off lies in the class $A' = A - B$. *However*, of course we can't go on as with cases 2 and 3 here. We work with evaluation maps. Define the moduli space of such configurations as

$$\mathcal{L}'(A, B, x, y, J) := \{(u, u') \in \mathcal{L}_2(A, J) \times \mathcal{L}_2(B, J) :\ u(-1) \in W^u(x),\ u(+1) = u'(-1),\ u(1) \in W^s(y)\}/\mathrm{Aut}(D^2)$$

and assume it is smooth. Denote by $ev^A_\pm$ and $ev^B_\pm$ the evaluations map introduced before for $\mathcal{L}_2(A, J)$ and $\mathcal{L}_2(B, J)$ respectively. Then

$$\mathcal{L}'(A, B, x, y, J) = (ev^A_- \times ev^A_+ \times ev^B_- \times ev^B_+)^{-1}(W^u(x) \times \Delta_L \times W^s(y))$$

and so

$$\begin{aligned}\dim \mathcal{L}'(A, B, x, y, J) &= \dim \mathcal{L}_2(A, J) + \dim \mathcal{L}_2(B, J) - \mathrm{codim}(W^u(x)) - \mathrm{codim}(\Delta_L) - \mathrm{codim}(W^s(y)) = \\ &= (n + \mu(A) - 1) + (n + \mu(B) - 1) - (n - |x|) - n - |y| = 0\end{aligned}$$

Hence (as we assumed every moduli space is smooth), configurations in case 4 *can* a priori exist, and in fact do by a gluing argument (see [BC07, Chapter 4]).

This problem is solved by introducing time parameters associated to a piece of Morse flowline joining two different pseudoholomorphic disks with boundary on L, and hence counting configurations such as the one in Figure 1.3 in the differential.

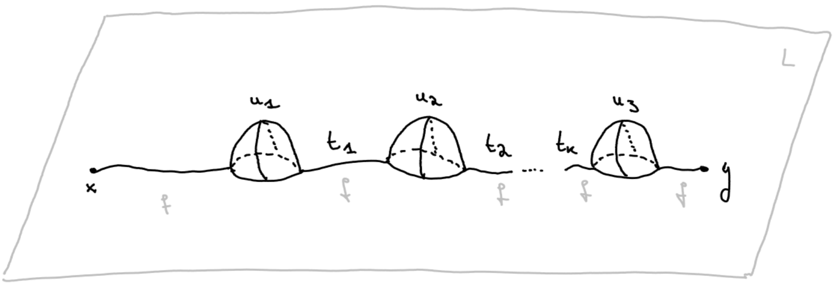

Figure 1.3: A pearly trajectory contributing to the differential of y.

By what above, it is then very easy to see that (assuming that everything is smooth) $d^2 = 0$ in this case, as configurations from Case 4 may happen in the limit both because of linear bubbling and because of shrinking of the finite-time Morse flowlines bewteen two different disks, so that they cancel out in the computation of d^2, and only Morse breaking become relevant. In fact, one uses the results of Lazzarini [Laz11] (see Proposition 1.4) combined with a count of dimension to show (see [BC07]) that the moduli spaces of such pearly configurations are generically smooth manifolds, whose dimension depends on the Morse index of x and y and the sum of the areas of the disks. The resulting cohomology may be shown not to depend on the choice of the parameters (f, g, J) and to be ismomorphic to the Lagrangian Floer cohomology of L.

2 — Immersed Lagrangian quantum cohomology: vector space structure

In this chapter, we aim to generalize Lagrangian quantum cohomology to nice Lagrangian immersions. We will work in the exact setting: in this case, it is easy to see that there are no smooth psuedoholomorphic disks with positive area, so that Floer homology for embedded Lagrangians is isomorphic to singular homology; this will not be true in general for immersions. Indeed, we have to be careful when dealing with immersions: for instance, considering the standard definition of Floer cohomology ([Oh93; AJ10; AB18]), trickier phenomena may happen obstructing $d^2 = 0$; indeed, a portion of a sequence of Floer strips may degenerate in the limit at a self-intersection point of the immersion to a disk with a singularity, a so-called *teardrop* (see Figure 2 2.1). On the other hand, in the Morse-Bott case we are going to focus on, one has to take teardrops into account, so that the resulting pearly cohomology may indeed not be isomorphic to Morse cohomology even in the exact case. We will mainly follow [AB18; AB19] with a few refinements. A more general approach to Floer homology for immersed Lagrangians was developed in [AJ10] using A_∞-perturbations and Kuranishi spaces, in the vein of [Fuk+09] for the embedded case. Here we will use an assumption that does the same job of monotonicity in the embedded case (see [Oh93]).

Our setup will be that of [Sei08]. Let (M, ω) be a compact exact symplectic $2n$-manifold with boundary and with vanishing first Chern class $c_1(M) = 0$. Fix a primitive $\lambda \in \Omega^1(M)$ of ω such that the Liouville vector field $X_\lambda \in \Gamma(TM)$ associated to λ points outward ∂M, and an almost complex structure $J_M \in \mathcal{J}_c(M, \omega)$ compatible with the symplectic form ω such that any J_M-holomorphic curve touching the boundary ∂M is completely contained in it.

2.1 Gradings

The assumption on the first Chern class of our symplectic manifold M allows us to grade Lagrangians in order to grade Floer cohomology; indeed, grading is usually problematic in Lagrangian Floer theory, whereas it is not in the Hamiltonian case. We follow the construction in [Sei00, Example 2.9]. It is easy to see that the map $\det^2 : U(n) \to S^1$ descends to a map $\det^2 : \frac{U(n)}{O(n)} \to S^1$ inducing an isomorphism on the π_1 level. Moreover, it is also easy to show that the Lagrangian Grassmannian $\mathrm{Gr}(\mathbb{R}^{2n}) = \{L \subset V : \ L \text{ is Lagrangian}\}$ is ~~isomorphic~~ to $\frac{U(n)}{O(n)}$, implying that there is a (non-canonical)

canonically homeomorphic

Figure 2.1: The formation of a teardrop in the limit of a sequence of pseudoholomorphic curves with some boundary condition on an immersed Lagrangian L.

homeo.

~~isomorphism~~ $\mathrm{Gr}(T_pM) \cong \frac{U(n)}{O(n)}$ for any point $p \in M$. Indeed, lagrangians in $\mathbb{R}^{2n}$ determine a splitting of the vector space with respect to J, so that $U(n)$ acts transitively on $\mathrm{Gr}(\mathbb{R}^{2n})$; moreover, for the same reason, g-orthogonal linear maps on L extends uniquely to h-unitary linear maps on $(\mathbb{R}^{2n}, J)$, which, as ω_0 vanishes on lagrangians by definition, leaves L invariant: therefore we conclude by the orbit-stabilzer theorem that the map associating h-unitary maps $f \in U(n)$ to $f(L) \in \mathrm{Gr}(\mathbb{R}^{2n})$ induces an isomorphism as claimed. The idea now is to globalize $\det^2$ to a map $\mathrm{Gr}(TM) \to S^1$, but due to the non-canonical situation, we need extra structure. This is provided by the fact that $c_1(M) = 0$: pick a nowhere vanishing top holomorphic form $\alpha \in \Omega^n(M, J_M)$ and define $\det^2_\alpha : \mathrm{Gr}(TM) \to S^1$ as follows: for any $p \in M$, $L \in \mathrm{Gr}(T_pM)$ pick a bais $v_1, ..., v_n$ of L and define

?

$$\overset{2}{\underset{\alpha}{\det}}(L) = \frac{\alpha_p(v_1 \wedge ... \wedge v_n)^{\otimes 2}}{|\alpha_p(v_1 \wedge ... \wedge v_n)|^2}$$

Let now $\imath : L \to M$ be a Lagrangian immersion. We lift $\imath$ to $\bar{\imath} : L \to \mathrm{Gr}(TM)$ and define $\det^2_L := \det^2_\alpha \circ \bar{\imath} : L \to S^1$. We define the Maslov class of L as $\mu_L := (\det^2_L)^*([S^1]) \in H^1(L)$, where $[S^1] \in H^1(S^1)$ is the standard positive generator.

Definition 2.1. *A grading of $\imath$ is a lift $\theta_L : L \to \mathbb{R}$ of $\det^2_L$. A graded Lagrangian immersion is a couple $(\imath, \theta_L)$.*

two ?

We will often omit the grading from the notation. Notice that there exists a grading for the Lagrangian immersion L if and only if $\mu_L = 0$, as $H^1(\mathbb{R})$ is trivial.

Assume now that the Lagrangian immersion $\imath$ is generic. The purpose of grading is that is allows one to assign an integer to Hamiltonian orbits connecting to Lagrangians or to intersection points of two Lagrangians, which means that the Floer complex is graded. In the immersed (and transverse), single Lagrangian case, we grade self-intersection points as follows. Let $(p, q) \in R$ with image $x \in M$

where is R defined?

and consider the transversely intersecting Lagrangian subspaces $D\imath(p)[T_pL]$ and $D\imath(q)[T_qL]$ of T_xM. We know there is a unique unitary matrix relating the two subspaces, and we pick a unitary basis $(u_1, ..., u_n)$ of $D\imath(p)[T_pL]$ such that there are $\alpha_1, ..., \alpha_n \in (0, \frac{1}{2})$ such that $(e^{2\pi i\alpha_1}u_1, ..., e^{2\pi i\alpha_n}u_n)$ is a unitary basis of $D\imath(q)[T_qL]$. We call $\alpha_1, ..., \alpha_n$ the Kähler angles between $D\imath(p)[T_pL]$ and $D\imath(q)[T_qL]$. We define the index $|p,q| \in \mathbb{Z}$ of the ordered self-intersection point $(p,q) \in R$ as

$$|p,q| := n + \theta_L(q) - \theta_L(p) - 2\sum_{i=1}^{n} \alpha_i$$

The easiest way to see that the index is an integer is to define it trought a Maslov index of a bundle pair (see [AB18; Oh15]) or via Fredholm operators (see [AJ10]).
Assume that $\imath$ is exact, and fix a primitive $h_L : L \to \mathbb{R}$, i.e. a smooth ~~map~~ funct. such that $\imath^*\lambda = dh_L$. In this case, we define the energy of an ordered double point $(p,q) \in R$ as

action?

$$\mathscr{A}(p,q) = h_L(q) - h_L(p)$$

We will often drop the primitive from the notation of an exact Lagrangian. Our main assumption troughout this work will be the following *positivity* assumption.

Assumption 2.2 (positivity)**.** *Let $(p,q) \in R$ be an ordered double point of a Lagrangian immersion $\imath$. We assume that if $\mathscr{A}(p,q) > 0$, then $|p,q| \geq 3$.*

2.2 Moduli spaces of pearly trajectories

Consider the setup described above and fix for the rest of the chapter an exact, compact, connected, generic and graded Lagrangian immersion $\imath : L \to M$ satisfying Assumption 2.2.
As we are now working in the exact case, the only configurations that appeared in [BC07] and that are relevant in this case are Morse flowlines joining critical points of a Morse function on L. However, to fully compute a cohomology which is isomorphic to standard Floer cohomology [AB18], we have to include double points and teardrops in some possible configuration. The purpose of this section is to introduce the relevant moduli spaces in order to improve the pearly construction to our immersed case.

Fix now two non-negative integers $k^-, k^+ \in \mathbb{Z}_{\geq 0}$, a finite subset $\Delta := \Delta^- \sqcup \Delta^+ \subset \partial D$ such that $|\Delta^-| = k^-$ and $|\Delta^+| = k^+$ and a map $\alpha : \{1, ..., k^- + k^+\} \to R$. Define the moduli space $\tilde{\mathcal{M}}_{k^-,k^+}(\Delta, \alpha, J)$ of parametrized α-marked pseudoholomorphic disks with boundary and corners on $\imath$ with corners at Δ as the set of curves as in Definition 1.6. Define also

$$\tilde{\mathcal{M}}_{k_-,k_+}(\alpha, J) := \bigsqcup_{|\Delta^-|=k^-,\ |\Delta^+|=k^+} \tilde{\mathcal{M}}_{k_-,k_+}(\Delta^- \sqcup \Delta^+, \alpha, J) \times \{\Delta^- \sqcup \Delta^+\}$$

There is an action of the 3-dimensional group $\mathrm{Aut}(D) \cong PSL(2,\mathbb{R})$ on $\tilde{\mathcal{M}}_{k_-,k_+}(\alpha, J)$ and we define the moduli space of unparametrized α-marked pseudoholomorphic disks with corners on $\imath$ as

$$\mathcal{M}_{k_-,k_+}(\alpha, J) := \frac{\tilde{\mathcal{M}}_{k_-,k_+}(\alpha, J)}{\mathrm{Aut}(D)}$$

It is known (see [Fuk+09; AJ10]) that

$|\alpha(j)|$?

?

$$\operatorname{virdim}(\mathcal{M}_{k_-,k_+}(\alpha, J)) = n - \sum_{-}(n - \alpha(j)) - \sum_{+} \alpha(j) + |\Delta| - 3$$

1where we denote by $\sum_-$ the sum over the indices of incoming branch jumps and by $\sum_+$ the sum over the indices of ougoing ones. For our purposes, we also define, for $d \geq 1$, the moduli space

$$\mathcal{M}^d_{k_-,k_+}(\alpha, J) := \frac{\bigsqcup_{|\Delta^-|=k^-,\ |\Delta^+|=k^+} \left(\tilde{\mathcal{M}}_{k^+,k^-}(\Delta^- \sqcup \Delta^+, \alpha, J) \times (\partial D - \Delta)^d\right) \times \{\Delta^- \cup \Delta^+\}}{\operatorname{Aut}(D)}$$

of virtual dimension

$$\operatorname{virdim}(\mathcal{M}^d_{k_-,k_+}(\alpha, J)) = n + d - \sum_{-}(n - \alpha(j)) - \sum_{+} \alpha(j) + |\Delta| - 3$$

In our setting, we consider constant disks if and only if $\Delta \neq 0$, due to stability conditions. If the domain of α is a single point and $\alpha(1) = \gamma = (p, q)$ we will often write γ or (p, q) in the notation of the moduli spaces. The almost complex structure will often be removed from the notation.

Remark 2.3. *In our setting, the virtual dimensions of moduli spaces of marked disks with corner do not depend on the homology class of the disks, as graded Lagrangian are Maslov zero.*

Moreover, we need some moduli spaces with a priori fixed corner points, in order to define pearly trajectories correctly while avoiding to introduce strips. Given an ordered set $\Delta = \Delta^- \sqcup \Delta^+ \subset \partial D$, we define for $z_1, ..., z_{k_1+k_2} \notin \Delta$ the ordered sets $\Delta_{z_1,...z_{k_1}} := \{z_1, ..., z_{k_1}\} \sqcup \Delta$, $\Delta^{z_{k_1+1},...,z_{k_1+k_2}} := \Delta \sqcup \{z_{k_1+1}, ..., z_{k_1+k_2}\}$ and $\Delta^{z_{k_1+1},...,z_{k_1+k_2}}_{z_1,...z_{k_1}} := \{z_1, ...z_{k_1}\} \sqcup \Delta \sqcup \{z_{k_1+1}, ..., z_{k_1+k_2}\}$, where we put $z_1, ...z_{k_1} \in \Delta^-$, $z_{k_1+1}, ..., z_{k_1+k_2} \in \Delta^+$ in all three definitions. For α associated to Δ, $z_0, ..., z_{k_1+k_2} \in \partial D - \Delta$ as above and $\gamma_1, ..\gamma_{k_1+k_2}$, we define

1. $\alpha_{\gamma_1,...,\gamma_{k_1}} : \{1, ..., |\Delta| + k_1\} \to R$ as $\alpha_{\gamma_1,...,\gamma_{k_1}}|_{\{k_1+1,...,|\Delta|+k_1\}} = \alpha$ and $\alpha_{\gamma_1,...,\gamma_{k_1}}(i) = \gamma_i$ for $i \in \{1, ..., k_1\}$;
2. $\alpha^{\gamma_{k_1+1},...,\gamma_{k_1+k_2}} : \{1, ..., |\Delta| + k_2\} \to R$ as $\alpha^{\gamma_{k_1+1},...,\gamma_{k_1+k_2}}|_{\{1,...,|\Delta|\}} = \alpha$ and $\alpha^{\gamma_1}(i) = \gamma_i$ for $i \in \{\Delta + 1, ..., \Delta + k_2\}$;
3. $\alpha^{\gamma_{k_1+1},...,\gamma_{k_1+k_2}}_{\gamma_1,...,\gamma_{k_1}} : \{0, ..., |\Delta| + k_1 + k_2\} \to R$ as $(\alpha_{\gamma_1,...,\gamma_{k_1}})^{\gamma_{k_1+1},...,\gamma_{k_1+k_2}}$.

Fix $k_-, k_+ \in \mathbb{Z}_{\geq 0}$, $\Delta = \Delta^- \sqcup \Delta^+ \subset \partial D$ finite subset of ∂D such that $-1, +1 \notin \Delta$, $\alpha : \{1, ..., k_- + k_+\} \to R$ and $\gamma_-, \gamma_+ \in R$. With this is hand, we define the following three classes of moduli spaces.

1. The moduli space of unparametrized α-marked J-holomorphic disks with corners on $\imath$, and a corner γ_- fixed at $-1 \in \partial D$ as

$$\mathcal{M}_{k_-,k_+}(\gamma_-, \emptyset, \alpha, J) := \bigsqcup_{\Delta} \frac{\tilde{\mathcal{M}}_{k_-+1,k_+}(\Delta_{-1}, \alpha_{\gamma_-}, J) \times \{\Delta_{-1}\}}{\operatorname{Aut}(D, -1, +1)}$$

which has virtual dimension

$$\operatorname{virdim}(\mathcal{M}_{k_-,k_+}(\gamma_-, \emptyset, \alpha, J)) = |\gamma_-| - \sum_{-}(n - \alpha(j)) - \sum \alpha(i) + k_- + k_+ - 1$$

2. The moduli space of unparametrized α-marked J-holomorphic disks with corners on $\imath$, and a corner γ_+ fixed at $+1 \in \partial D$ as

$$\mathcal{M}_{k_-,k_+}(\emptyset, \gamma_+, \alpha, J) := \bigsqcup_{\Delta} \frac{\tilde{\mathcal{M}}_{k_-,k_+1}(\Delta^{+1}, \alpha^{\gamma_+}, J) \times \{\Delta^{+1}\}}{\mathrm{Aut}(D, -1, +1)}$$

which has virtual dimension

$$\mathrm{virdim}(\mathcal{M}_{k_-,k_+}(\emptyset, \gamma_+, \alpha, J)) = n - |\gamma_+| - \sum_{-}(n - \alpha(j)) - \sum_{+} \alpha(j) + k_- + k_+ - 1$$

3. The moduli space of unparametrized α-marked J-holomorphic disks with corners on $\imath$, and corners γ_-, γ_+ fixed at $-1, +1 \in \partial D$ respectively, as

$$\mathcal{M}_{k_-,k_+}(\gamma_-, \gamma_+, \alpha, J) := \bigsqcup_{\Delta} \frac{\tilde{\mathcal{M}}_{k_-+1,k_++1}(\Delta_{-1}^{+1}, \alpha_{\gamma_-}^{\gamma_+}, J) \times \{\Delta_{-1}^{+1}\}}{\mathrm{Aut}(D, -1, +1)}$$

which has virtual dimension

$$\mathrm{virdim}(\mathcal{M}_{k_-,k_+}(\gamma_-, \gamma_+, \alpha, J)) = |\gamma_-| - |\gamma_+| - \sum_{-}(n - \alpha(j)) - \sum_{+} \alpha(j) + k_- + k_+ - 1$$

For coherence with what will follow, we will refer as configuration in this class of moduli spaces as α-marked **RR**-pearls of first kind (or $\mathbf{RR}_1$-pearls) with corners on $\imath$ joining γ_- to γ_+.

explain why you call it RR. And on the next page CC, RC etc.

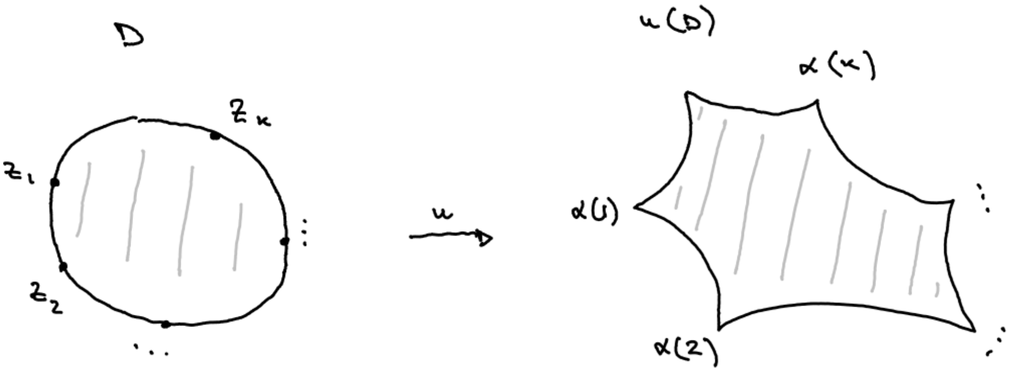

Figure 2.2: A sketch of an α-marked pseudoholomorphic disk with corners.

Let now $f : L \to \mathbb{R}$ be a Morse function on L and $g \in \Gamma(TL)$ be a Riemannian pseudogradient field such that the pair (f, g) is Morse-Smale (for a brief review of the basics of Morse theory, see [AD14]). We denote $\mathrm{Crit}(f)$ the set of critical points of f, $\varphi^{f,g}$ the flow of the negative gradient of

f, and for any critical point $x \in \mathrm{Crit}(f)$ of f we denote by $W^u(x)$ and $W^s(x)$ its unstable and stable manifolds and by

$$\mathcal{M}(y,x) := \frac{W^u(y) \cap W^s(x)}{\mathbb{R}}$$

the moduli space of unparametrized Morse trajectories joining the critical point y to the critical point x of f; for coherence with what will follow, we will also call them **CC**-flowlines. We will denote $|x| \in \mathbb{Z}$ the Morse index of a critical point $x \in \mathrm{Crit}(f)$ of f.

Let $x, y \in \mathrm{Crit}(f)$ be critical points of f, $\gamma_-, \gamma_+ \in R$ be ordered double points of $\imath$, $k_-, k_+ \in \mathbb{Z}_{\geq 0}$ be non-negative integers and $\alpha : \{1, ..., k_- + k_+\} \to R$ a map indexing ordered double points of $\imath$. We define the evaluation maps

$$\mathrm{ev}_{\mathbf{RC}}^{\gamma_-,\alpha} : \mathcal{M}_{k_-,k_+}(\gamma_-, \emptyset, \alpha, J) \to L, \quad \overline{u} \longmapsto l(+1)$$

and

$$\mathrm{ev}_{\mathbf{CR}}^{\gamma_+,\alpha} : \mathcal{M}_{k_-,k_+}(\emptyset, \gamma_+, \alpha, J) \to L, \quad \overline{u} \longmapsto l(-1)$$

to define the following moduli spaces of pearls:

1. the moduli space of α-marked **RC**-pearls with corners on $\imath$ joining γ_- to x as

   $$\mathcal{M}_{k_-,k_+}(\gamma_-, x, \alpha, f, g, J) := \mathrm{ev}_{\mathbf{RC}}^{\gamma_-,\alpha\,-1}(W^s(x))$$

   of virtual dimension

   $$\mathrm{virdim}(\mathcal{M}_{k_-,k_+}(\gamma_-, x, \alpha, f, g, J)) = |\gamma_-| - |x| - \sum_{-}(n - \alpha(j)) - \sum \alpha(i) + k_- + k_+ - 1$$

2. the moduli space of α-marked **CR**-pearls with corners on $\imath$ joining y to γ_+ as

   $$\mathcal{M}_{k_-,k_+}(y, \gamma_+, \alpha, f, g, J) := \mathrm{ev}_{\mathbf{CR}}^{\gamma_+,\alpha\,-1}(W^u(y))$$

   of virtual dimension

   $$\mathrm{virdim}(\mathcal{M}_{k_-,k_+}(y, \gamma_+, \alpha, f, g, J)) = |y| - |\gamma_+| - \sum_{-}(n - \alpha(j)) - \sum \alpha(i) + k_- + k_+ - 1$$

Given a couple $\alpha_1 : \{1, ..., k_-^1 + k_+^1\} \to R$ and $\alpha_2 : \{1, ..., k_-^2 + k_+^2\} \to R$ of maps indexing ordered double points we also define the moduli space of α-marked **RR**-pearls of second kind (or $\mathbf{RR}_2$-pearls) joining γ_- to γ_+ as

$$\mathcal{M}_{k_-^1,k_+^1;k_-^2,k_+^2}(\gamma_-, \gamma_+, \alpha_1, \alpha_2, f, g, J) := (\mathrm{ev}_{\mathbf{RC}}^{\gamma_-,\alpha_1} \times \mathrm{ev}_{\mathbf{CR}}^{\gamma_+,\alpha_2})^{-1}(Q^{f,g})$$

where $Q^{f,g}$ is the image of the embedding

$$(x,t) \in (L - \mathrm{Crit}(f)) \times \mathbb{R}_{\geq 0} \longmapsto (x, \varphi_t^{f,g}(x)) \in L \times L$$

This last moduli space has virtual dimension

$$\begin{aligned}
\mathrm{virdim}(\mathcal{M}_{k_-^1,k_+^1;k_-^2,k_+^2}(\gamma_-,\gamma_+,\alpha_1,\alpha_2,f,g,J)) = \\
(|\gamma_-| - \sum_{-}(n-\alpha_1(j)) - \sum \alpha_1(i) + k_-^1 + k_+^1 - 1) + \\
+ (n - |\gamma_+| - \sum_{-}(n-\alpha_2(j)) - \sum_{+} \alpha_2(j) + k_-^2 + k_+^2 - 1) - (n-1) = \\
= |\gamma_-| - |\gamma_+| - \sum_{-}(n-\alpha_i(j)) - \sum_{+} \alpha_i(j) + k_-^1 + k_+^1 + k_-^2 + k_+^2 - 1
\end{aligned}$$

as expected. In Figure 2.2 we sketched curves in the five classes of moduli spaces we just defined.

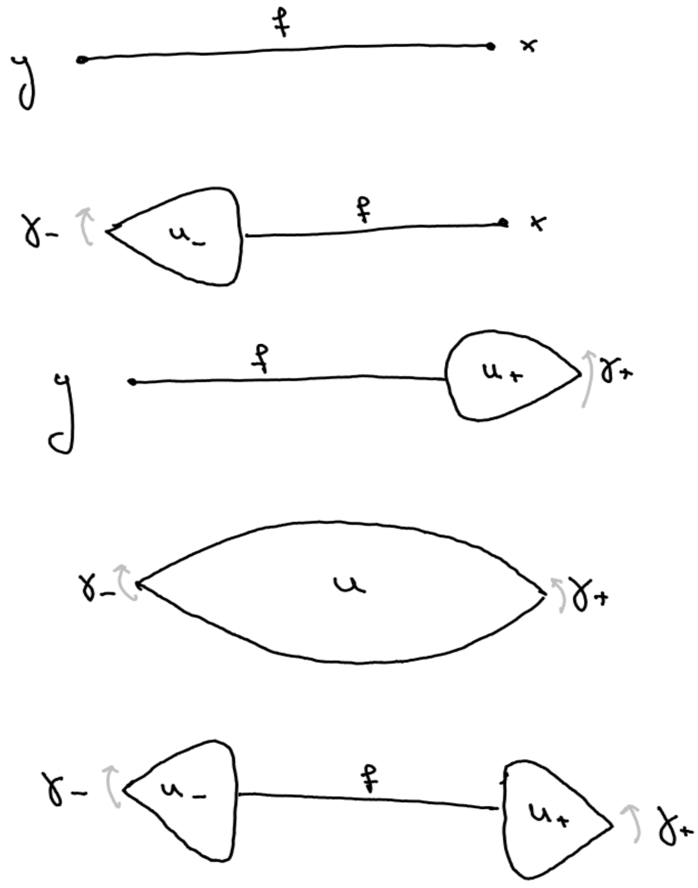

Figure 2.3: Sketches of curves in the five classes of moduli spaces we just defined, for $\alpha =\neq$. From above we have curves of type **CC**, **RC**, **CR**, $\mathbf{RR}^1$ and $\mathbf{RR}^2$.

We have the following definition which will be crucial for transversality of $\mathbf{RR}_2$-pearls.

Definition 2.4. *Consider a finite sequence of pseudoholomorphic disks $v_1, ..., v_k : (D, \partial D) \to (M, \imath(L))$ with corners on $\imath$. We say that $v_1, ..., v_k$ are absolutely distinct if $v_j(D) \not\subset \bigcup_{i\neq j} v_i(D)$ for any $j = 1, ..., k$.*

Given a moduli space $\mathcal{M}$ of some types of pearls, we denote by $\mathcal{M}^*$ the subspace of simple curves, $\mathcal{M}^{\text{abs}}$ the subspace of absolutely distinct curves and $\mathcal{M}^{*,\text{abs}}$ their intersection.

Remark 2.5. *Notice that the energy, or symplectic area, of an α-marked disk strongly depends on α in our exact case:*

$$E(u) = \int_{D-\Delta} u^*\omega = \int_{S^1-\Delta} (\imath \circ l)^*\lambda = \int_{S^1-\Delta} d(h_L \circ l)$$

is the sum of differences of $h_L \circ l$ on "smooth arcs" of ∂D, that is if $u \in \mathcal{M}_{k_-,k_+}(\gamma_-, \gamma_+, \alpha, J)$, then

$$E(u) = \mathscr{A}(\gamma_-) - \mathscr{A}(\gamma_+) + \sum_{-} \mathscr{A}(\alpha(i)) - \sum_{+} \mathscr{A}(\alpha(i))$$

i.e. energy is computed by summing action clockwise. In particular, if u is teardrop $\mathcal{M}_{0,0}(\emptyset, (q,p), J)$ we have

$$E(u) = \mathscr{A}(p,q)$$

Note that ~~what~~ the above directly implies that the finite-energy condition in Definition 1.6 is redundant.

Remark 2.6. *A pseudoholomorphic disk with only one branch jump can not be constant, as "its" l has to be a continous map connecting the two points of L corresponding to the two branches and $\imath$ is chosen to be generic. In particular, luckily, no constant teardrop may bubble off from a sequence of configurations contributing to the differential we are going to define in Section 2.6.*

In the remaining of this chapter, we will sometimes use the following corollaries to Proposition 1.7, which are analogue of some results in [BC07, Section 3] for the generic immersed case. The following statement is Proposition 1.3.1 in [Per19], is his proof right?. ?

Lemma 2.7. *Assume $n \geq 3$. Then for a generic choice of almost complex structure $J \in \mathcal{J}_c(M,\omega)$ and absolutely distinct and simple disks $\overline{u_1} \in \mathcal{M}^*_{k^\alpha_-,k^\alpha_+}(\alpha, J)$ and $\overline{u_2} \in \mathcal{M}^*_{k^\beta_-,k^\beta_+}(\beta, J)$, the set*

$$\{(z_1, z_2) \in \partial D^2 : \; u_1(z_1) = u_2(z_2)\}$$

is finite.

Proof. Let $n \geq 3$ and consider $\mathcal{M}^*_{k^\alpha_-,k^\alpha_+}(\alpha, \Delta_\alpha, J)$ and $\mathcal{M}^*_{k^\beta_-,k^\beta_+}(\beta, \Delta_\beta, J)$. Define $\mathcal{M}^{*,\text{abs}}(\alpha, \beta, \Delta_\alpha, \Delta_\beta, J) \subset \mathcal{M}^*_{k^\alpha_-,k^\alpha_+}(\alpha, \Delta_\alpha, J) \times \mathcal{M}^*_{k^\beta_-,k^\beta_+}(\beta, \Delta_\beta, J)$ to be the space of absolutely distinct elements of $\mathcal{M}^*_{k^\alpha_-,k^\alpha_+}(\alpha, \Delta_\alpha, J) \times \mathcal{M}^*_{k^\beta_-,k^\beta_+}(\beta, \Delta_\beta, J)$. Fix $k \geq 1$ and consider the map

$$\begin{aligned} \varphi : \bigsqcup \mathcal{M}^{*,\text{abs}}(\alpha, \beta, \Delta_\alpha, \Delta_\beta, J) \times (\partial D - \Delta_\alpha)^k \times (\partial D - \Delta_\beta)^k &\longrightarrow L^{2k} \\ (\overline{u_1}, \overline{u_2}, x_1, ..., x_k, y_1, ..., y_k) &\longmapsto (l_1(x_1), l_2(y_1), ..., l_1(x_k), l_2(y_k)) \end{aligned}$$

Then, according to standard theory [MS12], for a generic choice of $J \in J_c(M,\omega)$, $\varphi \pitchfork \Delta^k_L$. Note that

$$\dim(\varphi^{-1}(\Delta^k_L)) = 2n + \sum_{i=1}^{k^\alpha_-}(n - |\alpha(i)|) - \sum_{i=1}^{k^\beta_-}(n - |\beta(i)|) - \sum_{i=1}^{k^\alpha_+}|\alpha(i)| - \sum_{i=1}^{k^\beta_+}|\beta(i)| + k(2-n)$$

and, as $n \geq 3$, we have $\varphi^{-1}(\Delta_L^k) = \emptyset$ for k big enough. Hence fixed $(\overline{u_1}, \overline{u_2}) \in \mathcal{M}^{*,\text{abs}}(\alpha, \beta, \Delta_\alpha, \Delta_\beta, J)$ the set $\{(z_1, z_2) \in (\partial D - \Delta_\alpha) \times (\partial D - \Delta_\beta) : \ l_1(z_1) = l_2(z_2)\}$ is finite. Then, as R, Δ_α and Δ_β are finite sets and fibers of pseudoholomorphic curves are finite, we conclude the proof. ☐

We will often use this Corollary of Proposition 1.7 and Lemma 2.7.

Corollary 2.8 (Corollary 1.1.4 in [Per19]). *Assume that $n \geq 3$ and let $\imath : L \to M$ be a generic Lagrangian immersion. Then there is a generic family $\mathcal{J}_{\text{reg}} \subset \mathcal{J}(M, \omega)$ of compatible almost complex structures on (M, ω) such that for any $J \in \mathcal{J}_{\text{reg}}$ and any non-constant J-holomorphic disk $u : (D, \partial D) \to (M, \imath L)$ with corners and boundary on $\imath$ with finite energy we have the factorization*

$$u = v \circ \pi$$

where $\pi : D \to D$ is a branched covering with branch points in $\text{int}(D)$ and $v : (D, \partial D) \to (M, \imath(L))$ is a simple J-holomorphic disk with corners and boundary on $\imath$ with finite energy such that

Lemma 2.9. *Assume $n \geq 3$. Then for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$ we have that for any simple elements $u \in \mathcal{M}_{k_-, k_+}(\alpha, J)$ the intersection $u^{-1}(\imath(L)) \cap \text{int}(D)$ is finite.*

Proof. Consider the embedded submanifold $\tilde{L} := \imath(L - \pi_1(R))$ of M, where $\pi_1 : L \times L \to L$ is the first projection on L. Define the map

$$\begin{aligned} \phi : \bigsqcup \mathcal{M}^*(\alpha, \Delta, J) \times \text{int}(D)^k &\longrightarrow M^k \\ (\overline{u}, x_1, ..., x_k,) &\longmapsto (u(x_1), ..., u(x_k)) \end{aligned}$$

Then, according to standard theory [MS12], for a generic choice of $J \in \mathcal{J}_c(M, \omega)$, we have that ϕ is transverse to $\tilde{L}^k$. Note that

$$\dim(\phi^{-1}(\tilde{L}^k)) = n - \sum_{i=1}^{k_-} (n - |\alpha(i)|) - \sum_{i=1}^{k_+} |\alpha(i)| + k_- + k_+ 1 + k(2 - n)$$

and, as $n \geq 3$, we have $\phi^{-1}(\tilde{L}^k)$ for k big enough. As R is a finite set, we conclude the proof. ☐

Corollary 2.10. *Assume $n \geq 3$. Then, for a generic choice of almost complex structure $J \in \mathcal{J}_c(M, \omega)$ and for any couple of simple disks $u, v : (D, \partial D) \to (M, \imath(L))$ with corners and boundary on $\imath$ such that that $v(D) \cap u(D)$ is infinite, we have that either $v(D) \subset u(D)$ and $v(\partial D) \subset u(\partial D)$ or $u(D) \subset v(D)$ and $u(\partial D) \subset v(\partial D)$.*

Proof. We consider a compatible almost complex structure satisfying Lemma 2.7 and Lemma 2.9.According to Lemma 2.7, u and v are not absolutely distinct. Assume without loss of generality that $u(D) \subset v(D)$. According to Lemma 2.9, $u(\partial D) \cap v(\text{int}(D))$ is a finite set, so that $u(\partial D - \text{finite set}) \subset v(\partial D)$. We conclude by continuity of u. ☐

2.3 Transversality of pearls

From now on we will only consider maps α of the type $\{1, ..., k_+\} \to R$ indexing only outgoing corners which satisfy $\mathscr{A}(\alpha(i)) > 0$. The reason for that is that when proving that our pearly differential is well defined, we are confronted with bubbling of trees of pseudoholomorphic disks (with, of course, at least one branch jump) with positive area. When $k_+ = 0$ we write $\alpha = \emptyset$ and we drop α from the notation of the moduli spaces.

Case 1: $n \geq 3$

Lemma 2.11. *Assume $n \geq 3$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q) \in R$ be an ordered double point of $\imath$ such that $\mathscr{A}(p, q) > 0$. Then for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, any non-constant element of $\mathcal{M}_{0,k_+}((p, q), \emptyset, \alpha, J)$ and $\mathcal{M}_{0,k_+}(\emptyset, (q, p), \alpha, J)$ is simple.*

Proof. Pick $J \in \mathcal{J}_{\text{reg}}$ from Corollary 2.8 and consider $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), \emptyset, \alpha, J)$ non-constant. Notice that as $\mathscr{A}(p, q) > 0$ and (p, q) is seen as incoming, we have that $\alpha(i) \neq (q, p)$ for any i. u is multicovered by Corollary 2.8, i.e. there is a branched covering $\pi : D \to D$ of degree $d \geq 1$ and a simple disk $\overline{v} := (v, l_v, \Delta, \alpha)$ with corners and boundary on $\imath$ such that $u = v \circ \pi$. Then v has a corner at $\pi(-1)$. As π is a cover when restricted to ∂D, we have $\pi(d) = z^d$ for $z \in \partial D$; then, as $+1$ is a smooth point for u, it directly follows that d is odd. We show that $\Delta \cap \pi^{-1}(\pi(-1)) = \emptyset$, which implies that $\pi^{-1}(\pi(-1)) = \{1\}$ and $d = 1$. Write $\alpha(i) = (q_i, p_i)$. We have:

$$p = \lim_{\theta \to 0^+} l_u(e^{i\theta}(-1)) = \lim_{\theta \to 0^+} l_v(e^{2i\theta}(-1)) = \lim_{\theta \to 0^+} l_u(e^{i\theta}\pi(z_i)) = p_1$$

a contradiction. The case of $\mathcal{M}_{0,k_+}(\emptyset, (q, p), \alpha, J)$ is similar. □

Corollary 2.12. *Assume $n \geq 3$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q) \in R$ be an ordered double point of $\imath$ such that $\mathscr{A}(p, q) > 0$ and let $x, y \in \text{Crit}(f)$ be critical points of f. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, if $W^u(y) \cap \{p, q\} = \emptyset$, any element of $\mathcal{M}_{0,k_+}(y, (q, p), \alpha, J)$ is simple, while, if $\{p, q\} \cap W^s(y) = \emptyset$, then any element of $\mathcal{M}_{0,k_+}((p, q), x, \alpha, J)$ is simple.*

Proof. As $W^u(y) \cap \{p, q\} = \emptyset$, then any element of $\mathcal{M}_{0,k_+}(y, (q, p), \alpha, J)$ is non-constant. We conclude using Lemma 2.11. □

Lemma 2.13. *Assume $n \geq 3$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q), (r, s) \in R$ be ordered double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, any element of $\mathcal{M}_{0,k_+}((p, q), (r, s), \alpha, J)$ is simple.*

Proof. Pick $J \in \mathcal{J}_{\text{reg}}$ from Corollary 2.8 and consider $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), (r, s), \alpha, J)$. Then u is non-constant as $p, q, r, s \in L$ are pairwise distinct, and is hence multicovered by Corollary 2.8, that it, there is a branched covering $\pi : D \to D$ of degree $d \geq 1$ and a simple disk v with corners and boundary on $\imath$ such that $u = v \circ \pi$. In particular, $\pi(z) = z^d$ for $d \in \partial D$. We show that $d = 1$. Assume $d \geq 2$. We have

$$0 < E(u) = \mathscr{A}(p,q) - \mathscr{A}(r,s) - \sum_{i=1}^{k_+} \mathscr{A}(\alpha(i))$$

so that either $\mathscr{A}(p,q) > 0$ or $\mathscr{A}(r,s) < 0$. The first case implies that $\alpha(i) \neq (q,p)$ for any i by our assumptions on α, while the second one implies that $\alpha(i) \neq (r,s)$ for any i. Assume without loss of generality that $\mathscr{A}(p,q) > 0$. First, we show that $+1 \notin \pi^{-1}(\pi(-1))$: if that were the case, we would have d even and

$$p = \lim_{\theta \to 0^+} l_u(e^{i\theta}(-1)) = \lim_{\theta \to 0^+} l_v(e^{2i\theta}(+1)) = \lim_{\theta \to 0^+} l_v(e^{i\theta}(+1)) = s$$

a contradiction. Similarly, one shows $\Delta \cap \pi^{-1}(\pi(-1)) = \emptyset$, as $\alpha(i) \neq (q,p)$ for any i. It follows $|\pi^{-1}(\pi(-1))| = 1$ and hence $d = 1$. □

Case 2: $n \leq 2$

In the case $n \leq 2$ we are not a priori able to count corners to conclude the arguments about simplicity, as disks are not usually multicovered. To show simplicity of the disks we are interested to in this case, the idea is to consider only the dimensions one is interested to in order to define our cohomology, that is, 0 and 1, and then show that if a curve is not simple, new curves arise, which will lie in manifolds of either negative or too high dimension. To do that, we will analyze the possible shapes of connected components of the complement of the frame of a disk. There are two kinds of problematic behaviours of non-simple curves that we want to avoid:

Remark 2.14. *In general, a corner point of a disk may be a singular point for the frame. Indeed, consider the map $u(z) = z^{\frac{5}{2}}$ from the disk $D = \mathcal{H} \cup \{\infty\}$ to the sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and the immersed Lagrangian on the sphere given by the compactification of $\mathbb{R} \cup i\mathbb{R}$. Then, u is a disk with corners in $z = 0$ and $z = 0\infty$, which are singular points of the frame $W(u)$, see Figure 2.3. In this case, Proposition 1.7 gives us a decomposition of u into three simple disks with corners. Two are $u_{1,2}(z) = z^{\frac{1}{2}}$, which have the same type of corner as u, while the other, $u_3(z) = iz^{\frac{3}{2}}$ has the inverse corners than u.*

Remark 2.15. *In the decomposition into simple pieces of a pseudoholomorphic disk with corners, new (types of) corners may arise. Consider the teardrop $u \in \mathcal{M}_{0,0}((p,q), \emptyset, \alpha, J)$ in Figure 2.3, which is of course not simple. Then we can decompose u via Proposition 1.7 into three simple pieces u_1, u_2 and u_3. u_1 is a disk with the original incoming (p,q) corner plus the incoming corners $(a,b), (d,c), (d,c), (a,b)$, while $u_2 = u_3$ is a disk with outgoing corners of type $(a,b), (d,c)$.*

We now show that in our setup, the phenomena listed in Remark 2.14 and 2.15 can't happen (in the second case, we have to bound some Morse trajectory to the teardrop).

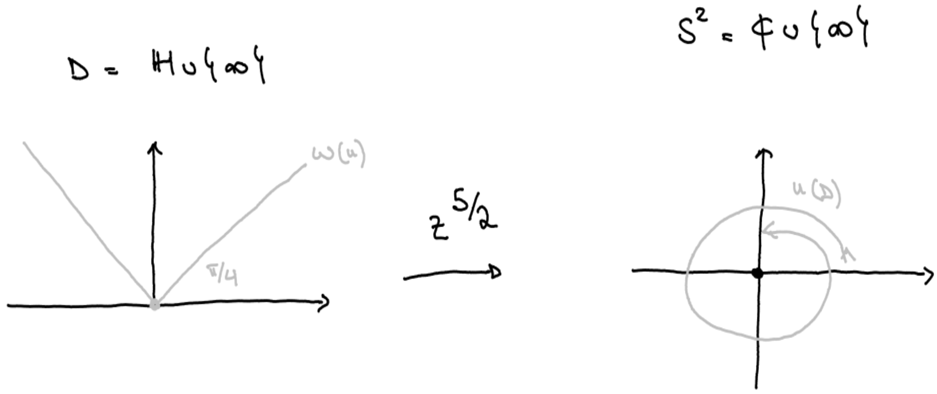

Figure 2.4: A schematic representation of what goes on with the curve $u(z) = z^{\frac{5}{2}}$ from the unit disk to the unit sphere. Note that the image is of cour~~n~~se not simple nor multiply covered and that the non-trivial frame passes trough the corner point $0 \in D$.

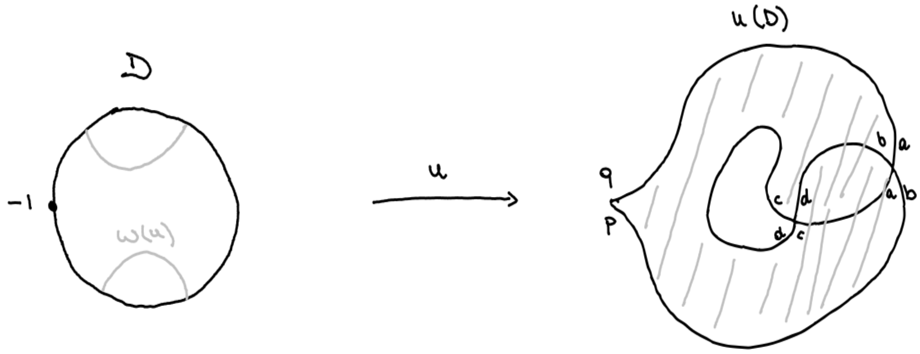

Figure 2.5: A teardrop $u \in \mathcal{M}_{0,0}(\gamma, \emptyset, J)$ which is non simple and whose simple decomposition will contain new types of corners. On the disk D we see $W(u) - \partial D$ in gray.

Lemma 2.16. *Assume $n \leq 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p,q), (r,s) \in R$ be ordered double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct and $|p,q| - |r,s| - 1 \leq 1$ and $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p,q),(r,s),\alpha, J)$. Then the elements of Δ_{-1}^{+1} are smooth points of the frame $\mathcal{W}(u)$ of u.*

Proof. As $p, q, r, s \in L$ are pairwise distinct, u is a non-constant disk, so that

$$E(u) = \mathscr{A}(p,q) - \mathscr{A}(r,s) - \sum_{i=1}^{k_+} \mathscr{A}(\alpha(i)) > 0$$

Notice that if $\mathscr{A}(p,q) > 0$ and $\mathscr{A}(r,s) < 0$ we have $|p,q| \geq 3$ and $-|r,s| = |s,r| - n \geq 3 - 2 = 1$, so that $|p,q| - |r,s| - 1 \geq 3$ which contradicts the assumption of the Lemma. We assume that $\mathscr{A}(p,q) > 0$

(and hence $\mathscr{A}(r,s) \geq 0$).
Assume that -1 is a singular point of the frame $\mathcal{W}(u)$: then, by Section 1.2.2 in [Per19], there is an open neighbourhood $\Omega \subset D$ of -1 such that $\mathcal{W}(u) \cap \Omega$ is a union of (counteclockwise ordered) arcs $r_1, ..., r_m$ (whose number and slope depend on the order of u near -1 and on the Kähler angles between the two branches of L meeting at $u(-1)$). In particular, r_1 and r_m correspond to $\partial D \subset \mathcal{W}(u)$. Always following [Per19], as -1 corresponds to an incoming corner of type (p,q), we have that the r_{2j}'s are arcs with image on the q-branch of $\imath(L)$, while the r_{2j+1}'s are arcs with image on the p-branch of L. We label by D_i the connected component of $D - W(u)$ which near -1 is bounded by the arcs r_i and r_{i+1}. In the decomposition of u into simple pieces, D_i induces a simple (and in particular non-constant) disk $v_i : (D, \partial D) \to (M, \imath(L))$ with incoming corner of type (p,q) if i is odd and incoming corner of type (q,p) is i is even. Consider v_i with i even. We analyze all the possible structures of v_i case by case.

1. Of course, v_i is not a teardrop, as in this case we would have $E(v_i) = \mathscr{A}(q,p) < 0$ by assumption, a contradiction.

2. v_i might have outgoing corners of type $\alpha(i)$, for some $i \in \{1, ..., k_+\}$, and of type (r,s); anyway, in all possible combinations of this type we would have $E(v_i) < 0$ as $\mathscr{A}(r,s) \geq 0$ and $\mathscr{A}(\alpha(i)) > 0$ by assumption, a contradiction.

3. v_i might have an outgoing corner of type (s,r) (this may e.g. be possible if $+1$ is also a singular point of $\mathcal{W}(u)$) (which implies it would have no outgoing corners of type (r,s)). In this case we would have
$$E(v_i) = \mathscr{A}(q,p) - \mathscr{A}(s,r) = -E(u) < 0$$
a contradiction. Similarly, writing $\alpha(i) = (q_i, p_i)$, v_i might have corners of type (p_i, q_i) (this might e.g. be possible if z_i is also a singular point of $\mathcal{W}(u)$); however, this also leads to a contradiction, as we would have
$$E(v_i) = \mathscr{A}(q,p) + \mathscr{A}(\alpha(i)) < 0$$
as from $E(u) > 0$ it follows $\mathscr{A}(p,q) > \mathscr{A}(\alpha(i))$.

4. v_i might have corners which do not involve p, q, r, s or components of the $\alpha(i)$'s as a result of the presence of a connected component of $D - \mathcal{W}(u)$ of the same kind as the one presented in Remark 2.15. However, also in this case we would end up with $E(v_i) < 0$, as those "new" corners bound a non-constant disk.

5. v_i might look like combinations of the cases above, still implying $E(v_i) < 0$.

We conclude that $E(v_i) < 0$, contradicting the fact that -1 is a singular point of $\mathcal{W}(u)$ if $\mathscr{A}(p,q) > 0$. The case $\mathscr{A}(r,s) < 0$ is very similar and is hence omitted. We also omit the proofs that $+1$ and $z_i \in \Delta$ are smooth point of $\mathcal{W}(u)$, as they proceed in the exact same manner. $\square$

Lemma 2.17. *Assume $n \leq 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q) \in R$ be an ordered double point of $\imath$ such that $\mathscr{A}(p, q) > 0$ and let $x, y \in \mathrm{Crit}(f)$ be critical points of f such that $\{p, q\} \cap W^s(y) = \emptyset$ and $\{p, q\} \cap W^s(y) = \emptyset$. Let $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), x, \alpha, J)$ and $\overline{\tilde{u}} = (\tilde{u}, \tilde{l}, \tilde{\Delta}, \alpha) \in \mathcal{M}_{0,k_+}(y, (q, p), \alpha, J)$. Then the elements of Δ_{-1} are smooth points of the frame $\mathcal{W}(u)$ of u, while the elements of $\tilde{\Delta}^{+1}$ are smooth points of the frame $\mathcal{W}(\tilde{u})$ of $\tilde{u}$.*

Proof. The proof is essentially a "sub-proof" of the proof of Lemma 2.16. □

Before handling the second pathological behaviour of the frame in full generality, we consider the example from Remark 2.15 with a Morse condition. Let $(p, q) \in R$ and $x \in \mathrm{Crit}(f)$ such that $|p, q| - |x| - 1 \leq 2$ and pick $u \in \mathcal{M}_{0,0}((p, q), x, J)$ with the frame as in Remark 2.15. Then, u decomposes into simple pieces $u_1 \in \mathcal{M}_{0,4}((p, q), x, \alpha, J)$ and $u_2 = u_3 \in \mathcal{M}_{0,0}((c, d), (a, b), J)$, with α as described in Remark 2.15. This implies that

$$|p, q| - |x| - 2|b, a| - 2|c, d| + 4 - 1 \geq 0 \text{ and } |c, d| - |a, b| - 1| \geq 0$$

The last estimate tells us that $|c, d| + |b, a| \geq n+1 = 3$. It follows from $|p, q| - |x| - 2|b, a| - 2|c, d| + 4 - 1 \geq 0$ that $|p, q| - |x| - 1| \geq 2|b, a| + 2|c, d| - 4 \geq 2$, a contradiction. Notice that the key point for this observation is that the "new" corners are counted twice in one of the resulting curves.

Lemma 2.18. *Assume $n = 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q), (r, s) \in R$ be ordered double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct and $|p, q| - |r, s| - 1 \leq 1$ and $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), (r, s), \alpha, J)$. Consider the decomposition $v_1, ..., v_m : (D, \partial D) \to (M, \imath(L))$ of u into simple disks from Proposition 1.7. Then the corners of $v_1, ..., v_k$ are induced by corners of u.*

Proof. For simplicity we assume $\alpha = \emptyset$, as the proof differs only in notation. The possibilities of anomaly fall into two cases and combinations of them up to reordering of corners and singular points of the frame: the terminal part of a disk may be in the image of the interior (much like in the example in Remark 2.15) or not, see Figure 2.6 and 2.7. We show that both of these cases can not occour by sticking to the orderings displayed in Figure 2.6 and 2.7, but the computations for the other cases are very similar.

Consider $u \in \mathcal{M}_{0,0}((p, q), (r, s), J)$ with an image of the form sketched in Figure 2.6. Then, u factors via Proposition 1.7 into simple disks $u_1 \in \mathcal{M}_{0,4}((p, q), (r, s), \alpha, J)$ and $u_2 = u_3 \in M_{0,0}((d, c), (b, a), J)$, where $\alpha(1) = (a, b)$, $\alpha(2) = (d, c)$, $\alpha(3) = (d, c)$ and $\alpha(4) = (a, b)$. It follows that

$$|p, q| - |r, s| - 2|a, b| - 2|d, c| + 4 - 1 \geq 0 \text{ and } |d, c| - |b, a| - 1 \geq 0$$

From the last estimate it follows that $|d, c| + |a, b| \geq 1 + n = 3$. Combining it with the first estimate we get

$$|p, q| - |r, s| - 1 \geq 2|a, b| + 2|d, c| - 4 \geq 6 - 4 = 2$$

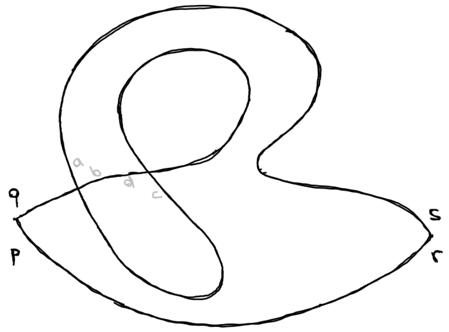

Figure 2.6: In this configuration, we see the terminal part of the disk being part of the image of the interior. The double points in gray are corners for the disks in the decomposition of this configuration into simple pieces.

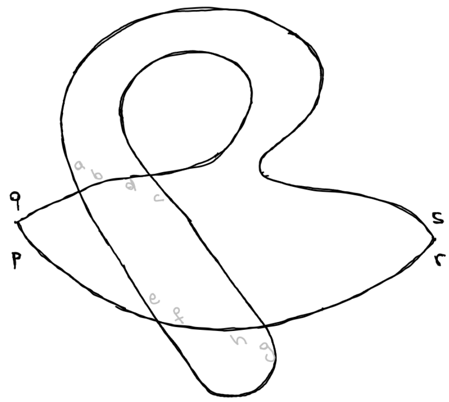

Figure 2.7: In this configuration, we see the terminal part of the disk not being part of the image of the interior. The double points in gray are corners for the disks in the decomposition of this configuration into simple pieces.

contradicting our assumption.
Consider $u \in \mathcal{M}_{0,0}((p,q),(r,s),J)$ with an image of the form sketched in Figure 2.7. Then, u factors via Proposition 1.7 into simple disks $u_1 \in \mathcal{M}_{1,2}(\alpha_1, J)$, $u_2 \in \mathcal{M}_{0,4}(\alpha_2, J)$, $u_3 \in \mathcal{M}_{0,2}(\alpha_3, J)$ and $u_4 \in \mathcal{M}_{0,5}(\alpha_4, J)$, where

1. $\alpha_1(1) = (p,q)$, $\alpha_1(2) = (f,e)$ and $\alpha_1(3) = (a,b)$;
2. $\alpha_2(1) = (e,f)$, $\alpha_2(2) = (h,g)$, $\alpha_2(3) = (c,d)$ and $\alpha_2(4) = b,a)$;
3. $\alpha_3(1) = (g,h)$ and $\alpha_3(2) = (f,e)$;
4. $\alpha_4(1) = (r,s)$, $\alpha_4(2) = (a,b)$, $\alpha_4(3) = (d,c)$, $\alpha_4(4) = (d,c)$ and $\alpha_4(5) = (g,h)$.

This implies that

$$|p,q| - |f,e| - |a,b| \geq 0, \quad n - |e,f| - |h,g| - |c,d| - |b,a| + 1 \geq 0$$
$$n - |g,h| - |f,e| - 1 \geq 0, \quad n - |r,s| - |a,b| - 2|d,c| - |g,h| + 2 \geq 0$$

implying $|h,g| + |e,f| \geq 1 + n$ and $|d,c| + |a,b| \geq n - 1 + |e,f| + |h,g|$. Moreover, summing the first and the fourth inequalities we get $|p,q| - |r,s| - 1 \geq 2|a,b| + 2|d,c| + |g,h| + |f,e| - 4 - 1 \geq 2n - 2 + 2n + 1 + n - 5 = 5n - 6 \geq 4$, contradicting our assumption. $\square$

Remark 2.19. *Notice that the proof of Lemma 2.18 does not work for $n = 1$.*

In the same way as Lemma 2.18, one proves that:

Lemma 2.20. *Assume $n = 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q) \in R$ be an ordered double point of $\imath$ such that $\mathscr{A}(p, q) > 0$ and let $x \in \mathrm{Crit}(f)$ be a critical point of f such that $\{p, q\} \cap W^s(y) = \emptyset$. Let $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), x, \alpha, J)$ and consider the decomposition $v_1, ..., v_m : (D, \partial D) \to (M, \imath(L))$ of u into simple disks from Proposition 1.7. Then the corners of $v_1, ..., v_k$ are induced by corners of u.*

Lemma 2.21. *Assume $n = 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q), (r, s) \in R$ be ordered double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct and $|p, q| - |r, s| - 1 \leq 1$. Then any element of $\mathcal{M}_{0,k_+}((p, q), (r, s), \alpha, J)$ is simple.*

Proof. Let $\overline{u} = (u, l, \Delta, \alpha) \in \mathcal{M}_{0,k_+}((p, q), (r, s), \alpha, J)$, then u is non-constant. We consider the connected components $D_1, ..., D_m$ of $D - W(u)$. If there is $j \in \{1, ..., m\}$ such that $\Delta^{+1}_{-1} \cap \overline{D_j} = \emptyset$ then D_j induces a smooth non-constant pseudoholomorphic disk by Proposition 1.7 and Lemma 2.16 and 2.18. Assume without loss of generality that $\mathscr{A}(p, q) > 0$, so that $\mathscr{A}(r, s) \geq 0$. Then, $-1 \in \overline{D_i}$ for any $i \in \{1, ..., m\}$, as otherwise we would have disks with corners with negative energy. Then, by Lemma 2.16, $m = 1$ and u is multicovered. Exactly as in the case $n \geq 3$, we immediately get that u is simple. □

In the same way as Lemma 2.21, one proves that:

Lemma 2.22. *Assume $n = 2$. Let $k_+ \geq 0$ and $\alpha : \{1, ..., k_+\} \to R$ a map indexing ordered double points of $\imath$, $(p, q) \in R$ be an ordered double point of $\imath$ such that $\mathscr{A}(p, q) > 0$ and let $x, y \in \mathrm{Crit}(f)$ be critical points of f such that $|p, q| - |x| - 1 \leq 1$ and $|y| - |q, p| - 1 \leq 1$. Then if $W^u(y) \cap \{p, q\} = \emptyset$, any element of $\mathcal{M}_{0,k_+}(y, (q, p), \alpha, J)$ is simple, while, if $\{p, q\} \cap W^s(y) = \emptyset$, then any element of $\mathcal{M}_{0,k_+}((p, q), x, \alpha, J)$ is simple.*

$\mathbf{RR}_2$-pearls are made of absolutely distinct elements for any n

According to standard theory of transversality of moduli spaces of pseudoholomorphic curves curves (see [MS12]), tuples of curves have to be absolutely distinct in order to prove transversality of their moduli spaces (see also [BC07]). We now prove that generically $\mathbf{RR}_2$-pearls are absolutely distinct for any $\cancel{n \in \mathbb{Z}_{\geq 1}}$. $n \geq 1$.

Lemma 2.23. *Let $k^1_+, k^2_+ \geq 0$ and $\alpha_1 : \{1, ..., k^1_+\} \to R$, $\alpha_2 : \{1, ..., k^2_+\} \to R$ maps indexing double point of $\imath$, $(p, q), (s, r) \in R$ double points of $\imath$ such that $p, q, s, r \in L$ are pairwise distinct and $|p, q| - |s, r| - 1 \leq 2$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, $\mathcal{M}_{0,k^1_+;0,k^2_+}((p, q), (r, s), \alpha, f, g, J)$ is made of absolutely distinct pearls.*

Proof. Let $(\overline{u_1}, \overline{u_2}) \in \mathcal{M}_{0,k^+_1;0,k^+_2}((p, q), (r, s), \alpha, f, g, J)$. If $n \geq 3$ we use Corollary 2.10: assume that u_1 and u_2 are not absolutely distinct, then without loss of generality $u_1(\partial D) \subset u_2(\partial D)$ which directly leads to a contradiction in any dimension, as this would mean that u_2 has an incoming corner of type (p, q), which can not happen as $\mathscr{A}(p, q) > 0$ and p, q, s, r are pairwise distinct. Assume now $n \leq 2$. The existence of u_1 and u_2 implies that $\mathscr{A}(p, q) > 0$ and $\mathscr{A}(r, s) > 0$. Then,

$$|p, q| - |s, r| - 1 = |p, q| + |r, s| - n - 1 \geq 6 - 2 - 1 = 3$$

contradicting the assumptions and hence proving the claim. □

The case of (p, q) to (q, p) pearls

We still have to deal with **RR** (both of type 1 and 2) pearls from a double point $(p, q) \in R$ to itself or to $(q, p) \in R$. As $\mathscr{A}(p, q) = -\mathscr{A}(q, p)$ it is immediate that there are no non-constant **RR**-pearls from a double point to itself. Moreover, as the next lemma points out, there are no **RR**-pearls in interesting virtual dimensions (that is, 0 and 1 as usual) whenever $n \leq 3$, which makes our task a bit easier as it allows us to use the dichotomy simple-multiply covered from Corollary 2.8. We will only work with pearls with no α-markings, as those are the ones we need to define the differential.

Lemma 2.24. *Assume $n \leq 3$. Then, if $(p, q) \in R$ satisfies $|p, q| - |q, p| - 1 \leq 1$, the moduli spaces $\mathcal{M}_{0,0}((p, q), (q, p), J)$ and $\mathcal{M}_{0,0;0,0}((p, q), (q, p), f, g, J)$ are empty.*

Proof. Assume that $\mathcal{M}_{0,0}((p, q), (q, p), J) \neq \emptyset$ and pick $\overline{u} = (u, l) \in \mathcal{M}_{0,0}((p, q), (q, p), J)$. Then u is non-constant by definition of the moduli space, hence $\mathscr{A}(p, q) > 0$, so that $1 \geq |p, q| - |q, p| - 1 = 2|p, q| - n - 1 \geq 5 - n$, which directly implies $n \geq 4$. The same for $\mathcal{M}_{0,0;0,0}((p, q), (q, p), f, g, J)$. □

Remark 2.25. *A very similar proof shows that under the same conditions of the last lemma the moduli spaces $\mathcal{M}_{0,k_1^+;0,k_2^+}((p, q), (q, p), \alpha_1, \alpha_2, f, g, J)$ are empty while the moduli spaces $\mathcal{M}_{0,k_+}((p, q), (q, p), \alpha, J)$ only contain constant disks.*

Assume $n \geq 4$. Recall that by the results of Section 2.3, teardrops are simple in this case. Consider a non absolutely distinct element

$$(\overline{u_1}, \overline{u_2}) \in \mathcal{M}_{0,0;0,0}((p, q), (q, p), f, g, J) - \mathcal{M}^{\text{abs}}_{0,0;0,0}((p, q), (q, p), f, g, J)$$

Then, as $u_1(D) \cap u_2(D)$ is infinite, we have either $u_1(\partial D) \subset u_2(\partial D)$ and $u_1(D) \subset u_2(D)$ or $u_2(\partial D) \subset u_1(\partial D)$ and $u_2(D) \subset u_1(D)$. In any of either cases we have $u_1(D) = u_2(D)$ and $u_1(\partial D) = u_2(\partial D)$ as u_1 and u_2 have the same symplectic area. In particular, Theorem 4.13 in [Laz11], which according to [Per19] translates without modification to the immersed case, tells us that u_1 and u_2 are reparametrisation of each other, that is, $\overline{u_1}$ and $\overline{u_2}$ can be seen as the same element in $\mathcal{M}_{0,0}(\emptyset, (q, p), J)$. In particular, an element $\overline{u_2} \in \mathcal{M}_{0,0}(\emptyset, (q, p), J)$ determines a unique element $(\overline{u_1}, \overline{u_2}) \in \mathcal{M}_{0,0;0,0}((p, q), (q, p), f, g, J) - \mathcal{M}^{\text{abs}}_{0,0;0,0}((p, q), (q, p), f, g, J)$. We then conclude that the map

$$\begin{aligned} \psi_1 : \mathcal{M}_{0,0;0,0}((p, q), (q, p), f, g, J) - \mathcal{M}^{\text{abs}}_{0,0;0,0}((p, q), (q, p), f, g, J) &\longrightarrow \mathcal{M}_{0,0}(\emptyset, (q, p), J) \\ (\overline{u_1}, \overline{u_2}) &\longmapsto \overline{u_2} \end{aligned}$$

is a bijection.

Consider a non simple element

$$\overline{u} \in \mathcal{M}_{0,0}((p, q), (q, p), J) - \mathcal{M}^*_{0,0}((p, q), (q, p), J)$$

By assumption u is non-constant, so that $\mathscr{A}(p,q) > 0$ and u is multicovered by a branched covering of degree 2. Hence we write any element $\overline{u} \in \mathcal{M}_{0,0}((p,q),(q,p),J)$ as $u = v_u \circ \pi_u$, where $\pi_i : D \to D$ is a branched covering of degree 2 and v_u is a simple teardrop. Notice that, as $v(D) = u(D)$ and $v(\partial D) = u(\partial D)$, the class $\overline{v_u} \in \mathcal{M}_{0,0}(\emptyset,(q,p),J)$ does not depend on the choice of v_u again by Theorem 4.13 in [Laz11]. We then conclude that the map

$$\begin{aligned} \psi_2 : \mathcal{M}_{0,0}((p,q),(q,p),J) - \mathcal{M}^*_{0,0}((p,q),(q,p),J) &\longrightarrow \mathcal{M}_{0,0}(\emptyset,(q,p),J) \\ \overline{u} &\longmapsto \overline{v_u} \end{aligned}$$

is a bijection. In particular it follows that

$$\psi_2^{-1} \circ \psi_1 : \mathcal{M}_{0,0;0,0} - \mathcal{M}^{\mathrm{abs}}_{0,0;0,0}((p,q),(q,p),f,g,J) \longrightarrow \mathcal{M}_{0,0} - \mathcal{M}^*_{0,0}((p,q),(q,p),J)$$

is a bijection. This tells us that bad pearls joining (p,q) to (q,p) come in pairs in dimension 0.

Let's discuss this further

here I realized too late that my proof to show that $\mathcal{M}_{0,0}(\emptyset,(q,p),J)$ is finite (altough it it $n-1/2$-dimensional), and hence that everything with the differential is well-defined, is wrong. One possibility is to assume that $\mathscr{A}(p,q) > 0$ implies $|p,q| \geq \frac{n+2}{2}$, but this is quite strong and a priori eliminates $\mathbf{RR}_2$ configurations here and other configurations in the following structures (PSS, product).

Summary of transversality

In summary, combining the result of this section and standard theory on transversality of moduli spaces of simple curves in [MS12], we get the following.

Proposition 2.26. *Let $k_+, k_+^1, k_+^2 \geq 0$ and $\alpha : \{1,...,k_+\} \to R$, $\alpha_1 : \{1,...,k_+^1\} \to R$ and $\alpha_2 : \{1,...,k_+^2\} \to R$ maps indexing double point of $\imath$, $(p,q),(r,s) \in R$ double points of $\imath$ such that $p,q,r,s \in L$ are pairwise distinct and $x,y \in \mathrm{Crit}(f)$ critical points of f. Then there is a generic family $\mathcal{J}_{\mathrm{tr}} \subset \mathcal{J}_c(M,\omega)$ such that for any $J \in \mathcal{J}_{\mathrm{tr}}$ the following points hold:*

- *Assume $|p,q| - |r,s| - 1 \leq 1$. Then the moduli space $\mathcal{M}_{0,k_+}((p,q),(r,s),\alpha,J)$ is either empty or a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*
- *Assume $|p,q| - |x| - 1 \leq 1$ and $W^s(x) \cap \{p,q\} = \emptyset$. Then the moduli space $\mathcal{M}_{0,k_+}((p,q),x,\alpha,f,g,J)$ is either empty or a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*
- *Assume $|y| - |r,s| - 1 \leq 1$ and $W^u(y) \cap \{r,s\} = \emptyset$. Then the moduli space $\mathcal{M}_{0,k_+}(y,(r,s),\alpha,f,g,J)$ is either empty or a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*
- *Assume $|p,q| - |r,s| - 1 \leq 1$. Then the moduli space $\mathcal{M}_{0,k_+^1;0,k_+^2}((p,q),(r,s),\alpha_1,\alpha_2,f,g,J)$ is either empty of a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*

- *Assume $|p,q| - |q,p| - 1 \leq 1$. Then the moduli space $\mathcal{M}^*_{0,0}((p,q),(q,p),J)$ is either empty of a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*

- *Assume $|p,q| - |q,p| - 1 \leq 1$. Then the moduli space $\mathcal{M}^{\text{abs}}_{0,0;0,0}((p,q),(q,p),f,g,J)$ is either empty of a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*

2.4 Compactness of moduli spaces of pearls in dimension 0

In this section we prove that the moduli spaces of pearly trajectories which will play a role in the definition of quantum homology are compact when the virtual dimension is zero.

Lemma 2.27. *Let $x \in \mathrm{Crit}(f)$ be a critical point of f and $(p,q) \in R$ be a double point of $\imath$ such that $|x| - |p,q| = 1$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M,\omega)$, the moduli space $\mathcal{M}_{0,0}(x,(p,q),f,g,J)$ is compact.*

Proof. By Gromov compactness for pseudoholomorphic curves with boundary and corners on totally real immersions (see [IS02]) and the fact that we are working in the exact setting, a sequence in $\mathcal{M}_{0,0}(x,(p,q),f,g,J)$ converges to a broken pearly trajectory of the form

$$\big(u_1, ..., u_{d-1}, ([\overline{u_d}], v_d), ([\overline{u_{d+1}}], v_{d+1}), ..., ([\overline{u_{d+m}}], v_{d+m})\big)$$

for $d \geq 1, m \geq 0$, where

1. for $1 \leq i \leq d-1$, $u_i \in \mathcal{M}(x_i, x_{i+1}, f, g)$ is a Morse trajectory between critical points $x_i, x_{i+1} \in \mathrm{Crit}(f)$; moreover, $x_1 = x$;
2. $[\overline{u_d}] \in \mathcal{M}_{0,k^d_+}(x_d, \gamma_{d+1}, \alpha_d, f, g, J)$ is a **CR**-pearl and v_d is a family of pseudoholomorphic trees attached to u_d along the associated Δ_d;
3. for $d+1 \leq i \leq d+m$, either $[\overline{u_i}] \in \mathcal{M}_{0,k^i_+}(\gamma_i, \gamma_{i+1}, \alpha_i, J)$ or $[\overline{u_i}] = ([\overline{u_{i,1}}], [\overline{u_{i,2}}])$, where $[\overline{u_{i,1}}] \in \mathcal{M}_{0,k^{i,1}_+}(\gamma_i, \emptyset, \alpha_{i,1}, J)$ and $[\overline{u_{i,2}}] \in \mathcal{M}_{0,k^{i,2}_+}(\emptyset, \gamma_{i+1}, \alpha_{i,2}, J)$ such that $u_{i,1}(+1) = u_{i,2}(-1)$, both with trees of pseudoholomorphic disks attached along the associated Δ_i. Moreover, $\gamma_{d+m+1} = (p,q)$.

Remark 2.28. *It may seem like the third case above does not follow the geometric picture of the degeneracy in the limit: the point is that once a sequence of disks degenerate at a double point of $\imath$ to get two disks, one has the original counterclockwise orientation on the boundary, while the other is oriented clockwise (see Figure 2.8).*

First of all, we show that non of the $[\overline{u_i}]$ is a constant disk. Pick $d+1 \leq i \leq d+m$ and assume $\gamma_i = \gamma_{i+1} = (p_i, q_i)$ and $[\overline{u_i} \in \mathcal{M}_{0,k^i_+}((p_i,q_i),(p_i,q_i),\alpha,J)$ is constant. It directly follows $\alpha \neq \emptyset$ as otherwise u_i would not be stable. Consider the "upper" part of the disk u_i. In order for u_i to be constant, we must have that k^i_+ is even, and half of the branch jumps indexed by α are of type (p_i,q_i), while the other half of type (q_i,p_i). As remarked above, the trees v_i are non-constant, as they have a unique branch jump, hence $\mathscr{A}(q_i,p_i)$ and $\mathscr{A}(p_i,q_i)$ have to be both positive, a contradiction. Pick

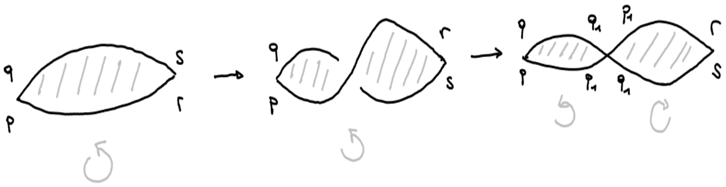

Figure 2.8: A possible degeneration of a disk in $\mathcal{M}_{0,0}((p,q),(r,s),J)$ at the double point (p_1,q_1): in the limit, the disk on the left is an element of $\mathcal{M}_{0,0}((p,q),(p_1,q_1),J)$, while the one on the right is oriented clockwise and can hence be seen as an element of $\mathcal{M}_{0,0}((p_1,q_1),(r,s),J)$.

now $d+1 \leq i \leq d+m$ and assume $\gamma_i = (p_i,q_i)$, $\gamma_{i+1} = (q_i,p_i)$ and $[\overline{u_i} \in \mathcal{M}_{0,k_+^i}((p_i,q_i),(q_i,p_i),\alpha,J)$ is constant. For u_i to be constant, α must index k branch jumps of type (p_i,q_i) and $k+2$ branch jumps of type (q_i,p_i) for some $k \geq 0$. The argument just above implies $k=0$ and $\mathscr{A}(q_i,p_i) > 0$. This contradicts conservation of energy from Gromov compactness in [IS02].
By standard Morse theory, we have $|x_i| - |x_{i+1}| \geq 1$ for any $1 \leq i \leq d-1$, as (f,g) is Morse-Smale. From now on write $\gamma_i = (p_i,q_i)$.
$[\overline{u_d}]$. If $\{p_{d+1},q_{d+1}\} \cap W^u(x_d) = \emptyset$, then u_d is non constant and by Proposition 2.26 we have

$$|x_d| - |\gamma_{d+1}| \geq 1 + 2k_+^d$$

If $\{p_{d+1},q_{d+1}\} \cap W^u(x_d) \neq \emptyset$ we can assume without loss of generality that $|x_d| = n$ so that:

1. if $\mathscr{A}(\gamma_{d+1}) < 0$, then $n - |\gamma_{d+1}| \geq 3$ annd hence $|x_k| - |\gamma_{d+1}| \geq 3$;
2. if $\mathscr{A}(\gamma_{d+1}) = 0$, then
$$0 \leq E(u) = -\sum \mathscr{A}(\alpha_d(i))$$
so that if $\alpha = \emptyset$, u_k is constant, a contradiction, as moduli space do not contain constant disks for $\alpha = \emptyset$, and if $\alpha \neq \emptyset$, $0 \leq E(u) < 0$, another contradiction;
3. if $\mathscr{A}(\gamma_{d+1}) > 0$, then
$$0 \leq E(u) = \mathscr{A}(\gamma_{d+1}) - \sum \mathscr{A}(\alpha_d(i)) < 0$$
again a contradiction

Consider now $[\overline{u_i}]$ for $d+1 \leq i \leq d+m$.

- $\gamma_{i+1} \neq (q_i,p_i)$ and $[\overline{u_i}]$ cannot be an element of $\mathcal{M}_{0,k_+^{i,1}}(\gamma_i,\emptyset,\alpha_{i,1},J) \times \mathcal{M}_{0,k_+^{i,2}}(\emptyset,\gamma_{i+1},\alpha_{i,2},J)$. Indeed, suppose $[\overline{u_i}]$ is the first one of the elements $[u_{d+1}],...,[u_{d+m}]$ to be such that $\gamma_{i+1} =$

(q_i, p_i) or of the second kind listed above. Then we have that $\mathscr{A}(\gamma_i) > 0$, but then $\mathscr{A}(\gamma_j) > 0$ for all $j \geq i$, implying in particular that $\mathscr{A}(p, q) > 0$: a contradiction, since the energy of the original sequence was $\mathscr{A}(q, p) = -\mathscr{A}(p, q) > 0$.

- $p_i \neq p_{i+1}$ and $q_i \neq q_{i+1}$, as if $p_i = p_{i+1}$ and $q_i = q_{i+1}$ and $\alpha_i \neq \emptyset$, $E(u_i) < 0$ and if $p_i = p_{i+1}$ and $q_i = q_{i+1}$ and $\alpha_i = \emptyset$, u_i is constant but not stable (q.v. the discussion above).
- If $p_i, q_i, p_{i+1}, q_{i+1}$ are pairwise distinct then $|\gamma_i| - |\gamma_{i+1}| \geq 1$ by Proposition 2.26.

In the end, from

$$1 = |x| - |p, q| = \sum_{i=1}^{k-1} |x_i| - |x_{i+1}| + (|x_d| - |\gamma_{d+1}) + \sum_{i=d+1}^{d+1} |\gamma_i| - |\gamma_{i+1}|$$

it follows that $d = 1$, $k_+^d = 0$ and hence that $\mathcal{M}_{0,0}(x, \gamma, f, g, J)$ is compact. □

Similarly we have the following three results.

Lemma 2.29. *Let $x \in \mathrm{Crit}(f)$ be a critical point of f and $(p, q) \in R$ be a double point of $\imath$ such that $|p, q| - |x| = 1$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, the moduli space $\mathcal{M}_{0,0}((p, q), x, f, g, J)$ is compact.*

Lemma 2.30. *Let $(p, q), (r, s) \in R$ be double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct and $|p, q| - |r, s| = 1$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, the moduli space $\mathcal{M}_{0,0;0,0}((p, q), (r, s), f, g, J)$ is compact.*

Lemma 2.31. *Let $(p, q) \in R$ be a double point of $\imath$ such that $|p, q| - |q, p| - 1 \leq 1$. Then the moduli space $\mathcal{M}^{\mathrm{abs}}_{0,0;0,0}((p, q), (q, p), f, g, J)$ is compact*

Lemma 2.32. *Let $(p, q), (r, s) \in R$ be ordered double points of $\imath$ such that $p, q, r, s \in L$ are pairwise distinct and $|p, q| - |r, s| = 1$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M, \omega)$, the moduli space $\mathcal{M}_{0,0}((p, q), (r, s), \alpha, J)$ is compact.*

Proof. By Gromov compactness for pseudoholomorphic curves with boundary and corners on totally real immersions (see [IS02]) and the fact that we are working in the exact setting, a sequence in $\mathcal{M}_{0,0}((p, q), (r, s), \alpha, J)$ converges to a broken pearly trajectory of the form

$$\mathbf{u} = \big(([\overline{u_1}], v_1), ([\overline{u_2}], v_2), ..., ([\overline{u_m}], v_m)\big)$$

for $m \geq 1$, where there are $\gamma_1, ..., \gamma_{m+1}$ with $\gamma_1 = (p, q)$ and $\gamma_{m+1} = (r, s)$ such that either $[\overline{u_i}] \in \mathcal{M}_{0,k_+^i}(\gamma_i, \gamma_{i+1}, \alpha_i, J)$ or $[\overline{u_i}] = ([\overline{u_{i,1}}], [\overline{u_{i,2}}])$, where $[\overline{u_{i,1}}] \in \mathcal{M}_{0,k_+^{i,1}}(\gamma_i, \emptyset, \alpha_{i,1}, J)$ and $[\overline{u_{i,2}}] \in \mathcal{M}_{0,k_+^{i,2}}(\emptyset, \gamma_{i+1}, \alpha_{i,2}, J)$ such that $u_{i,1}(+1) = u_{i,2}(-1)$, both with trees of pseudoholomorphic disks v_i attached along the associated marked points in Δ_i. Write $\gamma_i = (p_i, q_i)$ for any $i \in \{1, ..., m + 1\}$.
First, we have that none of the $[\overline{u_i}]$'s can be constant, and the argument is exactly as in the proof of Lemma 2.27.

If $[\overline{u_i}] = ([\overline{u_{i,1}}], [\overline{u_{i,2}}])$, where $[\overline{u_{i,1}}] \in \mathcal{M}_{0,k_+^{i,1}}(\gamma_i, \emptyset, \alpha_{i,1}, J)$ and $[\overline{u_{i,2}}] \in \mathcal{M}_{0,k_+^{i,2}}(\emptyset, \gamma_{i+1}, \alpha_{i,2}, J)$ such that $u_{i,1}(+1) = u_{i,2}(-1)$, then if $|\gamma_i| - |\gamma_{i+1}| \leq 2$, $[\overline{u_i}]$ lies in a moduli space which is (up to reducing the generic set of almost complex structures we are dealing with) a manifold of dimension $|\gamma_i| - |\gamma_{i+1}| - \sum |\alpha_{i,1}(j)| - \sum |\alpha_{i,2}(j)| + k_{i,1+} + k_{i,2+} - 2 \geq 0$, as it could be easily computed via evaluation maps plus the results from Section 2.3, which implies $|\gamma_i| - |\gamma_{i+1}| \geq 2$.

Hence, if $|\gamma_i| - |\gamma_{i+1}| - 1 \leq 1$ we have $|\gamma_i| - |\gamma_{i+1}| \geq 1 + 2k_i^+$ in any case (writing $k_+^i := k_+^{i,1} + k_+^{i,2}$ for $[\overline{u_1}] \in \mathcal{M}_{0,k_+^{i,1}}(\gamma_i, \emptyset, \alpha_{i,1}, J) \times \mathcal{M}_{0,k_+^{i,2}}(\emptyset, \gamma_{i+1}, \alpha_{i,2}, J)$) by the results in Section 2.3. This observation plus the fact that

$$1 = |p,q| - |r,s| = \sum_{i=1}^{m} |\gamma_i| - |\gamma_{i+1}|$$

leads to $|\gamma_i| - |\gamma_{i+1} \leq 1$ for any $i \in \{1, ..., m\}$, so that $m = 1$ and $k_+^1 = 0$, proving the lemma. $\square$

Similarly, we have the following result.

Lemma 2.33. *Let $(p,q) \in R$ be a double point of $\imath$ such that $|p,q| - |q,p| - 1 \leq 1$. Then the moduli space $\mathcal{M}^*_{0,0}((p,q),(q,p),J)$ is compact.*

2.5 Compactifications of moduli space of pearls in dimension 1

At this stage of a construction of a Morse-like cohomology, *gluing* of relevant configurations usually enters the game. What happens in dimension 1 is that we can not rule out broken configurations which arises from Gromov compactness from the Gromov compactification of moduli spaces by counting dimension. Instead, we want to prove that compactifications are smooth manifolds of dimension 2 whose boundary is made of exactly those broken configurations. Gromov compactness basically tells us that the boundary is contained in the space of broken configurations, but does not provide equality nor charts. Gluing is a technique introduced by Floer which goes to study what happens near those broken configurations by building an open embedding that geometrically corresponds to the literal gluing of two configurations via a real parameter, which coupled with the so-called surjectivity of the gluing map (which can be interpreted as uniqueness of those embeddings), directly give charts for the compactification in the expected way. A brief summary about gluing of Floer strips in the Hamiltonian case may be found in [Amb19], while [AD14] provides a complete exposition with a lot of details. Here, we will skip most of the details.
Gluing of smooth pseudoholomorphic disks with boundary on embedded Lagrangian is studied in detail in [BC07, Section 4], while gluing of Morse trajectories is nowadays a standard fact (see for instance [AD14, Chapter 1]). To handle the immersed case, we also have to study gluing of pseudoholomorphic disks at corners: however, it turns out that, in the correct framework, this is just a special case of what is described in [BC07], see Figure 2.9. From ~~this~~ these considerations, Gromov compactness and the same tricks we used in Section 2.4 we get the following statements about compactifications of our preferred moduli space in dimension 1.

Give ref. to F03 in the 1'st place.

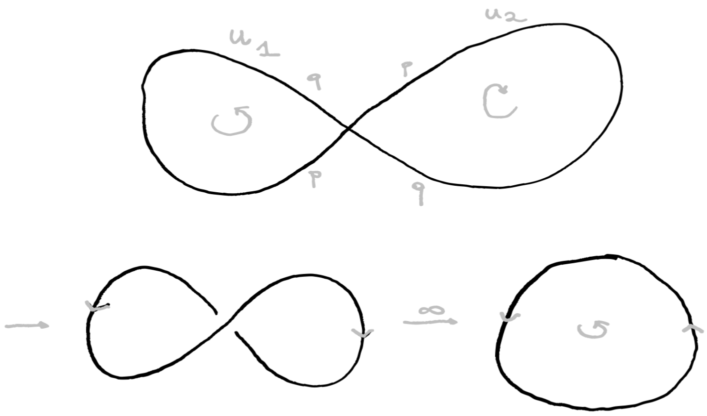

Figure 2.9: The gluing process of a teardrop $u_1 \in \mathcal{M}_{0,0}(\emptyset,(p,q),J)$ and a teardrop $u_2 \in \mathcal{M}_{0,0}((p,q),\emptyset,J)$ at the corners.

Lemma 2.34. *Let $x \in \mathrm{Crit}(f)$ be a critical point of f and $(p,q) \in R$ be a double point of $\imath$ such that $|x| - |p,q| = 2$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M,\omega)$, the moduli space $\mathcal{M}_{0,0}(x,(p,q),f,g,J)$ admits a Gromov-type compactification into a 1-dimensional manifold with boundary $\overline{\mathcal{M}}_{0,0}(x,(p,q),f,g,J)$ such that*

$$\begin{aligned}\partial\overline{\mathcal{M}}_{0,0}(x,(p,q),f,g,J) = &\bigcup_{y:\ |x|-|y|=1} \mathcal{M}(x,y) \times \mathcal{M}_{0,0}(y,(p,q),f,g,J) \cup \\ &\bigcup_{(r,s)\neq(q,p):\ |x|-|r,s|=1} \mathcal{M}_{0,0}(x,(r,s),f,g,J) \times \mathcal{M}_{0,0}((r,s),(p,q),J)\end{aligned}$$

Lemma 2.35. *Let $x \in \mathrm{Crit}(f)$ be a critical point of f and $(p,q) \in R$ be a double point of $\imath$ such that $|p,q| - |x| = 2$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M,\omega)$, the moduli space $\mathcal{M}_{0,0}((p,q),x,f,g,J)$ admits a Gromov-type compactification into a 1-dimensional manifold with boundary $\overline{\mathcal{M}}_{0,0}((p,q),x,f,g,J)$ such that*

$$\begin{aligned}\partial\overline{\mathcal{M}}_{0,0}((p,q),x,f,g,J) = &\bigcup_{y:\ |y|-|x|=1} \mathcal{M}_{0,0}((p,q),y,f,g,J) \times \mathcal{M}(y,x) \cup \\ &\bigcup_{(r,s)\neq(q,p):\ |p,q|-|r,s|=1} \mathcal{M}_{0,0}((p,q),(r,s),J) \times \mathcal{M}_{0,0}((r,s),x,f,g,J)\end{aligned}$$

Lemma 2.36. *Let $(p,q),(r,s) \in R$ be double points of $\imath$ such that $p,q,r,s \in L$ are pairwise distinct and $|p,q| - |r,s| = 2$. Then, for a generic choice of compatible almost complex structure $J \in \mathcal{J}_c(M,\omega)$, the moduli space $\mathcal{M}_{0,0;0,0}((p,q),(r,s),f,g,J)$ admits a Gromov-type compactification into a 1-dimensional* *manifold with boundary* $\overline{\mathcal{M}}_{0,0;0,0}((p,q),(r,s),f,g,J)$ *such that*

$$\begin{aligned}
\partial\overline{\mathcal{M}}_{0,0;0,0}((p,q),(r,s),f,g,J) = &\bigcup_{y:\ |p,q|-|y|=1} \mathcal{M}_{0,0}((p,q),y,f,g,J)\times\mathcal{M}_{0,0}(y,(r,s),J)\cup\\
&\bigcup_{(a,b)\neq(q,p):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0}((p,q),(a,b),J)\times\mathcal{M}_{0,0;0,0}((a,b),(r,s),f,g,J)\\
&\bigcup_{(a,b)\neq(q,p):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0;0,0}((p,q),(a,b),f,g,J)\times\mathcal{M}_{0,0}((a,b),(r,s),J)\\
&\bigcup \left\{(\overline{u_1},\overline{u_2})\in\mathcal{M}_{0,0}((p,q),\emptyset,J)\times\mathcal{M}_{0,0}(\emptyset,(r,s),J):\ l_1(1)=l_2(-1)\right\}
\end{aligned}$$

Lemma 2.37. *Let* $(p,q)\in R$ *be a double point of* $\imath$ *such that* $|p,q|-|q,p|=2$*. Then, for a generic choice of compatible almost complex structure* $J\in J_c(M,\omega)$*, the moduli space* $\mathcal{M}^{\mathrm{abs}}_{0,0;0,0}((p,q),(q,p),f,g,J)$ *admits a Gromov-type compactification into a* 1*-dimensional manifold with boundary* $\overline{\mathcal{M}}^{\mathrm{abs}}_{0,0;0,0}((p,q),(q,p),f,g,J$ *such that*

$$\begin{aligned}
\partial\overline{\mathcal{M}}^{\mathrm{abs}}_{0,0;0,0}((p,q),(q,p),f,g,J) = &\bigcup_{y:\ |p,q|-|y|=1} \mathcal{M}_{0,0}((p,q),y,f,g,J)\times\mathcal{M}_{0,0}(y,(q,p),J)\cup\\
&\bigcup_{(a,b):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0}((p,q),(a,b),J)\times\mathcal{M}_{0,0;0,0}((a,b),(q,p),f,g,J)\\
&\bigcup_{(a,b):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0;0,0}((p,q),(a,b),f,g,J)\times\mathcal{M}_{0,0}((a,b),(q,p),J)\\
&\bigcup \left\{(\overline{u_1},\overline{u_2})\in\mathcal{M}_{0,0}((p,q),\emptyset,J)\times\mathcal{M}_{0,0}(\emptyset,(q,p),J):\ l_1(1)=l_2(-1)\right\}
\end{aligned}$$

Lemma 2.38. *Let* $(p,q),(r,s)\in R$ *be ordered double points of* $\imath$ *such that* $p,q,r,s\in L$ *are pairwise distinct and* $|p,q|-|r,s|=2$*. Then, for a generic choice of compatible almost complex structure* $J\in\mathcal{J}_c(M,\omega)$*, the moduli space* $\mathcal{M}_{0,0}((p,q),(r,s),\alpha,J)$ *admits a Gromov-type compactification into a* 1*-dimensional manifold with boundary* $\overline{\mathcal{M}}_{0,0}((p,q),(r,s),J)$ *such that*

$$\begin{aligned}
\partial\overline{\mathcal{M}}_{0,0}((p,q),(r,s),J) = &\bigcup_{(a,b):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0}((p,q),(a,b),J)\times\mathcal{M}_{0,0}((a,b),(r,s),J)\\
&\bigcup \left\{(\overline{u_1},\overline{u_2})\in\mathcal{M}_{0,0}((p,q),\emptyset,J)\times\mathcal{M}_{0,0}(\emptyset,(r,s),J):\ l_1(1)=l_2(-1)\right\}
\end{aligned}$$

Lemma 2.39. *Let* $(p,q)\in R$ *be a double point of* $\imath$ *such that* $|p,q|-|q,p|=2$*. Then, for a generic choice of compatible almost complex structure* $J\in J_c(M,\omega)$*, the moduli space* $\mathcal{M}^*_{0,0}((p,q),(q,p),J)$ *admits a Gromov-type compactification into a* 1*-dimensional manifold with boundary* $\overline{\mathcal{M}}^*_{0,0}((p,q),(q,p),J)$ *such that*

$$\begin{aligned}
\partial\overline{\mathcal{M}}^*_{0,0}((p,q),(q,p),J) = &\bigcup_{(a,b):\ |p,q|-|a,b|=1} \mathcal{M}_{0,0}((p,q),(a,b),J)\times\mathcal{M}_{0,0}((a,b),(q,p),J)\\
&\bigcup \left\{(\overline{u_1},\overline{u_2})\in\mathcal{M}_{0,0}((p,q),\emptyset,J)\times\mathcal{M}_{0,0}(\emptyset,(q,p),J):\ l_1(1)=l_2(-1)\right\}
\end{aligned}$$

2.6 The pearl complex

Recall that we have fixed an exact, compact, connected, generic and graded Lagrangian immersion $\imath : L \to M$ satisfying Assumption 2.2, a compatible almost complex structure $J \in \mathcal{J}_c(M,\omega)$, a Morse function $f : L \to \mathbb{R}$ on L and a pseudogradient field $g \in \Gamma(TL)$ on L such that the couple (f,g) is Morse-Smale.

We define the graded vector space $QC^*(\imath; f, g, J) := \mathbb{Z}_2\mathrm{Crit}(f) \oplus \mathbb{Z}_2 R$, where $\mathrm{Crit}(f)$ is graded by the Morse index, and R is graded as in Section 2.1. We denote $\mathbf{C} := \mathbb{Z}_2\mathrm{Crit}(f)$ and $\mathbf{R} := \mathbb{Z}_2 R$.

Remark 2.40. *In the embedded counterpart of our construction (see [BC07]), the vector space QC is defined only using critical points of the chosen Morse function. A nice way to see that we have to add ordered double points in the definition of the vector space is to analyze what happens to the generator of the standard Floer complex (see [AB18]), when we let the Hamiltonian diffeomorphism tend to the identity.*

We define a map $d : QC^*(\imath; f, g, J) \to QC^*(\imath; f, g, J)[-1]$ by counting elements in the moduli spaces of pearls we constructed in Section 2.2. Let $x \in \mathrm{Crit}(f)$ and $\gamma \in R$. We define

1. $d_{\mathbf{CC}} : \mathbf{C} \longrightarrow \mathbf{C}$ by linearly extending
$$d_{\mathbf{CC}} x := \sum_{y \in \mathrm{Crit}(f):\ |y|-|x|=1} |\mathcal{M}(y,x)|_2 \cdot y$$
2. $d_{\mathbf{CR}} : \mathbf{R} \to \mathbf{C}$ by linearly extending
$$d_{\mathbf{CR}} \gamma := \sum_{y \in \mathrm{Crit}(f):\ |x|-|\gamma|=1} |\mathcal{M}_{0,0}(y,\gamma,f,g,J)|_2 \cdot y$$
3. $d_{\mathbf{RC}} : \mathbf{C} \longrightarrow \mathbf{R}$ by linearly extending
$$d_{\mathbf{RC}} x := \sum_{\gamma' \in R:\ |\gamma'|-|x|=1} |\mathcal{M}_{0,0}(\gamma',x,f,g,J)|_2 \cdot \gamma'$$
4. $d_{\mathbf{RR}} : \mathbf{R} \longrightarrow \mathbf{R}$ by linearly extending
$$d_{\mathbf{RR}^1} \gamma := \sum_{\gamma' \in R:\ |\gamma'|-|\gamma|=1} |\mathcal{M}_{0,0}(\gamma',\gamma,J) \sqcup \mathcal{M}_{0,0;0,0}(\gamma',\gamma,f,g,J)|_2 \cdot \gamma'$$

We then pack those four maps matrixwise as
$$d := \begin{pmatrix} d_{\mathbf{CC}} & d_{\mathbf{CR}} \\ d_{\mathbf{RC}} & d_{\mathbf{RR}} \end{pmatrix}$$

We will write
$$d_{\mathbf{RR}_1}(\gamma) := \sum_{\gamma' \in R:\ |\gamma'|-|\gamma|=1} |\mathcal{M}^*_{0,0}(\gamma',\gamma,J)|_2 \cdot \gamma'$$

and

$$d_{\mathbf{RR}_2} := \sum_{\gamma' \in R:\ |\gamma'|-|\gamma|=1} |\mathcal{M}^{\text{abs}}_{0,0;0,0}(\gamma', \gamma, f, g, J)|_2 \cdot \gamma'$$

so that $d_{\mathbf{RR}_1} + d_{\mathbf{RR}_2} = d_{\mathbf{RR}}$.
We have to prove that d is well defined and that $d^2 = 0$.

Proposition 2.41. *For a generic choice of the almost complex structure $J \in \mathcal{J}_c(M, \omega)$, the map d is a well-defined differential.*

Proof. First of all, d is well defined by the results in Section 2.4. We have that

$$d^2 := \begin{pmatrix} d^2_{\mathbf{CC}} + d_{\mathbf{CR}} d_{\mathbf{RC}} & d_{\mathbf{CC}} d_{\mathbf{CR}} + d_{\mathbf{CR}} d_{\mathbf{RR}^1} + d_{\mathbf{CR}} d_{\mathbf{RR}^2} \\ d_{\mathbf{RC}} d_{\mathbf{CC}} + d_{\mathbf{RR}^1} d_{\mathbf{RC}} + d_{\mathbf{RR}^2} d_{\mathbf{RC}} & d_{\mathbf{RC}} d_{\mathbf{CR}} + d^2_{\mathbf{RR}^1} + d_{\mathbf{RR}^1} d_{\mathbf{RR}^2} + d_{\mathbf{RR}^2} d_{\mathbf{RR}^1} + d^2_{\mathbf{RR}^2} \end{pmatrix}$$

Note that $d_{\mathbf{CR}} d_{\mathbf{RC}}$, $d_{\mathbf{CR}} d_{\mathbf{RR}^2}$, $d^2_{\mathbf{RR}^2}$ vanish by area reasons. For instance, if there is $\overline{u} \in \mathcal{M}_{0,0}((p,q), x, J)$ contributing to $d_{\mathbf{RC}} x$ for $x \in \mathrm{Crit}(f)$, then u is non-constant by assumption and hence $\mathscr{A}(p,q) > 0$, implying that for any $y \in \mathrm{Crit}(f)$, the moduli space $\mathcal{M}_{0,0}(y, (p,q), J)$ is empty so that

$$d_{\mathbf{CR}} d_{\mathbf{RC}} x = \sum_{|y|-|x|=2} \sum_{|p,q|-|x|=1} |\mathcal{M}_{0,0}(y, (p,q), J)|_2 |\mathcal{M}_{0,0}((p,q), x, J)|_2 \cdot y = 0$$

The other cases are similar. Hence we remain with

$$d^2 := \begin{pmatrix} d^2_{\mathbf{CC}} & d_{\mathbf{CC}} d_{\mathbf{CR}} + d_{\mathbf{CR}} d_{\mathbf{RR}^1} \\ d_{\mathbf{RC}} d_{\mathbf{CC}} + d_{\mathbf{RR}^1} d_{\mathbf{RC}} & d_{\mathbf{RC}} d_{\mathbf{CR}} + d^2_{\mathbf{RR}^1} + d_{\mathbf{RR}^1} d_{\mathbf{RR}^2} + d_{\mathbf{RR}^2} d_{\mathbf{RR}^1} \end{pmatrix}$$

The fact that the Morse differential squares to zero is well known. We work out the details for the other entries. Consider $(p,q) \in R$, then

$$d_{\mathbf{CC}} d_{\mathbf{CR}} + d_{\mathbf{CR}} d_{\mathbf{RR}^1}(p,q) = \sum_{|x|-|p,q|=2} |\partial \overline{\mathcal{M}}_{0,0}(x, (p,q), f, g, J)|_2 \cdot x = 0$$

by classification of 1-dimensional manifolds with boundary (see for instance [AD14]). Similarly, for $x \in \mathrm{Crit}(f)$ we have

$$d_{\mathbf{RC}} d_{\mathbf{CC}} + d_{\mathbf{RR}^1} d_{\mathbf{RC}} x = \sum_{|p,q|-|x|=2} |\partial \overline{\mathcal{M}}_{0,0}((p,q), x, f, g, J)|_2 \cdot x = 0$$

and

$$d_{\mathbf{RC}} d_{\mathbf{CR}} + d^2_{\mathbf{RR}^1} + d_{\mathbf{RR}^1} d_{\mathbf{RR}^2} + d_{\mathbf{RR}^2} d_{\mathbf{RR}^1}(p,q) = \\ \sum_{|r,s|-|p,q|=2} \big(|\partial \overline{\mathcal{M}}_{0,0;0,0}((r,s), (p,q), f, g, J)|_2 + |\partial \overline{\mathcal{M}}_{0,0}((r,s), (p,q))|_2 \big) \cdot (r,s) = 0$$

concluding the proof. □

We will call the complex $(QC^*(\imath; f, g, J), d)$ the pearl complex of $\imath$ defined via the parameters f, g and J.

2.7 Invariance from parameters

In this section we will prove the following proposition via a geometric argument.

Proposition 2.42. *Two generic choices of parameters* (f_0, g_0, J_0) *and* (f_1, g_1, J_1) *define quasi-isomorphic pearl complexes.*

The situation that arises in the proof of Proposition 2.42 is very similar to the one one encounters when showing independence of Morse homology from the chosen Morse-Smale pair. The main differences are two: we have to take care of the almost complex structures, and the fact that the map relating two complexes is a chain map is a little more subtle to prove, as the configurations may break at double points.

We want to build a chain map

$$\psi : QC^*(\imath; f_1, g_1, J_1) \longrightarrow QC^*(\imath; f_0, g_0, J_0)$$

Which we express matrixwise as

$$\psi := \begin{pmatrix} \psi_{\mathbf{C}_0\mathbf{C}_1} & \psi_{\mathbf{C}_0\mathbf{R}} \\ \psi_{\mathbf{R}\mathbf{C}_1} & \psi_{\mathbf{R}\mathbf{R}} \end{pmatrix}$$

We will start from the ideas behind the construction in Morse theory, that is, the definition of $\psi_{\mathbf{C}_0\mathbf{C}_1} : \mathrm{Crit}(f_1) \to \mathrm{Crit}(f_0)$. Recall that in the case of Morse cohomology, it sufficies to show that it is isomorphic to singular cohomology in order to show independence from the chosen Morse-Smale pair; however, there is not much geometry behind that construction.
Pick a smooth function $F : M \times [0,1] \to \mathbb{R},\ (x,s) \longmapsto F_s(x)$ such that $F_s = f_0$ on $[0,\epsilon]$ and $F_s = f_1$ on $[1-\epsilon, 1]$ for some $\epsilon > 0$. Extend F to $M \times (-\epsilon, 1+\epsilon)$ by asking to be f_0 respectively f_1 near the on $(-\epsilon, 0)$ and $(1, 1+\epsilon)$.

Pick a Morse function $h : (-\epsilon, 1+\epsilon) \to \mathbb{R}$ which has a maximum at 0, a minimum at 1 and no other critical points. Assume further that h is increasing on $(-\epsilon, 0)$ and $(1, 1+\epsilon)$ and sufficiently decreasing on $(0,1)$ such that for any $s \in (0,1)$:

$$\frac{\partial F}{\partial s} + h' < 0$$

Note that

$$\frac{\partial F}{\partial s}(s) + h'(s) = 0$$

for $s = 0$ and $s = 1$. It follows that $\tilde{F} := F + (0, h) : M \times (-\epsilon, 1+\epsilon) \to \mathbb{R}$ has critical points

$$\mathrm{Crit}(\tilde{F}) = \mathrm{Crit}(f_0) \times \{0\} \cup \mathrm{Crit}(f_1) \times \{1\}$$

and is hence a Morse function.

Via a partition of unit argument, one constructs a pseudogradient field $\tilde{g} \in \Gamma(L \times [0,1])$ on $M \times (-\epsilon, 1+\epsilon)$ such that it is $g_0 - \nabla h$ on $M \times (-\epsilon, \epsilon)$ and $g_1 - \nabla h$ on $M \times (1-\epsilon, 1+\epsilon)$. We perturb $\tilde{g}$ such that $(\tilde{F}, \tilde{g})$ is Morse-Smale and the perturbation is small enough in C^1-sense (for details, see

[AD14, Proposition 3.4.3]). We call such a pair a Morse cobordism between the Morse-Smale pairs (f_0, g_0) and (f_1, g_1). Then, as 0 is a maximum of g and 1 is a minimum, i.e. they have respectively 1 and 0 as Morse indexes, it follows that

$$\mathrm{Crit}_k(\tilde{F}) = \mathrm{Crit}_{k-1}(f_0) \times \{0\} \cup \mathrm{Crit}_k(f_1) \times \{1\}$$

for any $k \geq 0$. Then, the Morse complex (over $M \times [0,1]$) of the Morse-Smale pair $(\tilde{F}, \tilde{g})$ is then given by

$$CM_k(\tilde{F}, \tilde{g}) := \mathrm{Crit}_{k-1}(f_0) \oplus \mathrm{Crit}_k(f_1)$$

for any $k \geq 0$ with differential $d_{\tilde{F}}$ written in matrix form as

$$d_{\tilde{F}} := \begin{pmatrix} d_{f_0} & \psi_{\mathbf{C}_0\mathbf{C}_1} \\ 0 & d_{f_1} \end{pmatrix}$$

where $\psi_{\mathbf{C}_0\mathbf{C}_1} : CM_*(f_1) \to CM_*(f_0)$ counts trajectories of $\tilde{g}$ in $M \times [0,1]$ connecting critical points of f_0 (seen in $\mathrm{Crit}(f_0) \times \{0\}$) to critical points of f_1 (seen in $\mathrm{Crit}(f_1) \times \{1\}$), that is, elements of $\mathcal{M}((x,0),(y,1))$ for $x \in \mathrm{Crit}(f_0)$ and $y \in \mathrm{Crit}(f_1)$ when $|x| - |y| = 0$.

We define the other entries of ψ by also considering some configurations in the cobordism $M \times [0,1]$. We define the following moduli spaces for $x \in \mathrm{Crit}(f_0)$, $y \in \mathrm{Crit}(f_1)$ and $(p,q),(r,s) \in R$:

1. $\mathcal{M}_{0,0}((x,0),(p,q),\tilde{F},\tilde{g},J_1) := \{\overline{u} \in \mathcal{M}_{0,0}(\emptyset,(p,q),J_1) : \ l(-1) \in W^s_{\tilde{F}}(x,0)\}$ whose virtual dimension is $|x| - |p,q|$;

2. $\mathcal{M}_{0,0}((r,s),(y,1),\tilde{F},\tilde{g},J_0) := \{\overline{u} \in \mathcal{M}_{0,0}((r,s),\emptyset,J_0) : \ l(+1) \in W^u_{\tilde{F}}(y,1)\}$ whose virtual dimension is $|r,s| - |y|$;

3. $\mathcal{M}_{0,0}((r,s),(p,q),\tilde{F},\tilde{g},J_0,J_1) := \{(\overline{u}_1,\overline{u}_2,t) \in \mathcal{M}_{0,0}((r,s),\emptyset,J_0) \times \mathcal{M}_{0,0}(\emptyset,(p,q),J_1) \times \mathbb{R}_{>0} : \varphi^t_{\tilde{F}}(l_1(1)) = l_2(-1)\}$ whose virtual dimension is $|r,s| - |p,q|$.

We prove that the virtual dimension of $\mathcal{M}_{0,0}((x,0),(p,q)),\tilde{F},\tilde{g},J_1)$ is indeed $|x| - |p,q|$. First, notice that $\dim(W^u_{\tilde{F}}(x,0)) = |x| + 1$, as 0 is a maximum of h, and $W^u_{\tilde{F}}(x,0) - L \times \{0\} \subset L \times \{1\}$. Define the evaluation map

$$\mathrm{ev}^{(p,q)} : \mathcal{M}_{0,0}(\emptyset,(p,q),J_1) \times (0,1] \longrightarrow L \times [0,1], \quad \overline{u} \longmapsto (l(-1), t)$$

then

$$\begin{aligned} \dim(\mathcal{M}_{0,0}((x,0),(p,q)),\tilde{F},\tilde{g},J_1)) &= \dim(\mathrm{ev}^{(p,q)})^{-1}(W^u(x,0)) = \\ & n - |p,q| - 1 + 1 - (n + 1 - |x| - 1) = |x| - |p,q| \end{aligned}$$

The computations of the remaining virtual dimensions follows very similarly.

We are now ready to construct the map $\Psi := \Psi^{\tilde{F},\tilde{g}} : QC^*(\imath; f_1, g_1, J_1) \longrightarrow QC^*(\imath; f_0, g_0, J_0)$. Let $x \in \mathrm{Crit}(f_1)$ and $\gamma \in R$. We define:

1. $\psi_{\mathbf{C}_0\mathbf{C}_1} : \mathrm{Crit}(f_1) \longrightarrow \mathrm{Crit}(f_0)$ as above, that is, by linearly extending
$$\psi_{\mathbf{C}_0\mathbf{C}_1} x := \sum_{y \in \mathrm{Crit}(f_0):\ |y|=|x|} |\mathcal{M}((y,0),(x,1))|_2 \cdot y$$

2. $\Psi_{\mathbf{C}_0\mathbf{R}} : R \longrightarrow \mathrm{Crit}(f_0)$ by linearly extending
$$\psi_{\mathbf{C}_0\mathbf{R}} \gamma := \sum_{y \in \mathrm{Crit}(f_0):\ |p,q|=|y|} |\mathcal{M}_{0,0}((y,0),\gamma,\tilde{F},\tilde{g},J_1)|_2 \cdot x$$

3. $\Psi_{\mathbf{R}\mathbf{C}_1} : \mathrm{Crit}(f_1) \to R$ by linearly extending
$$\Psi_{\mathbf{R}\mathbf{C}_1} x := \sum_{\gamma' \in R:\ |\gamma'|=|x|} |\mathcal{M}_{0,0}(\gamma',(x,1),\tilde{F},\tilde{g},J_0)|_2 \cdot \gamma'$$

4. $\psi_{\mathbf{R}\mathbf{R}^1} : R \to R$ as the identity map on R;

5. $\psi_{\mathbf{R}\mathbf{R}^2} : R \to R$ by linearly extending
$$\Psi_{\mathbf{R}\mathbf{R}^2} \gamma := \sum_{\gamma' \in R:\ |\gamma'|=|\gamma|} |\mathcal{M}_{0,0}(\gamma',\gamma,\tilde{F},\tilde{g},J_0,J_1)|_2 \cdot \gamma'$$

6. $\psi_{\mathbf{R}\mathbf{R}} := \psi_{\mathbf{R}\mathbf{R}^1} + \psi_{\mathbf{R}\mathbf{R}^2}$.

The fact that $\Psi_{\mathbf{C}_0\mathbf{C}_1}$ is well-defined follows from the fact that $(\tilde{F},\tilde{g})$ is Morse-Smale, while to show that the remaining three entries are well defined one can proceed as in Section 2.3 and Section 2.4 to show that the moduli spaces above are generically either empty or finite dimensional smooth manifolds whose dimension agrees with virtual dimension, whenever the latter is smaller or equal to 1, and are compact when it is 0.

We compute $(d_{f_0} \circ \psi + \psi \circ d_{f_1})(x,\gamma)$ as

$$\begin{pmatrix} d_{\mathbf{C}_0\mathbf{C}_0} & d_{\mathbf{C}_0\mathbf{R}} \\ d_{\mathbf{R}\mathbf{C}_0} & d_{\mathbf{R}\mathbf{R}} \end{pmatrix} \begin{pmatrix} \psi_{\mathbf{C}_0\mathbf{C}_1} & \psi_{\mathbf{C}_0\mathbf{R}} \\ \psi_{\mathbf{R}\mathbf{C}_1} & \psi_{\mathbf{R}\mathbf{R}} \end{pmatrix} \begin{pmatrix} x \\ \gamma \end{pmatrix} + \begin{pmatrix} \psi_{\mathbf{C}_0\mathbf{C}_1} & \psi_{\mathbf{C}_0\mathbf{R}} \\ \psi_{\mathbf{R}\mathbf{C}_1} & \psi_{\mathbf{R}\mathbf{R}} \end{pmatrix} \begin{pmatrix} d_{\mathbf{C}_1\mathbf{C}_1} & d_{\mathbf{C}_1\mathbf{R}} \\ d_{\mathbf{R}\mathbf{C}_1} & d_{\mathbf{R}\mathbf{R}} \end{pmatrix} \begin{pmatrix} x \\ \gamma \end{pmatrix} =$$
$$= \begin{pmatrix} d_{\mathbf{C}_0\mathbf{C}_0}\psi_{\mathbf{C}_0\mathbf{C}_1} + \psi_{\mathbf{C}_0\mathbf{C}_1} d_{\mathbf{C}_0\mathbf{C}_0} & d_{\mathbf{C}_0\mathbf{C}_0}\psi_{\mathbf{C}_0\mathbf{R}} + \psi_{\mathbf{C}_0\mathbf{C}_1} d_{\mathbf{C}_1\mathbf{R}} \\ d_{\mathbf{R}\mathbf{C}_0}\psi_{\mathbf{C}_0\mathbf{C}_1} + d_{\mathbf{R}\mathbf{R}^1}\psi_{\mathbf{R}\mathbf{C}_1} + \psi_{\mathbf{R}\mathbf{C}_1} d_{\mathbf{C}_1\mathbf{C}_1} & d_{\mathbf{R}\mathbf{C}_0}\psi_{\mathbf{C}_0\mathbf{R}} + d_{\mathbf{R}\mathbf{R}^1}\psi_{\mathbf{R}\mathbf{R}^2} + \psi_{\mathbf{R}\mathbf{C}_1} d_{\mathbf{C}_1\mathbf{R}} + \psi_{\mathbf{R}\mathbf{R}^2} d_{\mathbf{R}\mathbf{R}^1} \end{pmatrix} \begin{pmatrix} x \\ \gamma \end{pmatrix}$$

By Gromov-compactifying the relevant moduli spaces in dimension 1, similarly to what we did in Section 2.5, we directly get that Ψ is a chain map. When looking at the Gromov-compactifications, one has to notice that Morse flowlines of $\tilde{F}$ can not break in $L \times (0,1)$ by definition of $\tilde{F}$ and that Morse flowlines of $\tilde{F}$ joining critical points of the form $(x,i),(y,i)$ lie on $L \times \{i\}$ as $\tilde{F}$ equals f_i near $L \times \{i\}$.

Remark 2.43. *In the above computation we silently canceled out various terms: a lot of terms got canceled because of area reasons, exactly as in the proof of $d^2 = 0$, while the other because the identity $\psi_{\mathbf{R}\mathbf{R}^1}$ (of course) commutes with the other maps.* ?

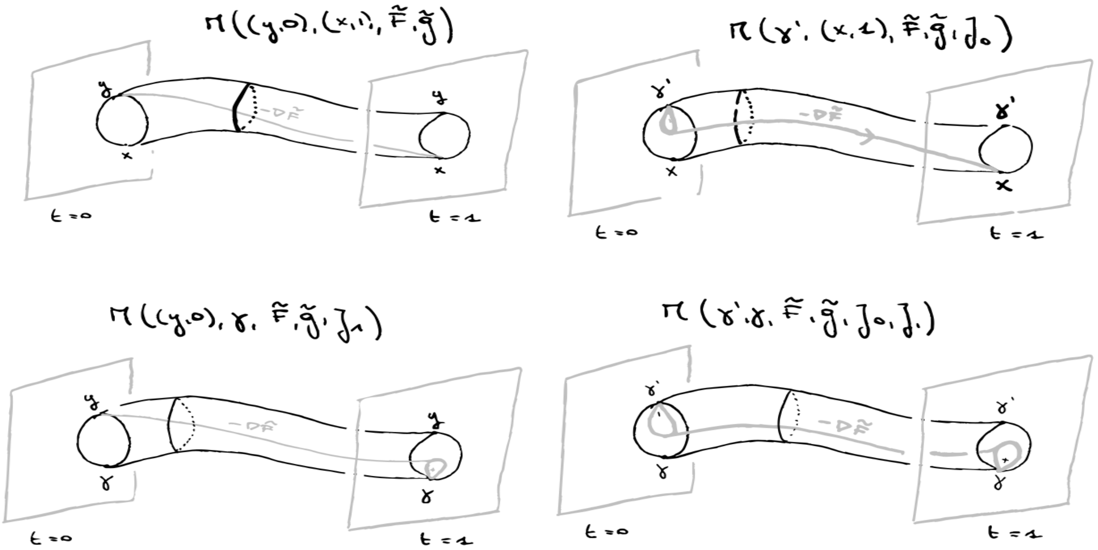

Figure 2.10: A sketch of the relevant configurations for the definition of ψ.

It remains to show that $\psi = \psi^{\tilde{F},\tilde{g}}$ is a chain isomorphism. First of all, we show that if $f_0 = f_1$, $g_0 = g_1$ $J_0 = J_1$, $\tilde{F} = (0, h)$ and $\tilde{g}$ is constantly $g_0 - \nabla h = g_1 - \nabla h$, then ψ is the identity map. This is rather easy since if $|x| = |y|$ for two critical points $x, y \in \mathrm{Crit}(f)$, then $\mathcal{M}(x, y)$ is a single point (the constant flowline at x) if $x = y$ and is empty if $x \neq y$, while on the other hand in this case $\psi_{\mathbf{RR}} = \psi_{\mathbf{RR}^1} = id$ as a configuration contributing to $\psi_{\mathbf{RR}^2}$ induces (by projecting on L) a configuration in a moduli space of negative dimension.

Now, it remains to show two things: first, that concatenation of two Morse-Smale pairs on $L\times[0,1]$ induce a map ψ which is chain homotopic to the composition of the ψ's induces by the two Morse-Smale pairs separately; second, that two Morse-Smale pairs on $L \times [0, 1]$ interpolating the same two Morse-Smale pairs on L induce chain homotopic ψ's (this is all quite standard, one finds proofs for Morse theory and Hamiltonian Floer homology in [Amb20; AD14] and for Lagrangian quantum homology for embedded Lagrangians in [BC07]). maybe add something.. This proves Proposition 2.42.

We are now ready to define the main object of this thesis.

Definition 2.44. *Consider a Morse-Smale pair* $(f, g) \in C^\infty(L) \times \Gamma(TL)$ *on the manifold* L *and a compatible almost complex structure* $J \in \mathcal{J}_c(M, \omega)$ *on* M *such that any object we have studied until now is regular. Then we define the Lagrangian quantum cohomology* $QH^*(\imath)$ *associated to the immersion* $\imath$ *with coefficients in* $\mathbb{Z}_2$ *as the homology of the complex* $(QC^*(\imath; f, g, J), d)$.

2.8 Lagrangian quantum cohomology is Floer cohomology

In this section, we exhibit an equivalence between the pearly definiton of Floer homology and the more standard one using Hamiltonian perturbations. This equivalence is based on a so called "PSS" construction, which was originally introduced by Piunikhin-Salamon-Schwarz [PSS96] in the context of quantum homology. We will follow [AB19], while adding the relevant configurations they do not consider in their paper, but first we quickly introduce Floer homology via Hamiltonian perturbations for Lagrangian immersions following [AB18].

The basic setup is the same which we described at the beginning of this chapter. We fix a compatible almost complex structure $J_M \in \mathcal{J}_c(M,\omega)$ and a smooth time-dependent compatible almost complex structure $J : [0,1] \to \mathcal{J}_c(M,\omega)$ which is equal to J_M in an open neighbourhood of the boundary ∂M. Consider a Hamiltonian $H : M \times [0,1] \to \mathbb{R}$ which vanishes in an open neighbourhood of the boundary ∂M: we call H an admissible Hamiltonian if $\varphi_1^H(\imath(L)) \cup (\varphi_1^H)^{-1}(\imath(L)$ is disjoint from $\imath(R)$ and $\varphi_1^H(\imath(L))$ is transverse to $\imath(L)$. We define Γ_H as the set of Hamiltonian orbits $\varphi : [0,1] \to M$ of H such that $\gamma(0), \gamma(1) \in \imath(L)$. We will call (H,J) as above an admissible couple.

Definition 2.45. *While the second hypotesis in the definition of admissible Hamiltonian is very standard (and necessary) in any flavour of Floer theory, as it implies that Γ_H is a finite set, the first hypothesis ensures that Hamiltonian orbits of H do not start and stop at double points of $\imath$, making our life easier as thing will look much more "embedded".*

We index Hamiltonian orbits in a very similar way to double points. Consider $\gamma \in \Gamma_H$ and the Lagrangian subspaces $D\imath[T_{\imath^{-1}(\gamma(1)}L]$ and $D(\varphi_1^H \circ \imath)[T_{\imath^{-1}(\gamma(0))}L]$ of TM. Then we define the index of γ as

$$|\gamma| := n + \theta_L(\imath^{-1}(\gamma(0)) - \theta_L(\imath^{-1}(\gamma(1)) - 2\sum_{i=1}^{n} \beta_i$$

where $\beta_1, ..., \beta_n \in (0, \frac{1}{2})$ are the Kähler angles between $D\imath[T_{\imath^{-1}(\gamma(1)}L]$ and $D(\varphi_1^H \circ \imath)[T_{\imath^{-1}(\gamma(0))}L]$.
As always in Floer theory, this is the moment to define the objects we will be counting to define the cohomology.

Definition 2.46. *Fix an admissible couple (H,J), two orbits $\gamma,\gamma_+ \in \Gamma_H$, an integer $k \in \mathbb{Z}_{\geq 0}$ and a map $\alpha : \{1,...,k\} \to R$ indexing double points of $\imath$. An α-marked Floer strip with boundary and corners on $\imath$ joining γ_- to γ_+ is a tuple $\overline{u} = (u, \Delta, \alpha, l)$ where:*

1. *$u : \mathbb{R}\times[0,1] \to M$ is a continous map which is smooth on $\mathbb{R}\times(0,1)$, satisfies $u(\mathbb{R}\times\{0,1\}) \subset \imath(L)$ and*
$$\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t)$$
uniformly with all derivatives;

2. *on $\mathbb{R} \times (0,1)$ u solves the so-called Floer equation*
$$\partial_s u + J_t(u)\partial_t u + \mathrm{grad}_{g_t} H_t(u) = 0 \tag{2.1}$$

where g_t is the time-dependent Riemannian metric induced by ω and J and grad *is the usual Riemannian gradient;*

3. *$\Delta := \{z_1, ..., z_n\} \subset \mathbb{R} \times \{0,1\}$ is an ordered subset of $\mathbb{R} \times \{0,1\}$;*
4. *$l : \mathbb{R} \times \{0,1\} - \Delta \to L$ is a countinous map lifting u to L, that is*

$$\imath \circ l = u$$

on $\mathbb{R} \times \{0,1\} - \Delta$ and such that for any $i \in \{1, ..., k\}$ we have

$$\left(\lim_{z \to z_i^-} l(z), \lim_{z \to z_i^+} l(z)\right) = \alpha(i)$$

that is, u has ougoing branch jumps of type $\alpha(i)$ at z_i;

5. *The energy $E(u) := \int_{\mathbb{R}\times[0,1]-\Delta} |\partial_s u|^2 \, dsdt$ is finite.*

We write the moduli space of parametrized α-marked Floer strip with boundary and corners on $\imath$ joining γ_- to γ_+ as $\tilde{M}_k(\gamma_-, \gamma_+, \alpha, H, J)$ and the moduli space of unparametrized ones as the orbit space

$$\mathcal{M}_k(\gamma_-, \gamma_+, \alpha, H, J) := \frac{\tilde{M}_k(\gamma_-, \gamma_+, \alpha, H, J)}{\mathbb{R}}$$

where the $\mathbb{R}$-action is given by horizontal translation. Moreover, if $\gamma_- = \gamma_+$ we define $\mathcal{M}_0(\gamma_-, \gamma, \alpha, H, J)$ to be empty if $\alpha = \emptyset$; note that in this case if $\alpha \neq \emptyset$, the moduli space does not contain any constant strips, as H is admissible. If $\alpha = \emptyset$ we often omit it from the notation.

Remark 2.47. *The Floer equation 2.1 describes the solutions to the negative gradient of the so called action functional, which is defined for $\gamma : [0,1] \to M$ such that $\gamma(0), \gamma(1) \in L$ as*

$$\mathscr{A}_H(\gamma) := h_L(\gamma(1)) - h_L(\gamma(0)) - \int_0^1 (\gamma^*\lambda + H_t(\gamma(t))) \, dt$$

which has Γ_H as set of critical points. In practice, Floer theory via Hamiltonian perturbation is an analogue of Morse theory in an infinite dimensional setting. Whereas people believed that there was no link between topological informations on M and L and the analysis of critical points of the action functional, it was Floer's breaktrough to use the ideas of Gromov to build one. Notice that the action functional as we defined is well-defined as we are working in an exact symplectic manifold: when working in other settings one has to be careful on how to define it (e.g. via covering spaces [Poz94] or via a fixed choice of capping for orbits [BC07]).

It turns out that for orbits $\gamma, \gamma_+ \in \Gamma_H$ and $\alpha : \{1, ..., k\} \to R$ indexing double points of $\imath$, $\mathcal{M}_k(\gamma_-, \gamma_+, \alpha, H, J)$ has virtual dimension

$$\operatorname{virdim}(\mathcal{M}_k(\gamma_-, \gamma_+, \alpha, H, J)) = |\gamma_-| - |\gamma_+| - \sum_{i=1}^k |\alpha(i)| + k - 1$$

(note that we are working with two different kinds of indexes). This can be seen by calculating the Maslov index of the bundle pair (see [Oh15]) (u^*TM, l^*TL) when extended to $\mathbb{R} \times [0,1] \cup \{-1,+1\} \times [0,1]$. This works similarly to the embedded case, with the only difference that one has to choose a counterclockwise path from $D\imath(p)[T_pL]$ to $D\imath(q)[T_qL]$ for any $(p,q) = \alpha(i)$.

At this point, one usually goes on proving transversality and compactness properties of the relevant moduli space. First, we have the following:

Proposition 2.48 ([AB18])**.** *Let $\gamma_-, \gamma_+ \in \Gamma_H$ be Hamiltonian orbits of H and $\alpha : \{1, ..., k\} \to R$ be a map indexing k double points of $\imath$. Then, for a generic choice of admissible couple (H, J) the moduli space $\mathcal{M}_k(\gamma_-, \gamma_+, \alpha, H, J)$ is a smooth finite dimensional manifold whose dimension agrees with its virtual dimension.*

An admissible couple (H, J) from the statement of 2.48 is said to be regular.
This kind of statement is pretty standard nowadays, and (after some years of troubles) is very well-known in the case of embedded Lagrangians. The only remarkable difference between the transverse immersed case and the embedded one is that one has to provide some new arguments to show that the solutions of the Floer equation decay fast eneough near branch jumps (this is done in [AB18, Appendix A]), while to model the moduli space on some Sobel space one needs to prove the existence of a metric which is totally geodesic on $\imath(L)$. We will not dive into the details here not to bring more hard machinery into the game; anyway, to have a grasp of what is going on here, one can find a very quick summary of how transversality works in the case of Floer cylinders (or double strips) used to construct Floer cohomology for ambient Hamiltonian diffeomorphisms in [Amb20], which summarizes the lenghty and detailed theory contained in [AD14].

We now define the Floer complex for a regular couple (H, J) as

$$CF^*(\imath; H, J) := \mathbb{Z}_2\Gamma_H$$

with the index for orbits we defined at the beginning of this section, with differential

$$d : CF^*(\imath; H, J) \longrightarrow CF^*(\imath; H, J)[-1], \quad \gamma_+ \longmapsto \sum_{\gamma_- \in \Gamma_H :\ |\gamma_-| - |\gamma_+| = 1} |\mathcal{M}_0(\gamma_-, \gamma_+, H, J)|_2 \cdot \gamma_-$$

As usual, one has now to prove some statements about compactness of the moduli spaces $\mathcal{M}_0(\gamma_-, \gamma_+, H, J)$ in dimensions 0 and 1 to really conclude that $(CF^*(\imath; H, J), d)$ is a well-defined chain complex. It is proved in [AB18] that such moduli spaces are generically compact when the dimension is 0, while they can be compactified by only adding broken strips when the dimension is 1. Again, as in the case of pseudoholomorphic disks, the keys to the proofs is that we are working in an exact environment where neither sphere nor disk bubbling may happen and that the index of a strips drops by 3 by Assumption 2.2 when a teardrop (or a tree of them) bubbles out, see Lemma 6.1 and Lemma 6.2 in [AB18] for details.

To define a nice cohomology, one then checks that couples of regular couples of parameters induce chain isomorphic chain complexes (see [AD14] for the case with Hamiltonian diffeomorphisms, [Oh93] for the embedded Lagrangian case and [AB18] for the immersed Lagrangian case). This is based on a standard argument with the so-called continuation data, i.e. homotopies between the different choices of parameters and the proof of the immersed case does not differ much from the embedded case at its core, also being quite similar to what we did for pearly homology. The idea is roughly as follows: using the chosen homotopies between regular datas we build a map counting Floer strips whose Floer equation also depends on the parameter s of $\mathbb{R} \times [0,1]$, which is the identity if we pick the trivial homotopy between the same couple of regular data; via transversality and compactness argument one shows that such a map is a chain map. Then one shows that the map induced by concatenation of homotopies is chain homotopic to the composition of the two maps induced by the two homotopies separately, still analyzing transversality and compactness of certain moduli spaces of solutions to a "higher" Floer equation (for a compact treatment with light analysis of the proof of invariance of Floer homology for Hamiltonian diffeomorphisms see Section 3.4 in [Amb20]). It goes without saying that, in our case, Assumption 2.2 still plays a central role in exluding configurations with teardrops attached.

The next definition makes now finally sense.

Definition 2.49. *We define the Lagrangian Floer cohomology $HF^*(\imath)$ of the immersion $\imath$ as the homology of the complex $CF^*(\imath; H, J)$ for a regular choice of parameters (H, J).*

One can show that $HF^*(\imath)$ is invariant under Hamiltonian diffeomorphisms and, more generally, from exact deformations of $\imath$.

For simplicity from now on we will appeal to the arguments contained in [FHS94] and consider regular couples (H, J) such that $J \in \mathcal{J}_c(M, \omega)$, that is, the almost complex structure is chosen to be autonomous.

Remark 2.50. *The question about time-dependence of the parameters (H, J) one has to request in order to achieve transversality of moduli space of Floer strips has been a subject of discussion in the years of developments of Floer theory, and the author (of course) is not aware of all the details. The point about time-dependence (or, said differently, domain-dependence) of the almost complex structure is that there may be pieces of (quasi-)pseudoholomorphic curves with different covering multiplicity in different regions of the curve (we have seen that in Proposition 1.4 and Proposition 1.7 for pseudoholomorphic disks), and, as different points of the surface are in particular mapped to the same point of the symplectic manifold, one (at least a priori) hence needs extra parameters in order to perturb the almost complex structure correctly to achieve transversality. Quite surprisingly (at least to the author), it seems like that the results contained in [Per19] (in particular Corollary 2.8) imply that one can do Floer homology of transversely intersecting embedded and immersed Lagrangians by considering a generic choice of trivial (in particular autonomous) Hamiltonian and autonomous almost complex structure in dimension bigger than* 3.

chain map ?

In the remaining of this section we will prove the following proposition.

Proposition 2.51. *Consider a regular tuple (f, g, J) in the sense of Proposition 2.41 and a regular couple (H, J') in the sense of Proposition 2.48 such that* $\mathrm{Crit}(f) \cap \mathrm{Crit}(H) \neq \emptyset$. *Then the complexes $QC^*(\imath; f, g, J)$ and $CF^*(\imath; H, J')$ are chain isomorphic via a chain isomorphism which is unique up to chain homotopy.*

? $\to$ perhaps quasi-iso. ?

No surprise here: Proposition 2.51 will be proved by analyzing transversality and compactness of moduli space of certain configurations.

Consider an smooth homotopy $(\mathbf{H}, \mathbf{J})$ connecting $(0, J)$ to (H, J'), that is smoooth maps $\mathbf{H} : \mathbb{R} \to C^\infty(M \times [0,1])$ and $\mathbf{J} : \mathbb{R} \to \mathcal{J}_c(M, \omega)$ such that there is $R > 0$ such that

$$\mathbf{J}(s) = \begin{cases} J, & \text{for } s \le -R \\ J', & \text{for } s \ge R \end{cases} \quad \text{and } \mathbf{H}(s) = \begin{cases} 0, & \text{for } s \le -R \\ H, & \text{for } s \ge R \end{cases}$$

We define the moduli space we are going to use. Consider an orbit $\gamma \in \Gamma_H$, an ordered double point $(p, q) \in R$, a critical point $x \in \mathrm{Crit}(f)$ and a map $\alpha : \{1, ..., k\} \to R$ indexing double points. We define the moduli space $\mathcal{M}_k(i, j, \alpha, \mathbf{H}, \mathbf{J})$ of α-marked Floer strips connecting $i \in \{\gamma, (p, q), x\}$ to $j \in \{\gamma, (p, q), x\}$ with $i \neq j$ (in the sense that if $i = x$, strips connect an element in $W^u(x)$ with j, while if $j = x$, strips connect i with an element in $W^s(x)$, much like in Section 2.2). We also allow $i = \emptyset$ or $j = \emptyset$ (but not both at the same time) by requiring that α-marked strips in such moduli spaces have removable singularities in $-\infty$ and $+\infty$ respectively. Notice that this time the Floer equation depends on the parameter s and that we again for simplicity omitted a priori the possibility of incoming branch jumps indexed by α. At this point, it is not hard to show that the virtual dimension of $\mathcal{M}_k(i, j, \alpha, \mathbf{H}, \mathbf{J})$ for $i, j \neq \emptyset$ is

$$\mathrm{virdim}(\mathcal{M}_k(i, j, \alpha, \mathbf{H}, \mathbf{J})) = |i| - |j| - \sum_{i=1}^{k} |\alpha(i)| + k$$

Notice that we have no -1 summand as we have not quotiented here by the action of the real numbers on strips.

Similarly to what we did Section 2.2 and Section 2.42 we need another type of configurations for index maps $\alpha_1 : \{1, ..., k_1\} \to R$ and $\alpha_2 : \{1, ..., k_2\} \to R$: we define

$$\mathcal{M}^p_{k_1;k_2}((p,q), \gamma, \alpha_1, \alpha_2, f, g, \mathbf{H}, \mathbf{J}) := \big\{(u, v, t) \in \mathcal{M}_{0,k_1}((p,q), \emptyset, \alpha_1, J) \times \mathcal{M}_{k_2}(\emptyset, \gamma, \alpha_2, \mathbf{H}, \mathbf{J}) \times \mathbb{R}_{>0} : \varphi^f_t(u(1)) = \lim_{s \to -\infty} v(s,t)\big\}$$

and viceversa

$$\mathcal{M}^p_{k_1;k_2}(\gamma, (p,q), \alpha_1, \alpha_2, f, g, \mathbf{H}, \mathbf{J}) := \big\{(v, u, t) \in \mathcal{M}_{k_2}(\emptyset, \gamma, \alpha_2, \mathbf{H}, \mathbf{J}) \times \mathcal{M}_{0,k_1}((p,q), \emptyset, \alpha_1, J) \times \mathbb{R}_{>0} : \lim_{s \to +\infty} v(s,t) = \varphi^f_{-t}(u(-1))\big\}$$

We write again $QC^*(\imath; f, g, J) = \mathbf{C} \oplus \mathbf{R}$ and define the following maps:

1. $\psi_{\mathbf{C}\Gamma} : CF^*(\imath; H, J') \to \mathbf{C}$ by linearly extending
$$\psi_{\mathbf{C}\Gamma}(\gamma) := \sum_{x \in \mathrm{Crit}(f):\ |x|=|\gamma|} |\mathcal{M}_0(x, \gamma, \mathbf{H}, \mathbf{J})|_2 \cdot x$$

2. $\psi_{\mathbf{R}\Gamma^1} : CF^*(\imath; H, J') \to \mathbf{R}$ by linearly extending
$$\psi_{\mathbf{R}\Gamma}(\gamma) := \sum_{(p,q) \in R:\ |p,q|=|\gamma|} |\mathcal{M}_0((p,q), \gamma, \mathbf{H}, \mathbf{J})|_2 \cdot x$$

3. $\psi_{\mathbf{R}\Gamma^2} : CF^*(\imath; H, J') \to \mathbf{R}$ by linearly extending
$$\psi_{\mathbf{R}\Gamma}(\gamma) := \sum_{(p,q) \in R:\ |p,q|=|\gamma|} |\mathcal{M}_{0;0}((p,q), \gamma, f, g, \mathbf{H}, \mathbf{J})|_2 \cdot x$$

4. $\psi_{\mathbf{R}\Gamma} := \psi_{\mathbf{R}\Gamma^1} + \psi_{\mathbf{R}\Gamma^2}$

and put all of this together in a map $\psi_{\mathrm{PSS}} : CF^*(\imath; H, J') \longrightarrow QC^*(\imath; f, g, J)$ as
$$\psi_{\mathrm{PSS}} := \begin{pmatrix} \psi_{\mathbf{C}\Gamma} \\ \psi_{\mathbf{R}\Gamma} \end{pmatrix}$$

Similarly we define $\phi_{\mathrm{PSS}} := \begin{pmatrix} \phi_{\Gamma\mathbf{C}} & \phi_{\Gamma\mathbf{R}} \end{pmatrix} : QC^*(\imath; f, g, J) \longrightarrow CF^*(\imath; H, J')$ by counting reverse configurations.

To show that ψ_{PSS} and ϕ_{PSS} are well-defined chain maps, one has to show that, as we are now used to, the relevant moduli spaces are generically regular in dimension 0 and 1 and have nice compactness properties. First, notice that all the strips we are dealing with in these moduli space can not be constant. Indeed, notice that as $H \neq 0$, $\mathbf{H}$ can not be independent from s: in particular there is no Hamiltonian orbit which is an orbit for all $H(s)$'s for a generic choice of homotopy $\mathbf{H}$. It goes without saying that the analysis of compactness for the moduli spaces above in dimension 0 and 1 is a combination of the results about compactness of α-marked strips with corners and of α-marked disks with corners: by counting dimensions it turns out that the moduli spaces above are compact in dimension 0 (implying that psi_{PSS} and ϕ_{PSS} are well defined) while one has to include configurations with breakings of Morse flowlines, breakings at double points, breakings of Floer and degenerations at embedded points (which however get counted twice, exactly as in Section 2.6 and Section 2.7) to compactify those moduli spaces in dimension 1 into one dimensional manifolds with boundary. With this in hand, to conclude that ψ_{PSS} and ϕ_{PSS} are chain maps one first has to show that certain terms vanish becuase of area resons, as for instance $d_{\mathrm{CR}}\psi_{\mathbf{R}\Gamma^2}$, exactly as in Section 2.6 and Section 2.7. We skip the details of step as we have already proved similar statements several times in this thesis. The fact that ψ_{PSS} and ϕ_{PSS} now follows by an argument which is exactly the same as in [AB19].

which fact ?

3 — Immersed Lagrangian quantum cohomology: A_∞ structure

In this chapter we describe a product structure for immersed Lagrangian quantum homology of generic Lagrangian immersions as well as higher operations endowing our immersion with the structure of an A_∞-algebra. Alston and Bao described in [AB18] an $\mathscr{A}_\infty$-structure for the immersed Lagrangian Floer cohomology we introduced in Section 2.8, which is practically identiacal to the one presented in detail in [Sei08], if not for some analytical details and the fact that one has to keep track of another kind of marked points when defining associahedra and in particular one has to work with open covers to define universal choices of strips like ends, as marked points where branch jumps happen may overlap. The A_∞ operations we introduce are highly inspired by [BC07] and heuristically inspired by the aforementioned trick of taking the identity as Hamiltonian diffeomorphism for the definition of Floer cohomology.

3.1 The pearly product

In this section we will prove the most of the following proposition.

Proposition 3.1. *The graded vector space $QH^*(\imath)$ admits the structure of associative ring with unit.*

Consider three Morse functions $f, f', f'' : L \longrightarrow \mathbb{R}$ in general position with the same critical points and such that all three form a Morse-Smale pair with the same pseudogradient field g. Denote $\mathbf{C} := \mathbb{Z}_2\mathrm{Crit}(f)$, $\mathbf{C'} := \mathbb{Z}_2\mathrm{Crit}(f')$ and $\mathbf{C''} := \mathbb{Z}_2\mathrm{Crit}(f'')$. At the chain level, the product will be a linear map $* : (\mathbf{C} \oplus \mathbf{R}) \otimes (\mathbf{C'} \oplus \mathbf{R}) \longrightarrow \mathbf{C''} \oplus \mathbf{R}$. Note that there is an unique isomorphism

$$(\mathbf{C} \oplus \mathbf{R}) \otimes (\mathbf{C'} \oplus \mathbf{R}) \longrightarrow (\mathbf{C} \otimes \mathbf{C'}) \oplus (\mathbf{C} \otimes \mathbf{R}) \oplus (\mathbf{R} \otimes \mathbf{C'}) \oplus (\mathbf{R} \otimes \mathbf{R})$$

We use it to define the product matrixwise as

$$* := \begin{pmatrix} \varphi^{\mathbf{C''}}_{\mathbf{CC'}} & \varphi^{\mathbf{C''}}_{\mathbf{CR}} & \varphi^{\mathbf{C''}}_{\mathbf{RC'}} & \varphi^{\mathbf{C''}}_{\mathbf{RR}} \\ \varphi^{\mathbf{R}}_{\mathbf{CC'}} & \varphi^{\mathbf{R}}_{\mathbf{CR}} & \varphi^{\mathbf{R}}_{\mathbf{RC'}} & \varphi^{\mathbf{R}}_{\mathbf{RR}} \end{pmatrix}$$

We will now define moduli space of Y-pearls of type abc for $a, b, c \in \{1, 2\}$. We leave α-markings away but at this point the reader can very well imagine how such markings would enter into play here.

We define the first some moduli spaces. Denote by $a_1 := -1 \in D$, $a_2 := e^{-\frac{i\pi}{3}} \in D$ and $a_3 := e^{\frac{i\pi}{3}} \in D$. We will need disks with a priori 3 marked points. Let $x \in \mathrm{Crit}(f), y \in \mathrm{Crit}(f')$, $z \in \mathrm{Crit}(f'')$ and $\gamma, \gamma', \gamma'' \in R$. Let $\alpha : \emptyset \to R$, we define $\mathcal{M}_0(\gamma'', \gamma, \gamma', J) := \mathcal{M}_3(\{a_1, a_2, a_3\}, \alpha_{\gamma''}^{\gamma,\gamma'}, J)$ and $\mathcal{M}_0(\emptyset, \gamma, \gamma', J) := \{\overline{u} \in \mathcal{M}_2(\{a_2, a_3\}, \alpha^{\gamma,\gamma'}, J) : \ l(a_1) \in L\}$ and so on.
We define moduli spaces of Y-shaped configurations of of type 111 as

1. $\mathcal{M}_0^{111}(z, x, y) := W^u(z) \cap W^s(x) \cap W^s(y)$
2. $\mathcal{M}_0^{111}(z, x, \gamma', J) := \{(u, t) \in \mathcal{M}_0(\emptyset, \gamma', J) \times \mathbb{R}_{>0} : \ \varphi_{-t}(u(-1)) \in W^u(z) \cap W^s(x)\}$
3. $\mathcal{M}_0^{111}(z, \gamma, y, J) := \{(u, t) \in \mathcal{M}_0(\emptyset, \gamma, J) \times \mathbb{R}_{>0} : \ \varphi_{-t}(u(-1)) \in W^u(z) \cap W^s(y)\}$
4. $\mathcal{M}_0^{111}(z, \gamma, \gamma', J) := \{(u_1, u_2, t_1, t_2) \in \mathcal{M}_0(\emptyset, \gamma, J) \times \mathcal{M}_0(\emptyset, \gamma', J) \times \mathbb{R}^2_{>0} : \ \varphi_{-t_1}(u_1(-1)) = \varphi_{-t_2}(u_2(-1)) \in W^u(z)\}$
5. $\mathcal{M}_0^{111}(\gamma'', x, y, J) := \{(u, t) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathbb{R}_{>0} : \ \varphi_t(u(1)) \in W^s(x) \cap W^s(y)\}$
6. $\mathcal{M}_0^{111}(\gamma'', x, \gamma', J) := \{(u_1, u_2, t_1, t_2) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma) \times \mathbb{R}^2_{>0} : \ \varphi_{t_1}(u_1(1)) = \varphi_{-t_2}(u_2(-1)) \in W^s(x)\}$
7. $\mathcal{M}_0^{111}(\gamma'', \gamma, y, J) := \{(u_1, u_2, t_1, t_2) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma, J) \times \mathbb{R}^2_{>0} : \ \varphi_{t_1}(u_1(1)) = \varphi_{-t_2}(u_2(-1)) \in W^s(y)\}$
8. $\mathcal{M}_0^{111}(\gamma'', \gamma, \gamma', J) := \{(u_1, u_2, u_3, t_1, t_2, t_3) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma, J) \times \mathcal{M}_0(\emptyset, \gamma', J) \times \mathbb{R}^3_{>0} : \ \varphi_{t_1}(u_1(1)) = \varphi_{-t_2}(u_2(-1)) = \varphi_{-t_3}(u_3(-1))\}$

we define moduli spaces of Y-shaped configurations of type 112 as

1. $\mathcal{M}_0^{112}(z, x, y) := \emptyset$
2. $\mathcal{M}_0^{112}(z, x, \gamma', J) := \{(v, t) \in \mathcal{M}_0(\emptyset, \emptyset, \gamma', J) : \ v(a_1) \in W^u(z), \ v(a_2) \in W^s(x)\}$
3. $\mathcal{M}_0^{112}(z, \gamma, y, J) := \emptyset$
4. $\mathcal{M}_0^{112}(z, \gamma, \gamma', J) := \{(v, u, t) \in \mathcal{M}_0(\emptyset, \emptyset, \gamma', J) \times \mathcal{M}_0(\emptyset, \gamma, J) \times \mathbb{R}_{>0} : \ v(a_1) \in W^u(z), \ \varphi_t(v(a_2)) = u(-1)\}$
5. $\mathcal{M}_0^{112}(\gamma'', x, y, J) := \emptyset$
6. $\mathcal{M}_0^{112}(\gamma'', x, \gamma', J) := \{(u, v, t) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \emptyset, \gamma') \times \mathbb{R}_{>0} : \ \varphi_t(u(1)) = v(a_1), \ v(a_2) \in W^s(x)\}$
7. $\mathcal{M}_0^{112}(\gamma'', \gamma, y, J) := \emptyset$
8. $\mathcal{M}_0^{112}(\gamma'', \gamma, \gamma', J) := \{(u_1, v, u_2, t_1, t_2) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \emptyset, \gamma', J) \times \mathcal{M}_0(\emptyset, \gamma, J) \times \mathbb{R}^2_{>0} : \ \varphi_{t_1}(u_1(1)) = v(a_1), \ \varphi_{-t_2}(u_2(-1)) = v(a_2)\}$

we define moduli spaces of Y-shaped configurations of type 121 as

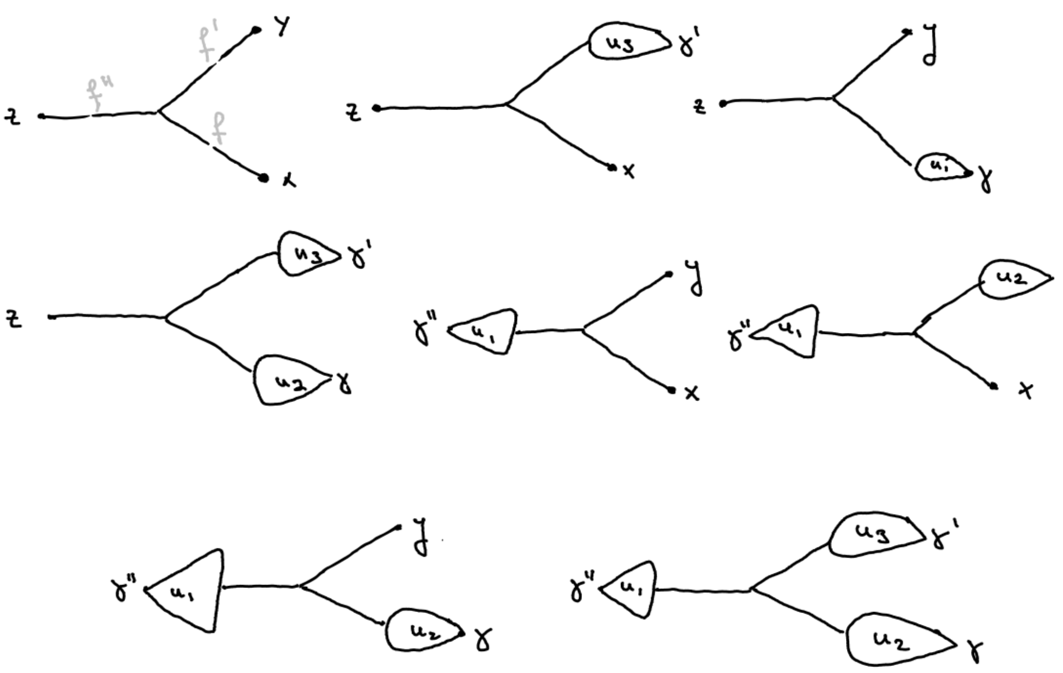

Figure 3.1: Sketch of Y-configurations of type 111.

1. $\mathcal{M}_0^{121}(z,x,y) := \emptyset$

2. $\mathcal{M}_0^{121}(z,x,\gamma',J) := \emptyset$

3. $\mathcal{M}_0^{121}(z,\gamma,y,J) := \{(v,t) \in \mathcal{M}_0(\emptyset,\gamma,\emptyset,J) :\ v(a_1) \in W^u(z),\ v(a_3) \in W^s(y)\}$

4. $\mathcal{M}_0^{121}(z,\gamma,\gamma',J) := \{(v,u,t) \in \mathcal{M}_0(\emptyset,\gamma,\emptyset,J) \times \mathcal{M}_0(\emptyset,\gamma',J) \times \mathbb{R}_{>0} :\ v(a_1) \in W^u(z),\ \varphi_t(v(a_3)) = u(-1)\}$

5. $\mathcal{M}_0^{121}(\gamma'',x,y,J) := \emptyset$

6. $\mathcal{M}_0^{121}(\gamma'',x,\gamma',J) := \emptyset$

7. $\mathcal{M}_0^{121}(\gamma'',\gamma,y,J) := \{(u,v,t) \in \mathcal{M}_0(\gamma'',\emptyset,J) \times \mathcal{M}_0(\emptyset,\gamma,\emptyset,J) \times \mathbb{R}_{>0} :\ \varphi_t(u(1)) = v(a_1),\ v(a_3) \in W^s(y)\}$

8. $\mathcal{M}_0^{121}(\gamma'',\gamma,\gamma',J) := \{(u_1,v,u_3,t_1,t_3) \in \mathcal{M}_0(\gamma'',\emptyset,J) \times \mathcal{M}_0(\emptyset,\gamma,\emptyset,J) \times \mathcal{M}_0(\emptyset,\gamma',J) \times \mathbb{R}^2_{>0} :\ \varphi_{t_1}(u_1(1)) = v(a_1),\ \varphi_{-t_3}(u_3(-1)) = v(a_3)\}$

we define moduli spaces of Y-shaped configurations of type 211 as

1. $\mathcal{M}_0^{211}(z,x,y) := \emptyset$

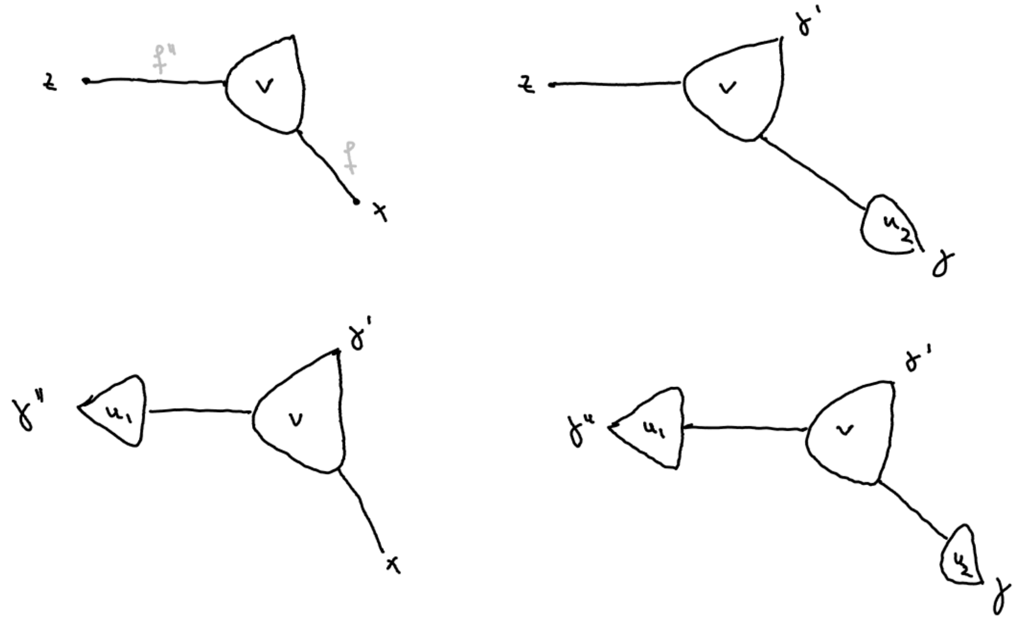

Figure 3.2: Sketch of Y-configurations of type 112.

2. $\mathcal{M}_0^{211}(z, x, \gamma', J) := \emptyset$

3. $\mathcal{M}_0^{211}(z, \gamma, y, J) := \emptyset$

4. $\mathcal{M}_0^{211}(z, \gamma, \gamma', J) := \emptyset$

5. $\mathcal{M}_0^{211}(\gamma'', x, y, J) := \{(v, t) \in \mathcal{M}_0(\gamma'', \emptyset, \emptyset, J) : \ v(a_2) \in W^s(x), \ v(a_3) \in W^s(y)\}$

6. $\mathcal{M}_0^{211}(\gamma'', x, \gamma', J) := \{(v, u, t) \in \mathcal{M}_0(', \emptyset, \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma', J) \times \mathbb{R}_{>0} : \ v(a_2) \in W^s(x), \ \varphi_t(v(a_3)) = u(-1)\}$

7. $\mathcal{M}_0^{211}(\gamma'', \gamma, y, J) := \{(v, u, t) \in \mathcal{M}_0(\gamma'', \emptyset, \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma, J) \times \mathbb{R}_{>0} : \ \varphi_t(u(2)) = v(a_2), \ v(a_3) \in W^s(y)\}$

8. $\mathcal{M}_0^{211}(\gamma'', \gamma, \gamma', J) := \{(v, u_2, u_3, t_2, t_3) \in \mathcal{M}_0(\gamma'', \emptyset, \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma, , J) \times \mathcal{M}_0(\emptyset, \gamma', J) \times \mathbb{R}^2_{>0} : \varphi_{-t_2}(u_2(-1)) = v(a_2), \ \varphi_{-t_3}(u_3(-1)) = v(a_3)\}$

we define moduli spaces of Y-shaped configurations of type 122 as

1. $\mathcal{M}_0^{122}(z, x, y) := \emptyset$

2. $\mathcal{M}_0^{122}(z, x, \gamma', J) := \emptyset$

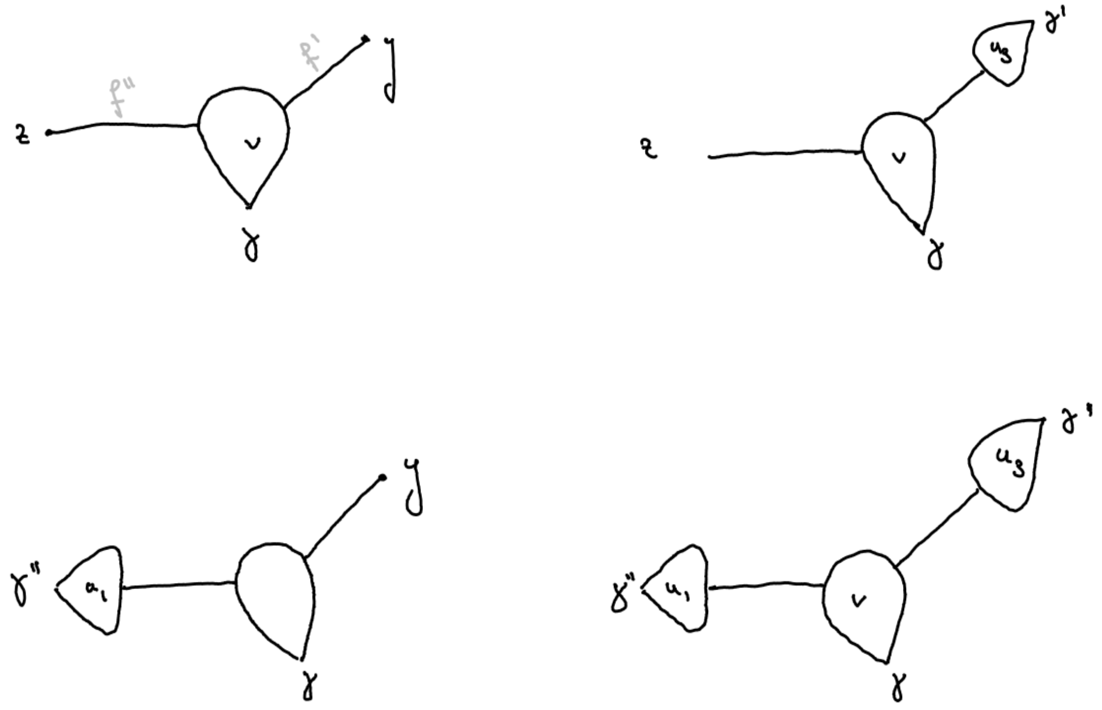

Figure 3.3: Sketch of Y-configurations of type 121.

3. $\mathcal{M}_0^{122}(z, \gamma, y, J) := \emptyset$
4. $\mathcal{M}_0^{122}(z, \gamma, \gamma', J) := \{v \in \mathcal{M}_0(\emptyset, \gamma, \gamma', J) : \ v(a_1) \in W^u(z)\}$
5. $\mathcal{M}_0^{122}(\gamma'', x, y, J) := \emptyset$
6. $\mathcal{M}_0^{122}(\gamma'', x, \gamma', J) := \emptyset$
7. $\mathcal{M}_0^{122}(\gamma'', \gamma, y, J) := \emptyset$
8. $\mathcal{M}_0^{122}(\gamma'', \gamma, \gamma', J) := \{(u_1, v, t_1) \in \mathcal{M}_0(\gamma'', \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma, \gamma', J) \times \mathbb{R}_{>0} : \ \varphi_{t_1}(u_1(1)) = v(a_1)\}$

we define moduli spaces of Y-shaped configurations of type 212 as

1. $\mathcal{M}_0^{212}(z, x, y) := \emptyset$
2. $\mathcal{M}_0^{212}(z, x, \gamma', J) := \emptyset$
3. $\mathcal{M}_0^{212}(z, \gamma, y, J) := \emptyset$
4. $\mathcal{M}_0^{212}(z, \gamma, \gamma', J) := \emptyset$
5. $\mathcal{M}_0^{212}(\gamma'', x, y, J) := \emptyset$

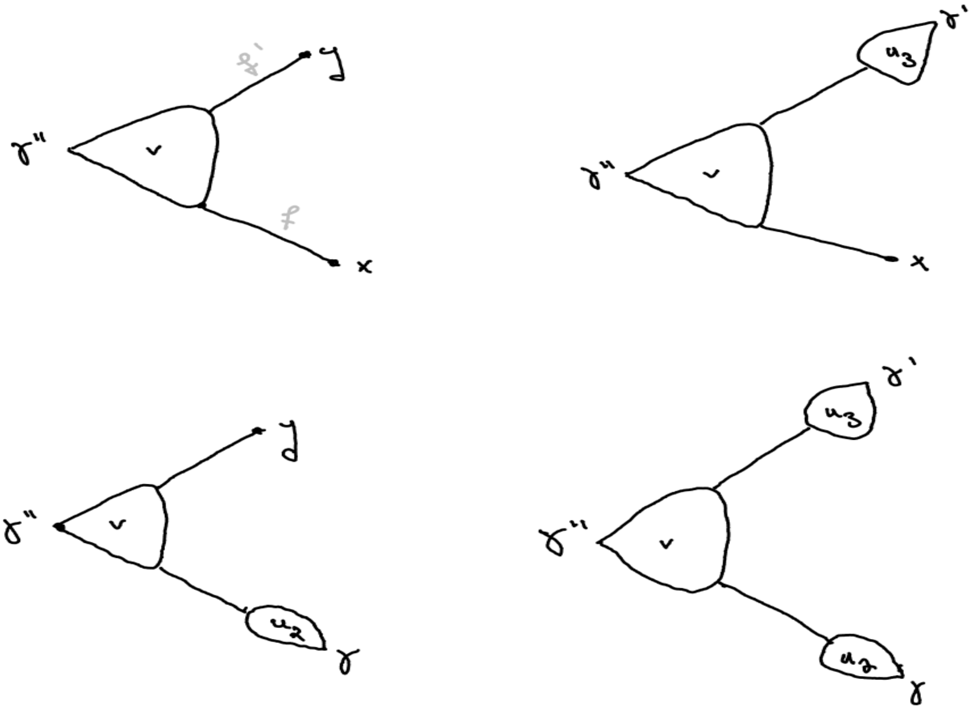

Figure 3.4: Sketch of Y-configurations of type 211.

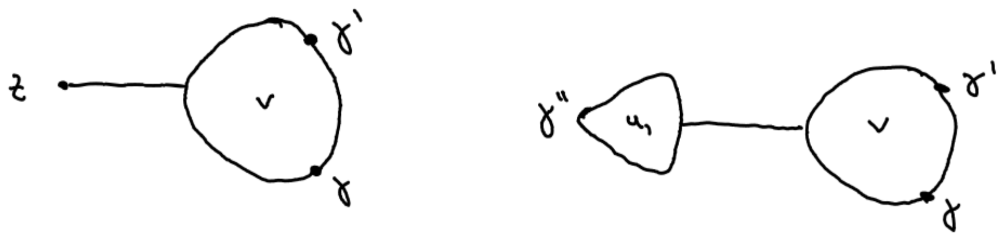

Figure 3.5: Sketch of Y-configurations of type 122.

6. $\mathcal{M}_0^{212}(\gamma'', x, \gamma', J) := \{v \in \mathcal{M}_0(\gamma'', \emptyset, \gamma') : \ v(a_2) \in W^s(x)\}$

7. $\mathcal{M}_0^{212}(\gamma'', \gamma, y, J) := \emptyset$

8. $\mathcal{M}_0^{212}(\gamma'', \gamma, \gamma', J) := \{(v, u_2, t_2) \in \mathcal{M}_0(\gamma'', \emptyset, \gamma', J) \times \mathcal{M}_0(\emptyset, \gamma, , J) \times \mathbb{R}_{>0} : \ \varphi_{-t_2}(u_2(-1)) = v(a_2)\}$

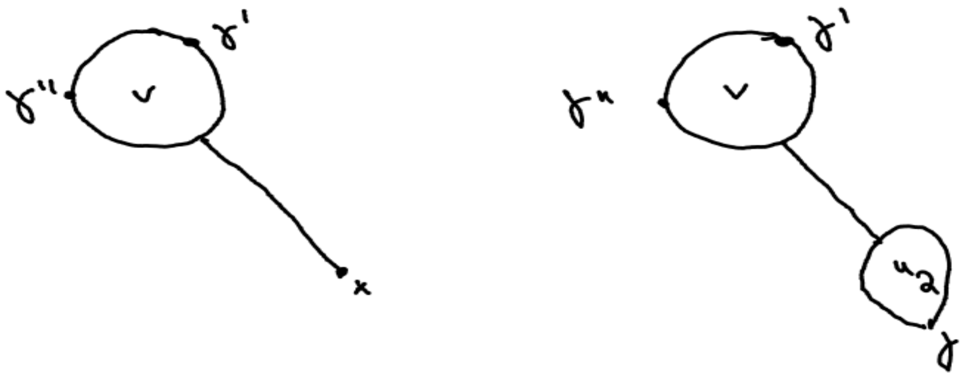

Figure 3.6: Sketch of Y-configurations of type 212.

we define moduli spaces of Y-shaped configurations of type 221 as

1. $\mathcal{M}_0^{221}(z, x, y) := \emptyset$

2. $\mathcal{M}_0^{221}(z, x, \gamma', J) := \emptyset$

3. $\mathcal{M}_0^{221}(z, \gamma, y, J) := \emptyset$

4. $\mathcal{M}_0^{221}(z, \gamma, \gamma', J) := \emptyset$

5. $\mathcal{M}_0^{221}(\gamma'', x, y, J) := \emptyset$

6. $\mathcal{M}_0^{221}(\gamma'', x, \gamma', J) := \emptyset$

7. $\mathcal{M}_0^{221}(\gamma'', \gamma, y, J) := \{v \in \mathcal{M}_0(\gamma'', \gamma, \emptyset, J) : \ v(a_3) \in W^s(y)\}$

8. $\mathcal{M}_0^{221}(\gamma'', \gamma, \gamma', J) := \{(v, u_3, t_3) \in \mathcal{M}_0(\gamma'', \gamma, \emptyset, J) \times \mathcal{M}_0(\emptyset, \gamma', , J) \times \mathbb{R}_{>0} : \ \varphi_{-t_3}(u_3(-1)) = v(a_3)\}$

we define moduli spaces of Y-shaped configurations of type 222 as

1. $\mathcal{M}_0^{222}(z, x, y) := \emptyset$

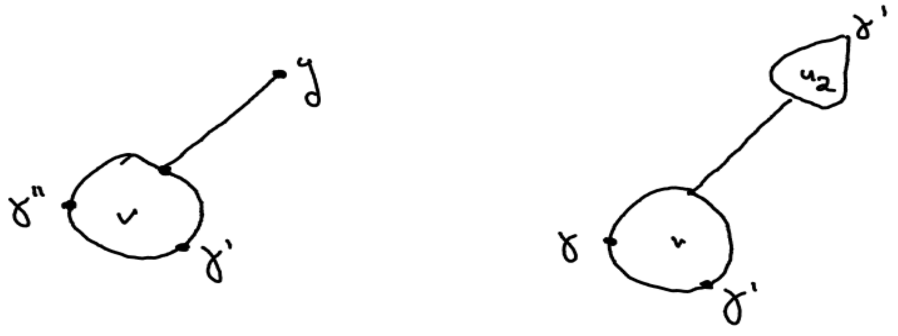

Figure 3.7: Sketch of Y-configurations of type 221.

2. $\mathcal{M}_0^{222}(z, x, \gamma', J) := \emptyset$
3. $\mathcal{M}_0^{222}(z, \gamma, y, J) := \emptyset$
4. $\mathcal{M}_0^{222}(z, \gamma, \gamma', J) := \emptyset$
5. $\mathcal{M}_0^{222}(\gamma'', x, y, J) := \emptyset$
6. $\mathcal{M}_0^{222}(\gamma'', x, \gamma', J) := \emptyset$
7. $\mathcal{M}_0^{222}(\gamma'', \gamma, y, J) := \emptyset$
8. $\mathcal{M}_0^{222}(\gamma'', \gamma, \gamma', J) := \mathcal{M}_0(\gamma'', \gamma, \gamma', J)$

Remark 3.2. *The author is sorry for such an eyesore.*

Remark 3.3. *We can view Y-configurations with no core disk in the definition as having a constant core disk.*

Notice that as we assumed f, f', f'' to be in general position it directly follows that $M_0^{111}(z, x, y)$ is a smooth manifold of dimension $|z| - |x| - |y|$. More generally, it is easy to see that all the other (non-empty) moduli spaces $\mathcal{M}_0^{abc}(i, j, k, J)$ have virtual dimension $|i| - |j| - |k|$, where the type of index depends on the nature of i, j, k respectively.

The argument for simpleness and absolute transversality of the moduli spaces above are quite similar to the arguments we provided in Section 2.3 and we get that as long as the virtual dimension is smaller equal than 1 and $\gamma \neq \gamma' \neq \sigma \circ \gamma''$, where σ is the non-trivial permutation of two elements, the moduli spaces above are finite dimensional smooth manifolds whose dimension agrees with the virtual dimension for a generic choice (of the same) almost complex structure.

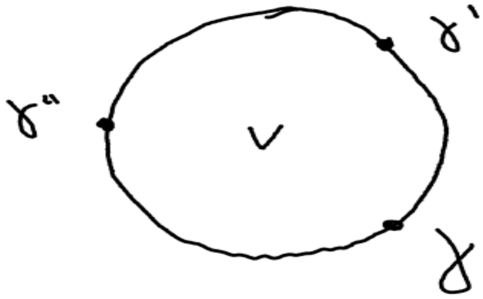

Figure 3.8: Sketch of Y-configurations of type 222.

The delicate configurations still as in Section 2.3 may arise when there are configurations containing couples of triples of the same double point. Again, this probem is solved by the fact that such configuration come in even classes of moduli spaces: consider for instance the eight (111, 112, 121, 211, 122, 212, 221 and 222) classes of moduli spaces connecting (q,p) to (p,q) and (p,q), then it is easy to see that the first seven only contain simple elements, which may however be not absolutely distinct, while the 222-class may contain 3-covered disks.

We define $* : QC^*(\imath; f, g, J) \otimes QC^*(\imath; f', g, J) \longrightarrow QC^*(\imath; f'', g, J)$ matrixwise by counting the elements of the above moduli spaces in dimension 0. Consider critical points $x \in \mathrm{Crit}(f), y \in \mathrm{Crit}(f')$ and double points $\gamma, \gamma' \in R$. We define

1. $\varphi^{\mathbf{C''}}_{\mathbf{CC'}} : \mathbf{C} \otimes \mathbf{C'} \to \mathbf{C''}$ by linearly extending
$$\varphi^{\mathbf{C''}}_{\mathbf{CC'}}(x,y) := \sum_{z \in \mathrm{Crit}(f''):\ |z|=|x|+|y|} |\mathcal{M}^{111}_0(z,x,y)|_2 \cdot z$$

2. $\varphi^{\mathbf{C''}}_{\mathbf{CR}} : \mathbf{C} \otimes \mathbf{R} \to \mathbf{C''}$ by linearly extending
$$\varphi^{\mathbf{C''}}_{\mathbf{CR}}(x,\gamma') := \sum_{z \in \mathrm{Crit}(f''):\ |z|=|x|+|\gamma'|} \sum_{a,b,c} |\mathcal{M}^{abc}_2(z,x,\gamma')|_2 \cdot z$$

3. $\varphi^{\mathbf{C''}}_{\mathbf{RC'}} : \mathbf{R} \otimes \mathbf{C'} \to \mathbf{C''}$ by linearly extending
$$\varphi^{\mathbf{C''}}_{\mathbf{RC'}}(\gamma,y) := \sum_{z \in \mathrm{Crit}(f''):\ |z|=|\gamma|+|y|} \sum_{a,b,c} |\mathcal{M}^{abc}_2(z,\gamma,y)|_2 \cdot z$$

4. $\varphi^{\mathbf{C''}}_{\mathbf{RR}} : \mathbf{R} \otimes \mathbf{R} \to \mathbf{C''}$ by linearly extending
$$\varphi^{\mathbf{C''}}_{\mathbf{RR}}(\gamma,\gamma') := \sum_{z \in \mathrm{Crit}(f''):\ |z|=|\gamma|+|\gamma'|} \sum_{a,b,c} |\mathcal{M}^{abc}_2(z,\gamma,\gamma')|_2 \cdot z$$

5. $\varphi^{\mathbf{R}}_{\mathbf{CC'}} : \mathbf{C} \otimes \mathbf{C'} \to \mathbf{R}$ by linearly extending

$$\varphi^{\mathbf{R}}_{\mathbf{CC'}}(x,y) := \sum_{\gamma'' \in R:\ |\gamma''|=|x|+|y|} \sum_{a,b,c} |\mathcal{M}_0^{abc}(\gamma'', x, y)|_2 \cdot \gamma''$$

6. $\varphi^{\mathbf{R}}_{\mathbf{CR}} : \mathbf{C} \otimes \mathbf{R} \to \mathbf{R}$ by linearly extending

$$\varphi^{\mathbf{R}}_{\mathbf{CR}}(x,\gamma') := \sum_{\gamma'' \in R:\ |\gamma''|=|x|+|\gamma'|} \sum_{a,b,c} |\mathcal{M}_2^{abc}(\gamma'', x, \gamma')|_2 \cdot \gamma''$$

7. $\varphi^{\mathbf{R}}_{\mathbf{RC'}} : \mathbf{R} \otimes \mathbf{C'} \to \mathbf{R}$ by linearly extending

$$\varphi^{\mathbf{R}}_{\mathbf{RC'}}(\gamma,y) := \sum_{\gamma'' \in R:\ |\gamma''|=|\gamma|+|y|} \sum_{a,b,c} |\mathcal{M}_2^{abc}(\gamma'', \gamma, y)|_2 \cdot \gamma''$$

8. $\varphi^{\mathbf{R}}_{\mathbf{RR}} : \mathbf{R} \otimes \mathbf{R} \to \mathbf{R}$ by linearly extending

$$\varphi^{\mathbf{R}}_{\mathbf{RR}}(\gamma,\gamma') := \sum_{\gamma'' \in R:\ |\gamma''|=|\gamma|+|\gamma'|} \sum_{a,b,c} |\mathcal{M}_2^{abc}(\gamma'', \gamma, \gamma')|_2 \cdot \gamma''$$

Again, to see that $*$ is a well-defined chain map, we have to look at compactness and compactification of the moduli spaces $\mathcal{M}_0^{a,b,c}(i,j,k,J)$. This is very similar to what we have already done in Section 2.4 and 2.5, and we skip the details (and the headaches coming from the analysis of α-markings). The only main difference with the compactness results which lead to the properties of the pearly differential, is bubbling off of disks from the core disk: those configuration (which may admit some a stable ghost disk if a couple of marked points overlap, as we defined the core disk to have three marked points) will cancel out in the modulo 2 sum with limit configurations coming from shrinkage of Morse flowlines, much like in the definition of the differential and of the product in [BC07]. In other words, following the notation above, we can express bubbling at the core disk in two ways by looking at the two values of $abc \in \{111, 112, 121, 211, 122, 212, 221, 222\}$ we are interested in (see Figure 3.1)

At this point it remains to prove that the product we defined in this section does not depend on choices of regular parameters $(f, , f', f'', g, J)$ at the cohomological level. The argument which one needs to prove this goes along the same lines of what we did in Section 2.7, while taking care of the bubbling configurations we just described.

Alough we did not elaborate any counterexample, it is expected that the product, even at the cohomological level, is not commutative, principally because the marked points on the core disks are fixed, so that the it can not be reparametrized.

3.2 Unit

We will now show that immersed Lagrangian quantum cohomology admist a canonical unit with respect to the product we defined above. What follows is very similar to the embedded case [BC07].

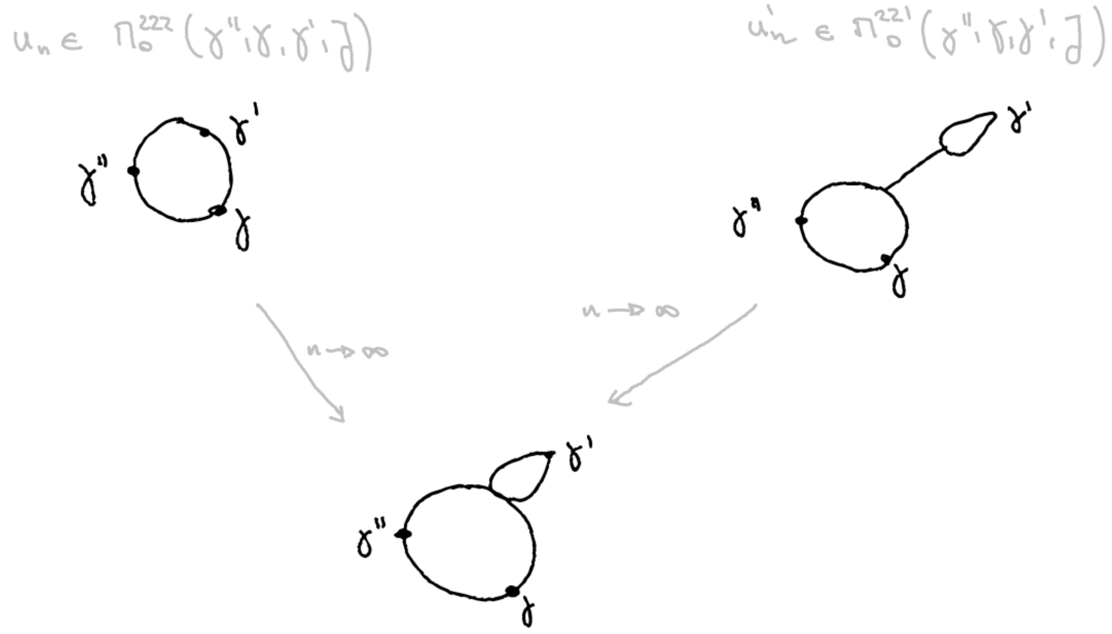

Figure 3.9: An example of how two sequences of Y-shaped configurations connecting the same double points but with different values of abc may compensate each other in the Gromov limit.

Lemma 3.4. *There is a canonical element $e \in QH^0(\imath)$ which is a unit with respect to the pearly product defined in Section 3.1.*

We will first work at the chain level. Assume that $f'' = f$ and that f' has a unique minimum $y \in \mathrm{Crit}(f')$ (these are generic assumptions). We claim that the generator $(y, 0) \in QC(\imath; f', g, J)$ is a unit for the pearly product defined using the above data. First, we show $d(y, 0) = 0$. We rewrite $d(y, 0)$ as

$$\begin{pmatrix} d_{\mathbf{C'C'}} & d_{\mathbf{C'R}} \\ d_{\mathbf{RC'}} & d_{\mathbf{RR}} \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix}$$

Clearly, $d_{\mathbf{C'R}}0 = 0$ and $d_{\mathbf{RR}}0 = 0$. Consider the Morse differential $d_{\mathbf{C'C'}}y$ of y: as $HM^0(L) \cong \mathbb{Z}_2$, and y is the only generator in degree 0, we have $\ker d_{\mathbf{C'C'}} \cong \mathbb{Z}_2$ in degree 0, so that $d_{\mathbf{C'C'}}y = 0$. Consider now $d_{\mathbf{RC'}}y$: it counts pearly configurations starting at double points $|p, q|$ with index equal to 1, as that of y vanishes. In particular, it follows that $\mathscr{A}(p, q) \leq 0$ and hence a teardrop u contributing to the differential would have energy $E(u) = \mathscr{A}(p, q) \leq 0$, a contradiction. This proves that $d_{\mathbf{RC'}}y = 0$ and hence that y is a cycle.

Let $x \in \mathrm{Crit}(f)$ and $(p,q) \in R$. We have

$$(x,\gamma) * (y,0) = \begin{pmatrix} \varphi^{\mathbf{C}}_{\mathbf{CC'}} & \varphi^{\mathbf{C}}_{\mathbf{CR}} & \varphi^{\mathbf{C}}_{\mathbf{RC'}} & \varphi^{\mathbf{C}}_{\mathbf{RR}} \\ \varphi^{\mathbf{R}}_{\mathbf{CC'}} & \varphi^{\mathbf{R}}_{\mathbf{CR}} & \varphi^{\mathbf{R}}_{\mathbf{RC'}} & \varphi^{\mathbf{R}}_{\mathbf{RR}} \end{pmatrix} \begin{pmatrix} x \otimes y \\ 0 \\ \overline{\gamma} \otimes y \\ 0 \end{pmatrix}$$

1. We claim that $\varphi^{\mathbf{C''}}_{\mathbf{CC'}}(x,y) = y$. This is an argument from Morse theory. Indeed, if for $z \in \mathrm{Crit}(f)$ there is a Morse Y-configurations joining z to x and y, then $|z| = |x|$. In particular, the existence of such an Y-configuration implies the existence of a Morse flowline joining z to x, which combined with the dimension above directly implies $x = z$.

2. We claim that $\varphi^{C}_{\mathbf{RC'}}(\gamma,y) = 0$. Assume there is $z \in \mathrm{Crit}(f)$ such that there exist an Y-configuration from z to γ and y in the case 111 or 121, then $|z| - |\gamma| = 0$. Then, as there is a reparametrization of D fixing -1 and taking $e^{-i\pi/3}$ to 1, the existence of such a configuration implies the existence of a CR-pearl joining z to γ, a contradiction to $|z| = |\gamma|$.

3. We claim that $\varphi^{R}_{\mathbf{CC'}}(x,y) = 0$. This is very similar to the case above.

4. We claim that $\varphi^{C}_{\mathbf{RC'}}(\gamma,y) = \gamma$. Note that there is no Y-configuration joining γ to γ and y of type which is not 221, as such a configuration would have a constant teardrop, which is not allowed. We conclude that the only configuration contributing to $\varphi^{C}_{\mathbf{RC'}}(\gamma,y) = \gamma$ is a constant disk with branch jumps which lies in the stable manifold of y (that is a configuration of type 221).

This proves that $(y,0) \in QC(\imath; f', g, J)$ is a unit for the pearly product at the chain level. Morover, again by simple Morse theoretic arguments, it is immediate to see that comparison morphisms ψ from Section 2.7 send unique minima to unique minima, so that $(y,0)$ induces a canonical unit $e := [(y,0)] \in QH(\imath)$. This concludes the proof of Lemma 3.4.

3.3 Associativity and higher operations

3.4 An example of computation

Alston [Als13] calculated Floer cohomology of some immersed Lagrangian submanifold in an affine symplectic submanifold of $\mathbb{C}^3$ using the machinery later developed in [AB19]. We mainly follow [Als13] while adding some details. Here, we will work with a symplectic manifold which is non-compact: all the machinery we developed in this thesis may be translated to a non-compact setting by asking that the convexity condition take place in some compact subset K and that the holomorphic curves having boundary on our immersion are contained in K; these are requirements that only affect the choice of the almost complex structure.

Definition of the immersion.

Let $N \in \mathbb{Z}_{>0}$ and consider the polynomial

$$F_N := x_1 x_2 - \prod_{i=1}^{N}(x_3 - k) \in \mathbb{C}[x_1, x_2, x_3]$$

We consider here $M_N := F_N^{-1}(0)$ endowed with the exact symplectic form ω and compatible complex structure J_{M_N} induced by the standard Kähler structure of $\mathbb{C}^3$. We moreover endow M_N with a nowhere vanishing holomorphic form $\Omega_{M_N} \in \Omega^{0,2}(M_N)$ by taking the Poincaré residue of the meromorphic volume form $\frac{1}{F_N} dx_1 \wedge dx_2 \wedge dx_3$ on $\mathbb{C}^3$.

Consider the functions $H, G : M_N \to \mathbb{R}$ given by $H(x_1, x_2, x_3) := \frac{1}{2}(|x_1| - |x_2|^2)$ and $G(x_+, x_2, x_3) := |x_3|^2$ for $(x_1, x_2, x_3) \in M_N$. For any $r \in \{1, ..., N\}$ define $L_{N,r} := (H, G)^{-1}(0, r)$. We construct an explicit immersion of S^2 into M_N parametrizing $L_{N,r}$ in cylindrical coordinates $(a, e^{ib}) \in (-\pi, \pi) \times S^1$:

$$i_{N,r} : (-\pi, \pi) \times S^1 \longrightarrow M_N, \quad (a, e^{ib}) \longmapsto (e^{ib}\xi(a), e^{-ib}\xi(a), -re^{ia})$$

where $\xi(a) := \prod_{k=1}^{N} \sqrt{-re^{ia} - k}$.

Proposition 3.5. *The map $i_{N,r}$ extends smoothly to a map $S^2 \to M_{N,r}$ is an exact Lagrangian immersion with one transverse double point and $L_{N,r}$ as image.*

Proof. One could use some result from theory of integrable system. We do the explicit computations. To show that the symplectic form vanishes on $L_{N,r}$ one simply shows that H and G Poisson commute, and this is straightforward. The fact that $i_{N,r}$ extends smoothly to S^2 is proved is also straightforward. $i_{N,r}$ is then an immersion as $i_{N,r}(q_{S^2}) = i_{N,r}(p_{S^2})$, where q_{S^2} and p_{S^2} are the south and north pole of S^2 respectively. Differentiating $i_{N,r}$ is not hard to compute that

$$Di_{N,r}(p_S^2)[T_{p_{S^2}} S^2] = \mathrm{span}_{\mathbb{R}}((v, v, 0), (iv, -iv, 0))$$

and

$$Di_{N,r}(q_S^2)[T_{q_{S^2}} S^2] = \mathrm{span}_{\mathbb{R}}((iv, iv, 0), (-v, v, 0))$$

where $v := \sqrt{2\pi i r \prod_{k=1}^{N}(r - k)}$. This shows that the only self-intersection is in fact transverse. $\square$

We grade $i_{N,r}$ by $\theta_{N,r} := \frac{1}{\pi} id$. Then

$$Di_{N,r}(q_{S^2})[T_{q_{S^2}} S^2] = \begin{pmatrix} e^{i\frac{\pi}{2}} & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix} Di_{N,r}(p_{S^2})[T_{p_{S^2}} S^2]$$

using the calculation in the above proof once modified to have unitary bases, so that $|(p_{S^2}, q_{S^2})| = -1$ and $|(q_{S^2}, p_{S^2})| = 3$.

We introduce the fibration

$$\pi : M_N \longrightarrow \mathbb{C}, \quad (x, y, z) \longmapsto z$$

One sees that under this fibration the p_{S^2}-branch of $L_{N,r}$ is an arc lying in the lower half-plane with endpoint r, while the q_{S^2}-branch of $L_{N,r}$ is an arc lying in the upper half-plane with endpoint r.

The Floer cohomological of $i_{N,r}$: vector space structure.

We now classify pseudoholomorphic disks with corners and boundary on $L_{N,1}$. Consider such a disk $u = (f, g, h) : D \to M_N$ with only outgoing branched jumps, that is, of the type (p_{S^2}, q_{S^2}). By definition, it is smooth on the interior $\mathrm{int}(D)$ of the disk, so that $h|_{\mathrm{int}(D)} : \mathrm{int}(D) \to \mathbb{C}$ is holomorphic with boundary on $\pi(L_{N,r}) = \{|z| = r\}$. By the maximum principle, h is then a map $D \to \{|z| \leq r\}$, which, by choice of $i_{N,r}$ (see discussion above), is holomorphic on all of D. Well-known complex analysis (see for instance [Eis71]) tells us that, if $r = 1$, h is a Blaschke product, of order given by the number of branched points of u. Then, it is not hard to see that there is $\theta \in [0, 2\pi)$ such that $f = e^{i\theta} \prod_{i=1}^{N} \sqrt{h - i}$ and $g = e^{-i\theta} \prod_{i=1}^{N} \sqrt{h - i}$.

For general $r \in \{1, ..., N\}$ the main difference with the case $r = 1$ is that we have more degrees of freedom, in the sense that there are some Blaschke products $h_1, ..., h_{r-1}$ contributing to f and $h'_1, ..., h'_{r-1}$ contributing to g such that $h_j h'_j = \frac{rh-j}{jh-r}$. Note that only such Blaschke products appearing in f and g determine the connected component in which u lies; in particular, in the case $r = 1$ the moduli space associated to fixed α is connected.

Pick a Morse function $f : S^2 \to \mathbb{R}$ with exactly a maximum at $p_M := (0, 1)$ and a minimum at $p_m := (0, -1)$ and a pseudogradient field g such that (f, g) is Morse-Smale. Then $QC^{-1}(i_{N,r}; f, g, J) = \mathbb{Z}_2(p_{S^2}, q_{S^2})$, $QC^0(i_{N,r}; f, g, J) = \mathbb{Z}_2 p_m$, $QC^2(i_{N,r}; f, g, J) = \mathbb{Z}_2 p_M$ and $QC^3(i_{N,r}; f, g, J) = \mathbb{Z}_2(q_{S^2}, p_{S^2})$, while in all the other degrees j $QC^j(i_{N,r}; f, g, J)$ is trivial. We have the following proposition on the structure of the Lagrangian quantum cohomology of $i_{N,r}$.

Proposition 3.6. *If $r = 1$, the Floer cohomology $QH(i_{N,r})$ of $i_{N,r}$ is trivial, while for $r > 1$ it is isomorphic $\mathbb{Z}_2$ in degrees -1, 0, 2, 3.*

to

Proof. Fix a regular choice of the almost complex structure J. We claim that

$$|\mathcal{M}_0(p_m, (p_{S^2}, q_{S^2}), J)| = |\mathcal{M}_0((q_{S^2}, p_{S^2}), p_M, J)| = 2^{r-1}$$

Consider the case of $\mathcal{M}_0(p_m, (p_{S^2}, q_{S^2}), J)$. Note that as p_m is the minimum of f, we have no Morse trajectories in our pearly trajectories.

Let $\overline{u} \in \mathcal{M}_0(\emptyset, (p_{S^2}, q_{S^2}), J)$ and write $u = (f, g, rh)$. Then, as we have only one branch jump, h is an automorphism of the disk fixing $1 \in D$, so that we can reparametrize D via an element of $\mathrm{Aut}(D, 1) \cong \mathbb{R} \times \mathbb{R}^*$ in a way such that h is the identity map. By the discussion above, the only other thing characterizing u is the choice of a phase $e^{i\theta} \in S^1$. Hence, as in the definition of the moduli space we quotient by $\mathrm{Aut}(D, 1, -1)$, we conclude that the moduli space of interesting configuration has components diffeomorphic to the cylinder $\mathbb{R} \times S^1$. Moreover, by the discussion above, it has 2^{r-1} such components. From this, it follows easily that the map $ev_{-1} : \mathcal{M}_0(\emptyset, (p_{S^2}, q_{S^2}), J) \to S^2$ sending u to $l(-1)$ restrict to a diffeomorphism between each component of our moduli space and the coordinate patch of S^2 in cylindrical coordinates. From this, we can conclude that exactly one element of u for each component of our moduli space of disks satisfies $ev_{-\infty}(u) = p_m$, implying that the cardinality of $\mathcal{M}_0(p_m, (p_{S^2}, q_{S^2}), J)$ is 2^{r-1} concluding the proof of the claim and of the proposition. $\square$

Floer cohomology of $i_{N,r}$: ring structure.

We use the machinery defined in this chapter to study the product structure on $QH(i_{N,r})$. Pick another Morse function f' on S^2 in general position with f and with the same critical points. Note that the only products that are a priori non-trivial are

$$QC^0(i_{N,r}, f) \otimes QC^i(i_{N,r}, f') \longrightarrow QC^i(i_{N,r}, f), \quad \text{for } i \in \{-1, 0, 2, 3\}$$

and

$$QC^{-1}(i_{N,r}, f) \otimes QC^3(i_{N,r}, f') \longrightarrow QC^2(i_{N,r}, f)$$

and the remaining with upper indices swapped. As $QC^0((i_{N,r}, f) = \mathbb{Z}_2 p_m$, the first map is just multiplication with zero and the unit. We investigate calculate $(p_{S^2}, q_{S^2}) * (q_{S^2}, p_{S^2})$. Note that as $\mathscr{A}(p_{S^2}, q_{S^2}) = -\mathscr{A}(q_{S^2}, p_{S^2}) < 0$, the only interesting configurations contributing to the product are those going from p_M to (p_{S^2}, q_{S^2}) and (q_{S^2}, p_{S^2}) of type 122. Let $\overline{u} \in \mathcal{M}_0^{122}(p_M, (p_{S^2}, q_{S^2}), (q_{S^2}, p_{S^2}), J)$, then u is constant as $E(u) = \mathscr{A}(p_{S^2}, q_{S^2}) + \mathscr{A}(q_{S^2}, p_{S^2} = 0$. It follows that

$$|\mathcal{M}_0^{122}(p_M, (p_{S^2}, q_{S^2}), (q_{S^2}, p_{S^2}), J)|_2 = 1$$

and hence

$$(p_{S^2}, q_{S^2}) * (q_{S^2}, p_{S^2}) = p_M$$

as conjectured in [Als13]. Similarly, one shows that

$$(q_{S^2}, p_{S^2}) * (p_{S^2}, q_{S^2}) = p_M$$

as well.

Floer cohomology of $i_{N,r}$: A_∞-structure.

We investigate on higher operations μ_k on $QH(i_{N,r})$. First of all, notice that the only non-trivial operations, i.e. those involving other generators than the identity element, always involve both (p_{S^2}, q_{S^2}) and (q_{S^2}, p_{S^2}) as (p_{S^2}, q_{S^2}) has negative action, (q_{S^2}, p_{S^2}) has index 3 which is relatively prime to the index of p_M. In particular, the only configurations contributing to μ_k for $k \geq 3$ are those with p_m, (p_{S^2}, q_{S^2}) and (q_{S^2}, p_{S^2}) as exits and p_M as entry. Pick another Morse function f'' on S^2 in general position with f and f' and with the same critical points as them. We claim that

$$\mu_3((q,p), p_m, (p,q)) = \mu_3((p,q), p_m, (q,p)) = p_M$$

while

$$\mu_3(p_m, (p,q), (q,p)) = \mu_3(p_m, (q,p), (p,q)) = \mu_3((q,p), (p,q), p_m) = \mu_3((p,q), (q,p), p_m) = 0$$

We start from the first line of equalities. $\mu_3((q,p), p_m, (p,q))$ counts configurations as in Figure 3.10. Note that v has to be constant constant as $E(v) = \mathscr{A}(p_{S^2}, q_{S^2}) + \mathscr{A}(q_{S^2}, p_{S^2}) = 0$, so that the first claim follows directly. The case of $\mu_3((p,q), p_m, (q,p))$ is identical. On the other hand, $\mu_3((p,q), (q,p), p_m)$ counts both configurations as in Figure 3.11 and 3.12, where again the disks in question are constant, so that the claim also follows directly. The remaining three cases are identical to the last one.

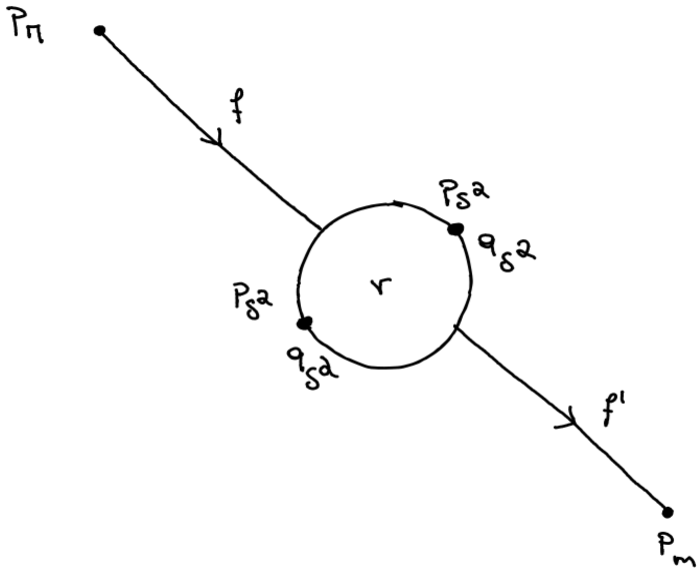

Figure 3.10: The only configuration contributing to $\mu_3((q,p), p_m, (p,q))$.

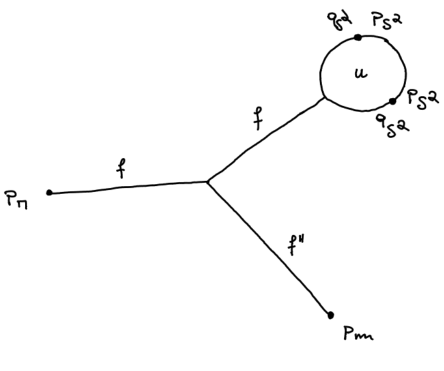

Figure 3.11: A configuration contributing to $\mu_3((p,q),(q,p),p_m)$.

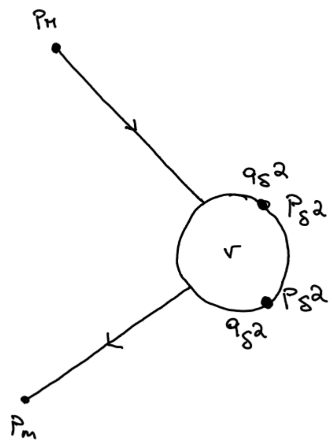

Figure 3.12: Another configuration contributing to $\mu_3((p,q),(q,p),p_m)$.

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