LEVEL-SET PERCOLATION OF THE GAUSSIAN FREE FIELD ON REGULAR GRAPHS I: REGULAR TREES

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Preliminary draft

Abstract

We study level-set percolation of the Gaussian free field on the infinite d-regular tree for fixed $d \geq 3$. Denoting by $h_\star$ the critical value, we obtain the following results: for $h > h_\star$ we derive estimates on conditional exponential moments of the size of a fixed connected component of the level set above level h; for $h < h_\star$ we prove that the number of vertices connected over distance k above level h to a fixed vertex grows exponentially in k with positive probability. Furthermore, we show that the percolation probability is a continuous function of the level h, at least away from the critical value $h_\star$. Along the way we also obtain matching upper and lower bounds on the eigenfunctions involved in the spectral characterisation of the critical value $h_\star$ and link the probability of a non-vanishing limit of the martingale used therein to the percolation probability. A number of the results derived here are applied in the accompanying paper [AČ19].

0 Introduction

In this article we investigate the Gaussian free field on d-regular trees with $d \geq 3$. We focus in particular on the level sets and their behaviour in connection with level-set percolation. The goal is to obtain a good description of the nature of the level sets for levels away from the critical value of level-set percolation.

Level-set percolation of the Gaussian free field is a significant representative of a percolation model with long-range dependencies and it has attracted attention for a long time, dating back to [MS83], [LS86] and [BLM87]. More recent developments can be found for instance in [RS13], [PR15], [Szn15], [DPR18b], [DPR18a], [Nit18] and [CN18]. The particular case of Gaussian free field on regular trees was studied before in [Szn16] and [Szn19], and on general transient trees in [AS18]. Compared to the present article, the emphasis in these three papers is put on a different aspect of the Gaussian free field, namely its connection with the model of random interlacements.

Studying level-set percolation of the Gaussian free field on regular trees specifically is of intrinsic interest. The case of the regular tree comes along with strong tools based on the structure and symmetry of the graph. These allow for often very exact computations which potentially lead to especially explicit, though not at all trivial, results. They also make it one of the most promising setups for understanding level-set percolation of the Gaussian free field near criticality.

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Besides the fact that the results obtained in this article are interesting in their own right, our prime motivation comes from a concrete application: in the accompanying paper [AČ19] we prove a phase transition in the behaviour of the level sets of the *zero-average* Gaussian free field on a certain class of *finite* d-regular graphs that are locally (almost) tree-like. This class includes d-regular expanders of large girth and typical realisations of random d-regular graphs. In a certain sense, the Gaussian free field on the d-regular tree provides the *local picture* of the zero-average Gaussian free field on these finite graphs, and its detailed understanding developed in the present article is a key ingredient for [AČ19].

We now describe our results more precisely. Let $d \geq 3$ and denote by $\mathbb{T}_d$ the infinite d-regular tree. On $\mathbb{T}_d$ we consider the Gaussian free field with law $\mathbb{P}^{\mathbb{T}_d}$ on $\mathbb{R}^{\mathbb{T}_d}$ and canonical coordinate process $(\varphi_{\mathbb{T}_d}(x))_{x \in \mathbb{T}_d}$ so that,

> under $\mathbb{P}^{\mathbb{T}_d}$, $(\varphi_{\mathbb{T}_d}(x))_{x \in \mathbb{T}_d}$ is a centred Gaussian field on $\mathbb{T}_d$ with covariance $\mathbb{E}^{\mathbb{T}_d}[\varphi_{\mathbb{T}_d}(x)\varphi_{\mathbb{T}_d}(y)] = g_{\mathbb{T}_d}(x,y)$ for all $x, y \in \mathbb{T}_d$, where $g_{\mathbb{T}_d}(\cdot,\cdot)$ is the Green function of simple random walk on $\mathbb{T}_d$ (see (1.3)). (0.1)

Our main interest lies in investigating properties of the level sets of $\varphi_{\mathbb{T}_d}$, i.e. of

$$E^{\geq h}_{\varphi_{\mathbb{T}_d}} \coloneqq \{x \in \mathbb{T}_d \,|\, \varphi_{\mathbb{T}_d}(x) \geq h\} \quad \text{for } h \in \mathbb{R}.$$

In particular, we are interested in the connected component of $E^{\geq h}_{\varphi_{\mathbb{T}_d}}$ containing a fixed vertex $\mathrm{o} \in \mathbb{T}_d$ (called *root*) and denoted by

$$\mathcal{C}^h_{\mathrm{o}} \coloneqq \big\{x \in \mathbb{T}_d \,\big|\, x \text{ is connected to o in } E^{\geq h}_{\varphi_{\mathbb{T}_d}}\big\}. \tag{0.2}$$

With this notation at hand we can define the critical value of level-set percolation of the Gaussian free field via

$$h_\star \coloneqq \inf\big\{h \in \mathbb{R} \,\big|\, \mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}^h_{\mathrm{o}}| = \infty\big] = 0\big\}. \tag{0.3}$$

We point out that there is *no* explicit formula for $h_\star$, even though we consider the Gaussian free field on a regular tree. However, the special structure of the underlying graph allows for a crucial spectral characterisation of the critical value, as derived in [Szn16]. We recall it in details in Section 1.2. Very roughly, in this spectral characterisation one associates to any level $h \in \mathbb{R}$ a self-adjoint, non-negative operator L_h on $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, where ν is a certain centred Gaussian measure. The operator L_h is naturally linked to the distribution of $\varphi_{\mathbb{T}_d}$ at a vertex conditioned on the value of $\varphi_{\mathbb{T}_d}$ at a neighbouring vertex and truncated below level h (see (1.12) and below it). One then considers the operator norms $(\lambda_h)_{h \in \mathbb{R}}$ of the operators $(L_h)_{h \in \mathbb{R}}$ and finds that (see [Szn16], Section 3)

> the map $h \mapsto \lambda_h$ is a decreasing homeomorphism from $\mathbb{R}$ to $(0, d-1)$ and $h_\star$ is the unique value in $\mathbb{R}$ such that $\lambda_{h_\star} = 1$. (0.4)

Additionally, for $h \in \mathbb{R}$ one has that λ_h is a simple eigenvalue of L_h which is associated to a unique, non-negative eigenfunction χ_h with unit L^2-norm, vanishing on $(-\infty, h)$ and positive elsewhere. So far one very important aspect of the eigenfunctions $(\chi_h)_{h \in \mathbb{R}}$ was unknown, namely the precise understanding of their asymptotic behaviour. Some care is applied in [Szn16] to circumvent this lack of control (see Remark 3.4 and Remark 4.4 therein).

As a first result in this paper, we close this gap and we obtain matching upper and lower bounds on the eigenfunctions $(\chi_h)_{h\in\mathbb{R}}$. In essence, we show in Proposition 2.1 that for every $h \in \mathbb{R}$ there exist $c_h, c'_h > 0$ such that

$$0 < c_h\, a^{1-\log_{d-1}(\lambda_h)} \leq \chi_h(a) \leq c'_h\, a^{1-\log_{d-1}(\lambda_h)} \quad \text{for all } a \in [h,\infty) \tag{0.5}$$

(see also Remark 2.2 (i)). Presumably, such exact bounds might be helpful when tackling level-set percolation questions of $\varphi_{\mathbb{T}_d}$ near the critical value $h_\star$. In this paper we will use the upper bound to show the exponential growth of $|\mathcal{C}_{\mathrm{o}}^h|$ for $h < h_\star$ (see (0.10)).

We also obtain another result related to the spectral characterisation of level-set percolation of $\varphi_{\mathbb{T}_d}$. It concerns the non-negative martingale $(M_k^{\geq h})_{k\geq 0}$ for $h \in \mathbb{R}$ on which the proof of the spectral characterisation of $h_\star$ in [Szn16] heavily relies and in which the eigenfunction χ_h and the associated eigenvalue λ_h appear (see (1.16) for the definition). We show in the present paper that for all $h \in \mathbb{R}\setminus\{h_\star\}$ the probability of a non-vanishing martingale limit is equal to the 'forward percolation probability' (see Proposition 4.2)

$$\mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+| = \infty\big] = \mathbb{P}^{\mathbb{T}_d}\big[M_\infty^{\geq h} > 0\big], \tag{0.6}$$

where $\mathbb{T}_d^+ \subseteq \mathbb{T}_d$ is the 'forward tree', that is, the subtree of $\mathbb{T}_d$ containing the root o and in which each vertex except for the root o has d neighbours and the root o has $d-1$ neighbours (the precise definition is given below (1.1)).

We moreover investigate the continuity properties in h of percolation probabilities like on the left hand side of (0.6) and as a third result we show in Theorem 4.1 that

the percolation probability $\eta(h) := \mathbb{P}^{\mathbb{T}_d}[|\mathcal{C}_{\mathrm{o}}^h| = \infty]$ for $h \in \mathbb{R}$ and the forward percolation probability $\eta^+(h) := \mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+| = \infty\big]$ for $h \in \mathbb{R}$ are continuous functions on $\mathbb{R}\setminus\{h_\star\}$. (0.7)

We then turn to $\mathcal{C}_{\mathrm{o}}^h$ and we obtain rather precise estimates of its cardinality in both the subcritical ($h > h_\star$) and supercritical ($h < h_\star$) phase.

We show that if $h > h_\star$, then there is some $\delta > 0$ such that for all $\gamma > 0$ we can find $c_{h,\gamma}, c'_{h,\gamma} > 0$ satisfying (see Theorem 5.1)

$$\mathbb{E}^{\mathbb{T}_d}\Big[(1+\delta)^{|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+|}\,\Big|\,\varphi_{\mathbb{T}_d}(\mathrm{o}) = a\Big] \leq c_{h,\gamma}\exp(c'_{h,\gamma}a^{1+\gamma}) \quad \text{for all } a \geq h. \tag{0.8}$$

In particular, this will imply that $|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+|$ has exponential moments. Incidentally, it also implies conditional exponential-tail estimates of $|\mathcal{C}_{\mathrm{o}}^h|$ of the form

$$\mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}_{\mathrm{o}}^h| \geq k\,\big|\,\varphi_{\mathbb{T}_d}(\mathrm{o}) = a\big] \leq c_{h,a}e^{-c'_h k} \quad \text{for all } k \geq 1 \text{ and } a \geq h, \tag{0.9}$$

with a control of the dependence of the constant $c_{h,a} > 0$ on the value $a = \varphi_{\mathbb{T}_d}(\mathrm{o})$ of the field at the root (see Remark 5.2).

Finally, for $h < h_\star$ we prove that the number of vertices connected over distance k above level h to the root $\mathrm{o} \in \mathbb{T}_d$ grows exponentially in k with positive probability. This can be shown by using our first result (0.5) in combination with (0.6). More precisely, with $S_{\mathbb{T}_d}(\mathrm{o},k)$ denoting the sphere of radius $k \geq 0$ around o in $\mathbb{T}_d$, we prove that (see Theorem 4.3)

$$\lim_{k\to\infty}\mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+ \cap S_{\mathbb{T}_d}(\mathrm{o},k)| \geq \tfrac{\lambda_h^k}{k^2}\big] = \eta^+(h) > 0. \tag{0.10}$$

We remind that here $\lambda_h > 1$ is the eigenvalue from (0.4).

As explained earlier, we will see in the accompanying paper [AČ19] that the Gaussian free field $\varphi_{\mathbb{T}_d}$ on $\mathbb{T}_d$ in essence plays the role of the local picture of the *zero-average* Gaussian free field on a specific class of *finite* d-regular graphs that are locally (almost) tree-like. By exploiting this feature, we establish in [AČ19] a phase transition for level-set percolation of the zero-average Gaussian free field on the finite graphs which is characterised by the critical value $h_\star$ *on the infinite tree*. Roughly, the strategy of [AČ19] is to use the local picture to transfer the problem from the finite graphs to $\mathbb{T}_d$ and then to use the new results developed in the present article. Specifically, we apply the estimate (0.8) in the proof of the subcritical phase ([AČ19], Theorem 3.1) and the two results (0.7) and (0.10) in the proof of the supercritical phase ([AČ19], Theorem 4.1). As an aside, let us mention that for the application in [AČ19] it is crucial that the exponent γ on the right hand side of (0.8) can be chosen strictly smaller than 2.

A similar approach as explained in the previous paragraph was carried out in [ČTW11] to describe a phase transition for the vacant set of simple random walk on the same class of finite graphs as considered in the accompanying paper [AČ19]. As shown in [ČTW11], the local picture in that case is given by the vacant set of random interlacements on $\mathbb{T}_d$. Thanks to the detailed understanding of random interlacements in the infinite model, the phase transition in the finite model can be established. An advantage of random interlacements on a tree is that the connected components of its vacant set can be described rather easily. Indeed, as observed in [Tei09], they are distributed as Galton-Watson trees with a binomial offspring distribution. Thus, properties like (0.9) or (0.10) are classical.

In contrast, the connected components of the level sets of $\varphi_{\mathbb{T}_d}$, which play the corresponding role in our setup, are not Galton-Watson trees. The situation is more complicated and obtaining results like (0.9) and (0.10) is not straightforward. Instead, by the domain Markov property of the Gaussian free field, one can view $\mathcal{C}_o^h$ for $h \in \mathbb{R}$ as a certain multi-type branching process with an uncountable type space. Some of the results in this paper are similar to classical results about branching processes, though to our knowledge they are not covered by the literature. We would like to stress that our arguments rely on the *special structure* of the Gaussian free field on regular trees. Let us also mention that, despite the connection between the Gaussian free field and random interlacements via isomorphism theorems, we are not aware of any technique allowing to transfer the results of [Tei09] and [ČTW11] directly to our context.

The structure of the article is as follows. In Section 1 we collect the main part of the notation and some known results about the Gaussian free field on $\mathbb{T}_d$. In particular, in Section 1.2 we recall the spectral description of the critical value $h_\star$ obtained in [Szn16]. In Section 2 we derive asymptotic bounds on the eigenfunctions appearing in the spectral description of $h_\star$. In Section 3 we give a recursive equation for the conditional non-percolation probability. Subsequently, we analyse the behaviour of the level sets of the Gaussian free field in the supercritical phase in Section 4. This includes the continuity of the percolation probability (Section 4.1) and the geometrical growth of level sets (Section 4.3). Finally, in Section 5 we investigate the subcritical phase and show that the cardinality of the connected component of the level set containing the root has exponential moments (and more).

Acknowledgements. The authors wish to express their gratitude to A.-S. Sznitman for suggesting the problem and for the valuable comments made at various stages of the project.

1 Notation and useful results

We start by introducing the notation and recalling known properties of the Green function and the Gaussian free field on $\mathbb{T}_d$. These include a recursive construction of $\varphi_{\mathbb{T}_d}$ (Section 1.1) and the spectral characterisation of $h_\star$ (Section 1.2).

As mentioned earlier, we consider for fixed $d \geq 3$ the d-regular tree $\mathbb{T}_d$ with root o. We endow $\mathbb{T}_d$ with the usual graph distance $d_{\mathbb{T}_d}(\cdot,\cdot)$. For any $R \geq 0$ and $x \in \mathbb{T}_d$ we let $B_{\mathbb{T}_d}(x,R) \coloneqq \{y \in \mathbb{T}_d \,|\, d_{\mathbb{T}_d}(x,y) \leq R\}$ and $S_{\mathbb{T}_d}(x,R) \coloneqq \{y \in \mathbb{T}_d \,|\, d_{\mathbb{T}_d}(x,y) = R\}$ denote the ball and the sphere of radius R around x, respectively. For $x, z \in \mathbb{T}_d$ a path from x to z is a sequence of vertices $x = y_0, y_1, \dots, y_m = z$ in $\mathbb{T}_d$ for some $m \geq 0$ such that y_i and y_{i-1} are neighbours for all $i = 1, \dots, m$ (if $m \geq 1$). It is a *geodesic* path from x to z if it is the path of shortest length.

For $x \in \mathbb{T}_d \setminus \{\mathrm{o}\}$ let $\overline{x}$ be the unique neighbour of x on the geodesic path from x to o. Moreover, let $\overline{\mathrm{o}} \in \mathbb{T}_d$ denote an arbitrary fixed neighbour of $\mathrm{o} \in \mathbb{T}_d$. For $x \in \mathbb{T}_d$ we define

$$U_x \coloneqq \{z \in \mathbb{T}_d \,|\, \text{the geodesic path from } z \text{ to } x \text{ does not pass through } \overline{x}\}. \tag{1.1}$$

In particular $\mathbb{T}_d = \{\mathrm{o}\} \cup \bigcup_{i=1}^d U_{x_i}$, where $\{x_1, \dots, x_d\}$ denote the neighbours of o. In the special case of $x = \mathrm{o}$ we write $\mathbb{T}_d^+ \coloneqq U_{\mathrm{o}}$. We also set $B_{\mathbb{T}_d}^+(\mathrm{o},R) \coloneqq \{y \in \mathbb{T}_d^+ \,|\, d_{\mathbb{T}_d}(\mathrm{o},y) \leq R\}$ and similarly $S_{\mathbb{T}_d}^+(\mathrm{o},R) \coloneqq \{y \in \mathbb{T}_d^+ \,|\, d_{\mathbb{T}_d}(\mathrm{o},y) = R\}$ for $R \geq 0$.

We write $P_x^{\mathbb{T}_d}$ for the canonical law of the simple random walk $(X_k)_{k\geq 0}$ on $\mathbb{T}_d$ starting at x as well as $E_x^{\mathbb{T}_d}$ for the corresponding expectation. Given $U \subseteq \mathbb{T}_d$ we write $T_U \coloneqq \inf\{k \geq 0 \,|\, X_k \notin U\}$ for the exit time from U and $H_U \coloneqq \inf\{k \geq 0 \,|\, X_k \in U\}$ for the entrance time in U (here we set $\inf \emptyset \coloneqq \infty$). In the special case of $U = \{z\}$ we use H_z in place of $H_{\{z\}}$. Recall that (see e.g. [Woe00], proof of Lemma 1.24)

$$P_x^{\mathbb{T}_d}\big[H_y < \infty\big] = \Big(\frac{1}{d-1}\Big)^{d_{\mathbb{T}_d}(x,y)} \quad \text{for } x, y \in \mathbb{T}_d. \tag{1.2}$$

The Green function $g_{\mathbb{T}_d}(\cdot,\cdot)$ of simple random walk on $\mathbb{T}_d$ is given by (see [Woe00], Lemma 1.24, for the explicit computation)

$$g_{\mathbb{T}_d}(x,y) \coloneqq E_x^{\mathbb{T}_d}\Big[\sum_{k=0}^{\infty} \mathbf{1}_{\{X_k = y\}}\Big] = \frac{d-1}{d-2}\Big(\frac{1}{d-1}\Big)^{d_{\mathbb{T}_d}(x,y)} \quad \text{for } x, y \in \mathbb{T}_d. \tag{1.3}$$

For $U \subseteq \mathbb{T}_d$ the Green function $g_{\mathbb{T}_d}^U(\cdot,\cdot)$ of simple random walk on $\mathbb{T}_d$ killed when exiting U is

$$g_{\mathbb{T}_d}^U(x,y) \coloneqq E_x^{\mathbb{T}_d}\Big[\sum_{0 \leq k < T_U} \mathbf{1}_{\{X_k = y\}}\Big] \quad \text{for } x, y \in \mathbb{T}_d.$$

The functions $g_{\mathbb{T}_d}(\cdot,\cdot)$ and $g_{\mathbb{T}_d}^U(\cdot,\cdot)$ are symmetric and finite, and $g_{\mathbb{T}_d}^U(\cdot,\cdot)$ vanishes whenever $x \notin U$ or $y \notin U$. They are related by the identity

$$g_{\mathbb{T}_d}(x,y) = g_{\mathbb{T}_d}^U(x,y) + E_x^{\mathbb{T}_d}\big[g_{\mathbb{T}_d}(X_{T_U}, y)\mathbf{1}_{\{T_U < \infty\}}\big] \quad \text{for } x, y \in \mathbb{T}_d, \tag{1.4}$$

which is an easy consequence of the strong Markov property of simple random walk at time T_U. In the particular case of $U \coloneqq U_x$ this implies that (by using (1.2), (1.3) and that $X_{T_U} = \overline{x}$ on $\{T_U < \infty\}$ under $P_x^{\mathbb{T}_d}$)

$$g_{\mathbb{T}_d}^{U_x}(x,x) = g_{\mathbb{T}_d}(x,x) - \frac{1}{d-1} g_{\mathbb{T}_d}(\overline{x},x) = \frac{d-1}{d-2}\Big(1 - \frac{1}{(d-1)^2}\Big) = \frac{d}{d-1}. \tag{1.5}$$

Recall from (0.1) that $(\varphi_{\mathbb{T}_d}(x))_{x\in\mathbb{T}_d}$ is the centred Gaussian field with covariance given by $g_{\mathbb{T}_d}(\cdot,\cdot)$. It satisfies the following *domain Markov property*: for $U \subseteq \mathbb{T}_d$ define the new field

$$\varphi^U_{\mathbb{T}_d}(x) := \varphi_{\mathbb{T}_d}(x) - E^{\mathbb{T}_d}_x[\varphi_{\mathbb{T}_d}(X_{T_U})\mathbf{1}_{\{T_U<\infty\}}] \quad \text{for } x \in \mathbb{T}_d.$$

Then,

$$\text{under } \mathbb{P}^{\mathbb{T}_d},\ (\varphi^U_{\mathbb{T}_d}(x))_{x\in\mathbb{T}_d} \text{ is a centred Gaussian field on } \mathbb{T}_d \text{ which is independent from } (\varphi_{\mathbb{T}_d}(x))_{x\in\mathbb{T}_d\setminus U} \text{ and has covariance } \mathbb{E}^{\mathbb{T}_d}[\varphi^U_{\mathbb{T}_d}(x)\varphi^U_{\mathbb{T}_d}(y)] = g^U_{\mathbb{T}_d}(x,y) \text{ for all } x,y\in\mathbb{T}_d. \tag{1.6}$$

The proof of this fact follows by an easy computation of covariances and (1.4).

1.1 Recursive construction of the Gaussian free field

Property (1.6) can be applied to obtain a useful recursive representation of the Gaussian free field on $\mathbb{T}_d$ that we introduce now. We point out that this description crucially relies on the special features of the Gaussian free field when considered on a (regular) tree.

Let $x \in \mathbb{T}_d$ and let $\{x_1,\dots,x_I\}$ be the neighbours of x not contained in the geodesic path from x to o. In particular, $I = d$ if $x =$ o and $I = d-1$ otherwise. We set $U := \bigcup_{i=1}^I U_{x_i}$. Since $\mathbb{T}_d$ is a tree, it can be easily seen that

- $g^U_{\mathbb{T}_d}(x_i,x_i) = g^{U_{x_i}}_{\mathbb{T}_d}(x_i,x_i)$ for $i\in\{1,\dots,I\}$,
- $g^U_{\mathbb{T}_d}(y,y') = 0$ for $y\in U_{x_i}$, $y'\in U_{x_j}$ where $i,j\in\{1,\dots,I\}$ with $i\neq j$,
- for any $y\in U$, one has $X_{T_U} = x$ on $\{T_U<\infty\}$ under $P^{\mathbb{T}_d}_y$.

Hence (1.6) together with (1.2) and (1.5) yields that

$$\text{under } \mathbb{P}^{\mathbb{T}_d}, \text{ conditionally on } \varphi_{\mathbb{T}_d}(x), \text{ the random variables } (\varphi_{\mathbb{T}_d}(x_i))_{1\le i\le I} \text{ are i.i.d. Gaussians with mean } \tfrac{1}{d-1}\varphi_{\mathbb{T}_d}(x) \text{ and variance } \tfrac{d}{d-1}. \text{ Furthermore, they are independent of } (\varphi_{\mathbb{T}_d}(y))_{y\in\mathbb{T}_d\setminus U}. \tag{1.7}$$

Let now $(Y_x)_{x\in\mathbb{T}_d}$ be a collection of independent centred Gaussian variables defined on some auxiliary probability space $(\Omega,\mathcal{A},\mathbb{P})$ such that $Y_o \sim \mathcal{N}(0, g_{\mathbb{T}_d}(\text{o},\text{o})) = \mathcal{N}(0,\frac{d-1}{d-2})$ and $Y_x \sim \mathcal{N}(0, g^{U_x}_{\mathbb{T}_d}(x,x)) = \mathcal{N}(0,\frac{d}{d-1})$ for $x\neq$ o. Define recursively

$$\widetilde{\varphi}(\text{o}) := Y_{\text{o}} \quad \text{and} \quad \widetilde{\varphi}(x) := \frac{1}{d-1}\widetilde{\varphi}(\overline{x}) + Y_x \quad \text{for } x\in\mathbb{T}_d\setminus\{\text{o}\}. \tag{1.8}$$

Then, by applying (1.7) iteratively, we see that

$$\text{under } \mathbb{P}, \text{ the law of } (\widetilde{\varphi}(x))_{x\in\mathbb{T}_d} \text{ is } \mathbb{P}^{\mathbb{T}_d}, \tag{1.9}$$

so that (1.8) can serve as an alternative construction of the Gaussian free field on $\mathbb{T}_d$.

The recursive representation (1.8) has many useful consequences and it will be used repeatedly throughout the paper. In particular, it gives a representation of the conditional distribution of $\varphi_{\mathbb{T}_d}$ given $\varphi_{\mathbb{T}_d}(\text{o}) = a\in\mathbb{R}$,

$$\mathbb{P}^{\mathbb{T}_d}_a\big[(\varphi_{\mathbb{T}_d}(y))_{y\in\mathbb{T}_d}\in\cdot\,\big] := \mathbb{P}^{\mathbb{T}_d}\big[(\varphi_{\mathbb{T}_d}(y))_{y\in\mathbb{T}_d}\in\cdot\;\big|\;\varphi_{\mathbb{T}_d}(\text{o}) = a\big], \tag{1.10}$$

with corresponding expectation $\mathbb{E}_a^{\mathbb{T}_d}$. Moreover, if we let $x_1, \dots, x_d$ denote the neighbours of the root $\mathrm{o} \in \mathbb{T}_d$, then from (1.8) and (1.9) it follows that for every $a \in \mathbb{R}$,

$$\begin{aligned}&\text{under } \mathbb{P}_a^{\mathbb{T}_d}, \text{ the random fields } (\varphi_{\mathbb{T}_d}(y))_{y\in U_{x_i}} \text{ for } i=1,\dots,d \text{ are independent. Furthermore, for any event } A \in \sigma(\varphi_{\mathbb{T}_d}(z), z \in \mathbb{T}_d^+) \text{ and } i=1,\dots,d\\ &\text{one has } \mathbb{P}_a^{\mathbb{T}_d}\big[(\varphi_{\mathbb{T}_d}(y))_{y\in U_{x_i}} \in A\big] = \mathbb{E}^Y\Big[\mathbb{P}^{\mathbb{T}_d}_{\frac{a}{d-1}+Y}\big[(\varphi_{\mathbb{T}_d}(y))_{y\in\mathbb{T}_d^+} \in A\big]\Big],\\ &\text{where } Y \sim \mathcal{N}(0, \tfrac{d}{d-1}) \text{ and } \mathbb{E}^Y \text{ is the expectation with respect to } Y.\end{aligned} \tag{1.11}$$

(In the equality in (1.11) we also use that the law of $\varphi_{\mathbb{T}_d}$ on U_{x_i} equals the law of $\varphi_{\mathbb{T}_d}$ on $\mathbb{T}_d^+$.)

Due to (1.8) and (1.9), the Gaussian free field on $\mathbb{T}_d$ can be related to a multi-type branching process with type space $\mathbb{R}$. Indeed, we can view every $x \in S_{\mathbb{T}_d}(\mathrm{o}, k)$ as an individual in the k-th generation of the branching process with an attached type $\varphi_{\mathbb{T}_d}(x) \in \mathbb{R}$. In this perspective (1.7) can be rephrased as: every individual x has $d-1$ children (d children if $x = \mathrm{o}$) whose types, conditionally on $\varphi_{\mathbb{T}_d}(x)$, are chosen independently according to the distribution $\mathcal{N}(\frac{1}{d-1}\varphi_{\mathbb{T}_d}(x), \frac{d}{d-1})$.

This point of view can easily be adapted to $\mathcal{C}_\mathrm{o}^h$ from (0.2) as well, namely by considering the same multi-type branching process but instantly killing all individuals whose type does not exceed h. In other words, $\mathcal{C}_\mathrm{o}^h$ can be viewed as a multi-type branching process with a barrier and the percolation of $\mathcal{C}_\mathrm{o}^h$ corresponds to the non-extinction of this branching process. However, while some of the results in this paper are reminiscent of classical results about branching processes, we would like to emphasise that the proofs make heavy use of the special structure of the Gaussian free field on a regular tree. We are going to recall one of the special features in the next section.

1.2 Spectral characterisation of the critical value

We now recall the spectral characterisation of the critical value $h_\star$ from [Szn16], which is central for our paper. Note that our d-regular tree $\mathbb{T}_d$ corresponds in the notation of [Szn16] to the $(\widetilde{d}+1)$-regular tree T with $\widetilde{d} := d-1$. Moreover in [Szn16], in the definition of the Green function $g_{\mathbb{T}_d}(\cdot,\cdot)$ on the tree, there is an extra normalising factor equal to the degree of the tree (see [Szn16], (3.1)). This explains the differences between the formulas to come and the formulas in [Szn16].

Let $\nu = \mathcal{N}(0, \frac{d-1}{d-2})$ be the centred Gaussian measure with variance $\frac{d-1}{d-2} \overset{(1.3)}{=} g_{\mathbb{T}_d}(\mathrm{o},\mathrm{o})$. For $h \in \mathbb{R}$ define the operator

$$\begin{aligned}&(L_h f)(a) := (d-1)\mathbf{1}_{[h,\infty)}(a)\,\mathbb{E}^Y\big[f(\tfrac{a}{d-1}+Y)\mathbf{1}_{[h,\infty)}(\tfrac{a}{d-1}+Y)\big]\\ &\text{for } f \in L^2(\nu) := L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu) \text{ and } a \in \mathbb{R},\end{aligned} \tag{1.12}$$

where $Y \sim \mathcal{N}(0, \frac{d}{d-1})$ and $\mathbb{E}^Y$ is the expectation with respect to Y. The operator L_h is closely linked to the Gaussian free field and its level set above level h. Indeed, one has $(L_h f)(a) = \mathbb{E}_a^{\mathbb{T}_d}\big[\sum_{x\in\mathcal{C}_\mathrm{o}^h\cap S^+_{\mathbb{T}_d}(\mathrm{o},1)} f(\varphi_{\mathbb{T}_d}(x))\big]$ for $a \geq h$ by (1.8).

The following proposition summarises the known properties of the operators $(L_h)_{h\in\mathbb{R}}$ and characterises the critical value $h_\star$.

Proposition 1.1 ([Szn16], Propositions 3.1 and 3.3)**.** *For every $h \in \mathbb{R}$ the operator L_h is self-adjoint, non-negative and its operator norm*

$$\lambda_h := \|L_h\|_{L^2(\nu)\to L^2(\nu)} \tag{1.13}$$

is a simple eigenvalue of L_h. Moreover, there is a unique, non-negative eigenfunction $\chi_h \in L^2(\nu)$ of L_h, corresponding to the eigenvalue λ_h, with $\|\chi_h\|_{L^2(\nu)} = 1$. The function χ_h is continuous and positive on $[h, \infty)$, and vanishing on $(-\infty, h)$. Additionally, the map $h \mapsto \lambda_h$ is a decreasing homeomorphism from $\mathbb{R}$ to $(0, d-1)$ and $h_\star$ is the unique value in $\mathbb{R}$ such that $\lambda_{h_\star} = 1$.

In Proposition 2.1 in Section 2 we will give matching upper and lower bounds on the eigenfunctions χ_h.

On the way, we recall the following hypercontractivity estimate which is a direct consequence of the hypercontractivity property of the Ornstein-Uhlenbeck semigroup (see [Szn16], (3.14)): for $1 < p < \infty$ and $q = (p-1)(d-1)^2 + 1$ one has (with $Y \sim \mathcal{N}(0, \frac{d}{d-1})$)

$$\left\| \mathbb{E}^Y \left[f\left(\tfrac{\cdot}{d-1} + Y\right) \right] \right\|_{L^q(\nu)} \leq \|f\|_{L^p(\nu)} \quad \text{for } f \in L^p(\nu). \tag{1.14}$$

We will use the estimate (1.14), with its precise relation between the parameters p and q, several times. Especially, it will be applied to prove Proposition 3.6 which computes the Fréchet derivative of a certain operator. This will be a key ingredient for showing the existence of conditional exponential moments of $|\mathcal{C}_o^h|$ in the subcritical phase in Section 5.

Furthermore, for every $h \in \mathbb{R}$ there is a martingale $(M_k^{\geq h})_{k \geq 0}$ closely related to L_h. Indeed, if we set

$$\mathcal{Z}_k^h := \mathcal{C}_o^h \cap S_{\mathbb{T}_d}^+(o, k) \quad \text{for } k \geq 0, \tag{1.15}$$

then (see [Szn16], (3.31) and (3.35))

$$M_k^{\geq h} := \lambda_h^{-k} \sum_{y \in \mathcal{Z}_k^h} \chi_h(\varphi_{\mathbb{T}_d}(y)) \quad \text{for } k \geq 0 \tag{1.16}$$

defines a non-negative martingale under $\mathbb{P}^{\mathbb{T}_d}$ with respect to the filtration $(\mathcal{F}_k)_{k \geq 0}$ given by

$$\mathcal{F}_k := \sigma(\varphi_{\mathbb{T}_d}(y), y \in B_{\mathbb{T}_d}^+(o, k)). \tag{1.17}$$

In particular, $M_k^{\geq h}$ converges $\mathbb{P}^{\mathbb{T}_d}$-almost surely to some $M_\infty^{\geq h} \geq 0$ as $k \to \infty$ and (see [Szn16], proof of Proposition 3.3)

$$\text{for } h < h_\star \text{ one has } \mathbb{P}^{\mathbb{T}_d}[M_\infty^{\geq h} > 0] > 0. \tag{1.18}$$

Note that there is a direct relation between the probability of a non-vanishing martingale limit $M_\infty^{\geq h}$ and the forward percolation probability

$$\eta^+(h) = \mathbb{P}^{\mathbb{T}_d}\left[|\mathcal{C}_o^h \cap \mathbb{T}_d^+| = \infty\right] \quad \text{for } h \in \mathbb{R} \tag{1.19}$$

from (0.7). We only need to observe that

$$\{|\mathcal{C}_o^h \cap \mathbb{T}_d^+| < \infty\} \subseteq \{\mathcal{Z}_k^h = \emptyset \text{ for } k \text{ large enough}\} \subseteq \{M_\infty^{\geq h} = 0\}.$$

Therefore $\mathbb{P}^{\mathbb{T}_d}[M_\infty^{\geq h} > 0] \leq \eta^+(h)$. In Section 4 we will see that this inequality is actually an equality, at least when $h \neq h_\star$ (Proposition 4.2). As a last observation, note that by a union bound and the symmetry of $\mathbb{T}_d$ we obtain that $\eta^+(h) \leq \eta(h) \leq d \cdot \eta^+(h)$ for the percolation probability η from (0.7). Hence by (0.3) one has $\eta^+(h) = 0$ for $h > h_\star$ and $\eta^+(h) > 0$ for $h < h_\star$.

A final word on the convention followed concerning constants: by $c, c', \ldots$ we denote positive constants with values changing from place to place and which only depend on the dimension d. The dependence of constants on additional parameters appears in the notation.

2 Asymptotic behaviour of the eigenfunctions

The main result of this section are the matching bounds on the eigenfunctions $(\chi_h)_{h\in\mathbb{R}}$ from Proposition 1.1 collected in Proposition 2.1 below (corresponding to (0.5)). The upper bound will be used later to show that connected components of supercritical level sets grow exponentially with positive probability (Theorem 4.3 in Section 4). The corresponding lower bound is not used further but it is included for completeness.

Proposition 2.1. (i) *There exists $c > 0$ (see in (2.8) below) such that for all $h \in \mathbb{R}$ one has*

$$\chi_h(a) \le c\, a^{1-\log_{d-1}(\lambda_h)} \quad \text{for all } a \ge d-1. \tag{2.1}$$

(ii) *For every $h \in \mathbb{R}$ there exists $c_h > 0$ such that*

$$\chi_h(a) \ge c_h a^{1-\log_{d-1}(\lambda_h)} \quad \text{for all } a \ge h. \tag{2.2}$$

Remark 2.2. (i) From Proposition 1.1 recall that χ_h is continuous and strictly positive on $[h, \infty)$. Therefore, by adjusting the constant c, Proposition 2.1 implies (0.5).

(ii) By Proposition 1.1 one has $\lambda_h \in (0, d-1)$. Hence, the exponent $\kappa_h := 1 - \log_{d-1}(\lambda_h)$ in (2.1) and (2.2) is positive for all $h \in \mathbb{R}$. Moreover, $\kappa_h \in (0, 1)$ for $h < h_\star$, $\kappa_h = 1$ for $h = h_\star$ and $\kappa_h > 1$ for $h > h_\star$. □

Proof of Proposition 2.1. (i) Let $(Y_i)_{i\ge1}$ be i.i.d. random variables with distribution $\mathcal{N}(0, \frac{d}{d-1})$. By iteratively using (1.12) and the fact that χ_h is the eigenfunction of L_h with eigenvalue λ_h, we obtain for every $a \in \mathbb{R}$ and $k \ge 1$

$$\begin{aligned}\chi_h(a) &= \frac{1}{\lambda_h}(L_h\chi_h)(a) \le \frac{d-1}{\lambda_h}\,\mathbb{E}^{Y_1}\Big[\chi_h\big(\tfrac{a}{d-1} + Y_1\big)\Big] = \frac{d-1}{\lambda_h}\,\mathbb{E}^{Y_1}\Big[\frac{1}{\lambda_h}(L_h\chi_h)\big(\tfrac{a}{d-1} + Y_1\big)\Big] \\ &\le \Big(\frac{d-1}{\lambda_h}\Big)^2 \mathbb{E}^{Y_1}\Big[\mathbb{E}^{Y_2}\big[\chi_h\big(\tfrac{a}{(d-1)^2} + \tfrac{1}{d-1}Y_1 + Y_2\big)\big]\Big] \\ &\le \ldots \le \Big(\frac{d-1}{\lambda_h}\Big)^k \mathbb{E}\Big[\chi_h\big(\tfrac{a}{(d-1)^k} + \tfrac{1}{(d-1)^{k-1}}Y_1 + \ldots + \tfrac{1}{d-1}Y_{k-1} + Y_k\big)\Big],\end{aligned} \tag{2.3}$$

where the expectation on the right hand side of (2.3) is taken with respect to $Y_1, \ldots, Y_k$. Note that for any $k \ge 1$ the random variable $\frac{1}{(d-1)^{k-1}}Y_1 + \cdots + \frac{1}{d-1}Y_{k-1} + Y_k$ appearing on the right hand side of (2.3) is centred Gaussian with variance

$$\sigma_k^2 := \frac{d}{d-1}\sum_{i=0}^{k-1}\frac{1}{(d-1)^{2i}} = \frac{d-1}{d-2}\Big(1 - \frac{1}{(d-1)^{2k}}\Big) \le \frac{d-1}{d-2} =: \sigma^2. \tag{2.4}$$

Hence, if we denote by f_{μ,τ^2} the density of the normal distribution $\mathcal{N}(\mu, \tau^2)$ and $a_k := \frac{a}{(d-1)^k}$ for $k \ge 1$, then (recall from above (1.12) that $\nu = \mathcal{N}(0, \sigma^2)$)

$$\begin{aligned}\mathbb{E}\Big[\chi_h\big(\tfrac{a}{(d-1)^k} + \tfrac{1}{(d-1)^{k-1}}Y_1 + \ldots + \tfrac{1}{d-1}Y_{k-1} + Y_k\big)\Big] &= \int_{\mathbb{R}}\chi_h(y) f_{a_k,\sigma_k^2}(y)\,dy \\ = \int_{\mathbb{R}}\chi_h(y)\frac{f_{a_k,\sigma_k^2}(y)}{f_{0,\sigma^2}(y)}\,\nu(dy) &\overset{(*)}{\le} \underbrace{\|\chi_h\|_{L^2(\nu)}}_{=1}\bigg(\int_{\mathbb{R}}\frac{f^2_{a_k,\sigma_k^2}(y)}{f_{0,\sigma^2}(y)}\,dy\bigg)^{\frac12},\end{aligned} \tag{2.5}$$

where in $(*)$ we apply the Cauchy-Schwarz inequality. Note that for all $k \ge 1$

$$\frac{f^2_{a_k,\sigma_k^2}(y)}{f_{0,\sigma^2}(y)} = \frac{\sqrt{\sigma^2}}{\sqrt{2\pi\sigma_k^2}}\exp\Big(-\frac{(y-a_k)^2}{\sigma_k^2} + \frac{y^2}{2\sigma^2}\Big) \le f_{2a_k,\sigma^2}(y)\cdot\frac{\sigma^2}{\sigma_1^2}\exp\Big(\frac{a_k^2}{\sigma^2}\Big), \tag{2.6}$$

where we use $\sigma_1^2 \le \sigma_k^2 \le \sigma^2$ (see (2.4)). By combining (2.3) with (2.5) and (2.6) we obtain for any $a \in \mathbb{R}$ and $k \ge 1$

$$\chi_h(a) \le \Big(\frac{d-1}{\lambda_h}\Big)^k \sqrt{\frac{\sigma^2}{\sigma_1^2}} \exp\Big(\frac{a_k^2}{2\sigma^2}\Big). \tag{2.7}$$

If $a \ge d-1$, we can apply (2.7) for $k = k(a) := \lfloor \log_{d-1}(a) \rfloor$, that is, for the unique $k \ge 1$ with $(d-1)^k \le a < (d-1)^{k+1}$. Since $\frac{d-1}{\lambda_h} > 1$ by Proposition 1.1 and $a_{k(a)} \le d-1$, we obtain from (2.7)

$$\chi_h(a) \le \Big(\frac{d-1}{\lambda_h}\Big)^{\log_{d-1}(a)} \underbrace{\sqrt{\frac{\sigma^2}{\sigma_1^2}} \exp\Big(\frac{(d-1)^2}{2\sigma^2}\Big)}_{=:\, c>0} = c\, a^{1-\log_{d-1}(\lambda_h)}, \tag{2.8}$$

which concludes the proof of part (i).

(ii) Let $K = K_h \ge 0$ be the smallest non-negative integer such that $(d-1)^K - 14 \ge h$. Define the intervals

$$\begin{aligned} I_0 &:= \big[h, (d-1)^{K+1} + 4 \cdot 2\big], \\ I_k &:= \big[(d-1)^{K+k} - 4(k+2), (d-1)^{K+k+1} + 4(k+2)\big] \quad \text{for } k \ge 1, \end{aligned} \tag{2.9}$$

which form a non-disjoint decomposition of $[h, \infty)$. (To see that all left boundaries of the intervals are larger than h use the assumption on K as well as $d \ge 3$.) The intervals are such that

$$\frac{a}{d-1} - (k+1) \in I_{k-1} \quad \text{and} \quad \frac{a}{d-1} + (k+1) \in I_{k-1} \quad \text{for } a \in I_k,\ k \ge 1. \tag{2.10}$$

Indeed, for $k \ge 1$ and $d \ge 3$ one has $4\frac{k+2}{d-1} + (k+1) \le 4(k+1)$ and hence for every $a \in I_k$

$$\begin{aligned} &\tfrac{a}{d-1} + (k+1) \le (d-1)^{K+k} + 4\tfrac{k+2}{d-1} + (k+1) \le (d-1)^{K+k} + 4(k+1), \\ &\tfrac{a}{d-1} - (k+1) \ge (d-1)^{K+k-1} - 4\tfrac{k+2}{d-1} - (k+1) \ge (d-1)^{K+k-1} - 4(k+1). \end{aligned}$$

By using once more that χ_h is the eigenfunction of L_h, we have for every $a \in I_k$ with $k \ge 1$ that

$$\chi_h(a) = \frac{1}{\lambda_h}(L_h \chi_h)(a) \overset{(1.12)}{\ge} \frac{d-1}{\lambda_h} \underbrace{\mathbf{1}_{[h,\infty)}(a)}_{=1} \mathbb{E}^Y\Big[\chi_h\big(\tfrac{a}{d-1} + Y\big)\mathbf{1}_{\{|Y| \le k+1\}}\Big]. \tag{2.11}$$

From (2.10) we have that on $\{|Y| \le k+1\}$ it holds $\frac{a}{d-1} + Y \in I_{k-1}$. Hence,

$$\chi_h(a) \overset{(2.11)}{\ge} \frac{d-1}{\lambda_h} \inf_{b \in I_{k-1}} \chi_h(b) \cdot \mathbb{P}\big[|Y| \le k+1\big] \quad \text{for } a \in I_k,\ k \ge 1. \tag{2.12}$$

The repeated application of (2.12) implies that

$$\chi_h(a) \ge \Big(\frac{d-1}{\lambda_h}\Big)^k \inf_{b \in I_0} \chi_h(b) \prod_{\ell=1}^k \big(1 - \mathbb{P}[|Y| > \ell+1]\big) \quad \text{for } a \in I_k,\ k \ge 1. \tag{2.13}$$

By the exponential Markov inequality

$$\prod_{\ell=1}^k \big(1 - \mathbb{P}[|Y| > \ell+1]\big) \ge \prod_{\ell=1}^k \big(1 - 2e^{-\frac{d-1}{2d}(\ell+1)^2}\big) \overset{d\ge3}{\ge} \prod_{\ell=1}^\infty \big(1 - 2e^{-\frac{1}{3}(\ell+1)^2}\big) \ge c \tag{2.14}$$

with $c \in (0,1)$ independent of $k \geq 1$.

The inequality (2.2) is now an easy consequence of (2.13) and (2.14). First note that $\chi_h(a) \geq \inf_{b \in I_0} \chi_h(b) > 0$ for $a \in [h, (d-1)^{K+1})$ since χ_h is continuous and strictly positive on $[h, \infty)$ by Proposition 1.1, and $I_0 \subseteq [h, \infty)$ is compact. Now, for $a \geq (d-1)^{K+1}$ let $k(a) := \lfloor \log_{d-1}(a) - K \rfloor \geq 1$ be the unique integer with $(d-1)^{K+k(a)} \leq a < (d-1)^{K+k(a)+1}$. In particular $a \in I_{k(a)}$ and therefore from (2.13) and (2.14) we obtain that

$$\chi_h(a) \geq c\Big(\frac{d-1}{\lambda_h}\Big)^{k(a)} \inf_{b \in I_0} \chi_h(b) \geq \underbrace{c\Big(\frac{\lambda_h}{d-1}\Big)^{K+1} \inf_{b \in I_0} \chi_h(b)}_{=: c_h > 0} \, a^{\log_{d-1}(\frac{d-1}{\lambda_h})}.$$

This shows (2.2) and concludes the proof of Proposition 2.1. □

3 Recursive equation for the non-percolation probability

In this section we adopt the perspective of multi-type branching processes and show that a certain function (see (3.1)) closely related to the forward percolation probability from (1.19) is the unique solution to a recursive equation (Theorem 3.1). This fact will be used to derive the results on the supercritical behaviour of the level sets of $\varphi_{\mathbb{T}_d}$ in Section 4, in particular the continuity of the percolation probability and its equality with the probability of a non-vanishing martingale limit (Theorem 4.1 and Proposition 4.2). At the end of the section we compute the Fréchet derivative of the operator involved in the recursive equation (Proposition 3.6). This will be an important ingredient to estimate exponential moments of $|\mathcal{C}_{\mathrm{o}}^h|$ in Section 5 (Theorem 5.1). The proofs in this section partially follow the lines from multi-type branching processes (see e.g. [Har63], Chapter III). However, a lot depends on the special structure of the Gaussian free field on regular trees.

We introduce for every $h \in \mathbb{R}$ the conditional forward extinction probability (see (1.10) for notation)

$$q_h(a) := \mathbb{P}_a^{\mathbb{T}_d}\big[|\mathcal{C}_{\mathrm{o}}^h \cap \mathbb{T}_d^+| < \infty\big] \quad \text{for } a \in \mathbb{R}. \tag{3.1}$$

The function q_h is closely related to the value of the forward percolation probability from (1.19) at h. Indeed, since the distribution ν above (1.12) is the distribution of $\varphi_{\mathbb{T}_d}(\mathrm{o})$ under $\mathbb{P}^{\mathbb{T}_d}$, one has

$$\int_{\mathbb{R}} q_h(a)\, d\nu(a) = \mathbb{E}^{\mathbb{T}_d}[q_h(\varphi_{\mathbb{T}_d}(\mathrm{o}))] \overset{(1.19)}{=} 1 - \eta^+(h). \tag{3.2}$$

In particular, this shows by the comment at the end of Section 1 that q_h is identically 1 if $h > h_\star$ and is not identically 1 if $h < h_\star$.

Now, recall the space $L^2(\nu)$ defined in (1.12) and let us define for every $h \in \mathbb{R}$ the (non-linear) operator R_h on $L^2(\nu)$ through:

$$\begin{aligned} &(R_h f)(a) := \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a)\mathbb{E}^Y\big[f(\tfrac{a}{d-1} + Y)\big]^{d-1} \\ &\text{for } f \in L^2(\nu) \text{ and } a \in \mathbb{R}, \end{aligned} \tag{3.3}$$

where $Y \sim \mathcal{N}(0, \frac{d}{d-1})$ as in (1.12). To see that indeed $R_h f \in L^2(\nu)$, abbreviate $\hat{f}(a) := \mathbb{E}^Y[f(\frac{a}{d-1} + Y)]$ for $a \in \mathbb{R}$ and apply the hypercontractivity estimate (1.14) with $p = 2$

and $q = (d-1)^2 + 1$ to find that $\|\hat{f}^{d-1}\|_{L^{q/d-1}(\nu)} = \|\hat{f}\|^{d-1}_{L^q(\nu)} \leq \|f\|^{d-1}_{L^2(\nu)} < \infty$ and thus $\hat{f}^{d-1} \in L^{\frac{q}{d-1}}(\nu)$. Since $\frac{q}{d-1} = (d-1) + \frac{1}{d-1} \geq 2$, this implies that $R_h f \in L^2(\nu)$.

We are actually only interested in the operator R_h for $h \in \mathbb{R}$ on the subset

$$\mathcal{S}_h \coloneqq \{f \in L^2(\nu) \,|\, 0 \leq f \leq 1 \text{ and } f = 1 \text{ on } (-\infty, h)\}.$$

By definition we directly have $R_h : \mathcal{S}_h \to \mathcal{S}_h$. In Theorem 3.1 we prove that q_h is essentially the unique solution in $\mathcal{S}_h$ to the equation $R_h f = f$.

From the multi-type branching process perspective, the operator R_h can be used to write recurrence relations for generating functionals related to $\mathcal{C}^h_{\mathrm{o}}$, cf. for example [Har63], Section III.7. In particular, by using the notation from (1.10), (1.15) and by applying (1.7), we see that

$$(R_h f)(a) = \mathbb{E}^{\mathbb{T}_d}_a \Big[\prod_{y \in \mathcal{Z}^h_1} f(\varphi_{\mathbb{T}_d}(y)) \Big] \quad \text{for } f \in \mathcal{S}_h \text{ and } a \in \mathbb{R},$$

where the empty product is interpreted as being equal to 1. This identity can be extended: define iteratively

$$R^0_h f \coloneqq f \quad \text{and} \quad R^k_h f \coloneqq R^{k-1}_h (R_h f) \quad \text{for } f \in L^2(\nu) \text{ and } k \geq 1.$$

Then one can prove by induction on $k \geq 0$ and using (1.7) that

$$(R^k_h f)(a) = \mathbb{E}^{\mathbb{T}_d}_a \Big[\prod_{y \in \mathcal{Z}^h_k} f(\varphi_{\mathbb{T}_d}(y)) \Big] \quad \text{for } f \in \mathcal{S}_h, k \geq 0, a \in \mathbb{R}. \tag{3.4}$$

We come to the main result of this section.

Theorem 3.1. *For every $h \in \mathbb{R}$ the function q_h is the smallest solution in $\mathcal{S}_h$ to the equation $f = R_h f$. More precisely, the only solutions in $\mathcal{S}_h$ to $R_h f = f$ are the function q_h and the constant 1 function. These two functions coincide if $h > h_\star$ and are distinct if $h < h_\star$.*

The proof of the theorem is broken into several steps stated as Lemmas 3.2–3.5. The first one is a classical observation from the theory of multi-type branching processes.

Lemma 3.2. *Let $h \in \mathbb{R}$. The function q_h satisfies $q_h \in \mathcal{S}_h$ and solves the equation $R_h f = f$.*

Proof. The fact that $q_h \in \mathcal{S}_h$ is clear. To prove the second statement, denote $S^+_{\mathbb{T}_d}(\mathrm{o}, 1) =: \{x_1, \dots, x_{d-1}\}$ and recall the notation from (1.1). Then, for every $a \in \mathbb{R}$,

$$\begin{aligned} q_h(a) &= \mathbb{P}^{\mathbb{T}_d}_a \big[|\mathcal{C}^h_{\mathrm{o}} \cap \mathbb{T}^+_d| < \infty, \varphi_{\mathbb{T}_d}(\mathrm{o}) < h \big] + \mathbb{P}^{\mathbb{T}_d}_a \big[|\mathcal{C}^h_{\mathrm{o}} \cap \mathbb{T}^+_d| < \infty, \varphi_{\mathbb{T}_d}(\mathrm{o}) \geq h \big] \\ &= \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a) \, \mathbb{P}^{\mathbb{T}_d}_a \big[|\mathcal{C}^h_{\mathrm{o}} \cap U_{x_i}| < \infty \text{ for } i = 1, \dots, d-1 \big] \\ &\overset{(1.11)}{=} \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a) \, \mathbb{E}^Y \Big[\mathbb{P}^{\mathbb{T}_d}_{\frac{a}{d-1}+Y} \big[|\mathcal{C}^h_{\mathrm{o}} \cap \mathbb{T}^+_d| < \infty \big] \Big]^{d-1} \underset{(3.1)}{\overset{(3.3)}{=}} (R_h q_h)(a), \end{aligned}$$

completing the proof. □

Next, we give various necessary properties of solutions to $R_h f = f$.

Lemma 3.3. *Let $h \in \mathbb{R}$. Assume that $f \in \mathcal{S}_h$ solves $R_h f = f$. Then f is continuous and positive on $[h, \infty)$. If additionally f is not identically 1, then $\sup_{a \in [h,\infty)} f(a) < 1$ and $\lim_{a \to \infty} f(a) = 0$.*

Proof. For the continuity and positivity we note that for $a \geq h$ one can write

$$f(a) = (R_h f)(a) \overset{(3.3)}{=} \left(\int_{\mathbb{R}} f\big(\tfrac{a}{d-1} + y\big) \tfrac{\sqrt{d-1}}{\sqrt{2\pi d}} e^{-\frac{(d-1)y^2}{2d}} \, dy \right)^{d-1}$$

$$= \left(\tfrac{\sqrt{d-1}}{\sqrt{2\pi d}} e^{-\frac{a^2}{2d(d-1)}} \int_{\mathbb{R}} f(z) e^{-\frac{d-1}{2d} z^2 + \frac{a}{d} z} \, dz \right)^{d-1}.$$

The right hand side is continuous in a by the dominated convergence theorem and it is also positive since f is non-negative and equal to 1 on $(-\infty, h)$.

If f is not identically 1, then there is some $b \geq h$ with $f(b) < 1$. Hence, by the continuity of f on $[h, \infty)$ previously shown, there is an interval of positive Lebesgue measure in $[h, \infty)$ on which f is strictly smaller than 1. Due to $f = R_h f$ and $0 \leq f \leq 1$, this implies that $f(a) < 1$ for all $a \geq h$ by the definition (3.3) of R_h.

We will now show that one even has $\sup_{[h,\infty)} f(a) < 1$. Consider the intervals $I_k \subseteq [h, \infty)$, $k \geq 0$, from (2.9). Since $f < 1$ and f is continuous on $[h, \infty)$ and I_0 is compact, we have $\Delta := \max\{\frac{1}{9}, \sup_{a \in I_0} f(a)\} < 1$. If we show by induction on $k \geq 0$ that $\sup_{a \in I_0 \cup \ldots \cup I_k} f(a) \leq \Delta$ for all $k \geq 0$, then $\sup_{[h,\infty)} f(a) \leq \Delta < 1$ follows since $\bigcup_{k=0}^{\infty} I_k = [h, \infty)$. Now for $k = 0$ the claim is true by definition of Δ. So assume it holds for $k \geq 0$. Let $Y \sim \mathcal{N}(0, \frac{d}{d-1})$ and define $\varepsilon := \mathbb{P}^Y[|Y| \geq 2]$. Observe that $\varepsilon < \frac{1}{4}$ because $d \geq 3$. For $a \in I_{k+1}$ we can estimate

$$f(a) = (R_h f)(a) \overset{(3.3)}{=} \mathbb{E}^Y\big[f\big(\tfrac{a}{d-1} + Y\big)\big]^{d-1} \leq \mathbb{E}^Y\big[f\big(\tfrac{a}{d-1} + Y\big)\big]^2$$

$$\leq \Big(\mathbb{E}^Y\big[f\big(\underbrace{\tfrac{a}{d-1} + Y}_{\in I_k \text{ by } (2.10)}\big)\mathbf{1}_{\{|Y| \leq 2\}}\big] + \mathbb{E}^Y\big[\mathbf{1}_{\{|Y| > 2\}}\big]\Big)^2 \leq \big(\Delta \cdot (1 - \varepsilon) + \varepsilon\big)^2$$

by induction hypothesis. Therefore,

$$\sup_{a \in I_{k+1}} f(a) - \Delta \leq \big(\Delta \cdot (1 - \varepsilon) + \varepsilon\big)^2 - \Delta = \underbrace{(\Delta - 1)}_{<0} \underbrace{\big(\Delta \cdot (1 - \varepsilon)^2 - \varepsilon^2\big)}_{\geq \frac{1}{9} \cdot (\frac{3}{4})^2 - (\frac{1}{4})^2 \geq 0} \leq 0.$$

This shows that $\sup_{a \in I_{k+1}} f(a) \leq \Delta$, which together with the induction hypothesis implies $\sup_{a \in I_0 \cup \ldots \cup I_{k+1}} f(a) \leq \Delta$ and completes the proof of $\sup_{a \in [h,\infty)} f(a) < 1$.

It remains to show $\lim_{a \to \infty} f(a) = 0$. The assumption $R_h f = f$ implies that

$$\limsup_{a \to \infty} f(a) = \limsup_{a \to \infty} (R_h f)(a) \overset{(3.3)}{=} \big(\limsup_{a \to \infty} \mathbb{E}^Y\big[f\big(\tfrac{a}{d-1} + Y\big)\big]\big)^{d-1}. \tag{3.5}$$

Since by Fatou's lemma (using $0 \leq f \leq 1$)

$$\limsup_{a \to \infty} \mathbb{E}^Y\big[f\big(\tfrac{a}{d-1} + Y\big)\big] \leq \mathbb{E}^Y\big[\limsup_{a \to \infty} f\big(\tfrac{a}{d-1} + Y\big)\big],$$

we have found

$$\ell := \limsup_{a \to \infty} f(a) \overset{(3.5)}{\leq} \mathbb{E}^Y\big[\limsup_{a \to \infty} f\big(\tfrac{a}{d-1} + Y\big)\big]^{d-1} = \mathbb{E}^Y[\ell]^{d-1} = \ell^{d-1}.$$

However, $\ell \in [0, 1)$ since $\sup_{a \in [h,\infty)} f(a) < 1$. Therefore, the only possibility is $\ell = 0$. Hence $\lim_{a \to \infty} f(a) = 0$ because f is non-negative. □

The third step of the proof of Theorem 4.1 is the following statement of 'transience'.

Lemma 3.4. *For $K \geq 1$ and $\Lambda \geq h$ let (see (1.15) for the notation)*

$$A_k^{K,\Lambda} := \big\{1 \leq |\mathcal{Z}_k^h| \leq K, \varphi_{\mathbb{T}_d}(y) \leq \Lambda \text{ for all } y \in \mathcal{Z}_k^h\big\} \quad \text{for } k \geq 0. \tag{3.6}$$

Then for every $a \in \mathbb{R}$, $K \geq 1$ and $\Lambda \geq h$ one has

$$\mathbb{P}_a^{\mathbb{T}_d}[\limsup_{k\to\infty} A_k^{K,\Lambda}] = 0.$$

Proof. Observe that the events $B_k := A_k^{K,\Lambda} \cap \bigcap_{n\geq k+1} (A_n^{K,\Lambda})^{\mathsf{c}}$ for $k \geq 0$ are disjoint. Furthermore, denoting $S_{\mathbb{T}_d}^+(\mathrm{o}, k+1) \cap U_y =: \{y_1, \ldots, y_{d-1}\}$ for $y \in \mathcal{Z}_k^h$ and recalling the definition of $\mathcal{F}_k$ from (1.17), it holds that for $a \in \mathbb{R}$ and $k \geq 0$

$$\begin{aligned}
\mathbb{P}_a^{\mathbb{T}_d}[B_k] \ &\geq\ \mathbb{P}_a^{\mathbb{T}_d}\big[A_k^{K,\Lambda},\, \mathcal{Z}_{k+1}^h = \emptyset\big] = \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{A_k^{K,\Lambda}}\, \mathbb{P}_a^{\mathbb{T}_d}\big[\mathcal{Z}_{k+1}^h = \emptyset \,\big|\, \mathcal{F}_k\big]\Big] \\
&=\ \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{A_k^{K,\Lambda}}\, \mathbb{P}_a^{\mathbb{T}_d}\big[\bigcap_{y\in\mathcal{Z}_k^h} \bigcap_{i=1}^{d-1} \{\varphi_{\mathbb{T}_d}(y_i) < h\} \,\big|\, \mathcal{F}_k\big]\Big] \\
&\overset{(1.7)}{=}\ \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{A_k^{K,\Lambda}} \prod_{y\in\mathcal{Z}_k^h} \mathbb{P}^Y\big[\tfrac{\varphi_{\mathbb{T}_d}(y)}{d-1} + Y < h\big]^{d-1}\Big] \\
&\geq\ \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{A_k^{K,\Lambda}} \underbrace{\mathbb{P}^Y\big[\tfrac{\Lambda}{d-1} + Y < h\big]^{K(d-1)}}_{=:c_{K,\Lambda}}\Big] = c_{K,\Lambda}\, \mathbb{P}_a^{\mathbb{T}_d}[A_k^{K,\Lambda}].
\end{aligned}$$

Thus for $a \in \mathbb{R}$ we have

$$\sum_{k=0}^{\infty} \mathbb{P}_a^{\mathbb{T}_d}[A_k^{K,\Lambda}] \leq \frac{1}{c_{K,\Lambda}} \sum_{k=0}^{\infty} \mathbb{P}_a^{\mathbb{T}_d}[B_k] = \frac{1}{c_{K,\Lambda}} \mathbb{P}_a^{\mathbb{T}_d}\Big[\bigcup_{k\geq 0} B_k\Big] < \infty.$$

The claim then follows by the Borel-Cantelli lemma. □

The next lemma proves Theorem 3.1. Before that, we introduce for every $h \in \mathbb{R}$ the functions

$$q_h^k(a) := \mathbb{P}_a^{\mathbb{T}_d}[\mathcal{Z}_k^h = \emptyset] = \mathbb{P}_a^{\mathbb{T}_d}\Big[|\mathcal{C}_{\mathrm{o}}^h \cap S_{\mathbb{T}_d}^+(\mathrm{o}, k)| = 0\Big] \quad \text{for } a \in \mathbb{R},\ k \geq 0. \tag{3.7}$$

It can be easily seen that $q_h^k \in \mathcal{S}_h$ for $k \geq 0$ and $\mathbf{1}_{(-\infty,h)}(a) = q_h^0(a) \leq q_h^1(a) \leq q_h^2(a) \leq \ldots$ for $a \in \mathbb{R}$. In particular, $\lim_{k\to\infty} q_h^k(a) = q_h(a)$ for all $a \in \mathbb{R}$ by (3.1). In addition, applying (3.4) to the function $f = \mathbf{1}_{(-\infty,h)}$ implies that

$$q_h^k = R_h^k \mathbf{1}_{(-\infty,h)} \quad \text{for } k \geq 0. \tag{3.8}$$

Lemma 3.5. *Let $h \in \mathbb{R}$. The only solutions in $\mathcal{S}_h$ to $R_h f = f$ are the function q_h and the constant 1 function. These two functions coincide if $h > h_\star$ and are distinct if $h < h_\star$.*

Proof. From Lemma 3.2 we know that $q_h \in \mathcal{S}_h$ and $R_h q_h = q_h$. The same is of course true for the constant 1 function. We first claim that every solution in $\mathcal{S}_h$ to $R_h f = f$ satisfies $f \geq q_h$. Indeed, if $f \in \mathcal{S}_h$ is such a solution, then $R_h^k f = f$ for all $k \geq 0$. Also, the fact that $f \in \mathcal{S}_h$ implies $f \geq \mathbf{1}_{(-\infty,h)}$. Hence $f = R_h^k f \geq R_h^k \mathbf{1}_{(-\infty,h)} = q_h^k$ for all $k \geq 0$ by (3.3) and (3.8). By letting k tend to infinity we find $f \geq q_h$, proving the claim. In particular, if $q_h \equiv 1$ (e.g. when $h > h_\star$, see below (3.2)), then we have $f \equiv 1$ and thus $R_h f = f$ has a unique solution.

Now assume that $q_h \not\equiv 1$ (e.g. when $h < h_\star$, see below (3.2)) and that $f \not\equiv 1$ is a solution to $R_h f = f$. We claim that $f = q_h$. As we have already shown $f \geq q_h$, it remains to prove $f \leq q_h$. To see this, observe that by Lemma 3.3 we know that $\delta := \sup_{a\in[h,\infty)} f(a) \in (0,1)$. Let $m \geq 0$ be such that $\delta := \sup_{a\in[h,\infty)} f(a) \in \big[\frac{1}{2^{m+1}}, \frac{1}{2^m}\big)$. Then for $a \in \mathbb{R}$ and $k \geq 0$ one has

$$\begin{aligned} f(a) \;&=\; (R_h^k f)(a) \overset{(3.4)}{=} \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{\{\mathcal{Z}_k^h=\emptyset\}} \prod_{y\in\mathcal{Z}_k^h} f(\varphi_{\mathbb{T}_d}(y))\Big] + \mathbb{E}_a^{\mathbb{T}_d}\Big[\mathbf{1}_{\{\mathcal{Z}_k^h\neq\emptyset\}} \prod_{y\in\mathcal{Z}_k^h} f(\varphi_{\mathbb{T}_d}(y))\Big] \\ &\overset{(3.7)}{\leq} q_h^k(a) + \sum_{n\geq m} \frac{1}{2^n} \mathbb{P}_a^{\mathbb{T}_d}\Big[\mathcal{Z}_k^h \neq \emptyset\,,\, \prod_{y\in\mathcal{Z}_k^h} f(\varphi_{\mathbb{T}_d}(y)) \in \big[\tfrac{1}{2^{n+1}}, \tfrac{1}{2^n}\big)\Big]. \end{aligned} \tag{3.9}$$

Note that for the events appearing on the right hand side of (3.9) one has

$$\begin{aligned} &\big\{\mathcal{Z}_k^h \neq \emptyset\,,\, \textstyle\prod_{y\in\mathcal{Z}_k^h} f(\varphi_{\mathbb{T}_d}(y)) \in \big[\frac{1}{2^{n+1}}, \frac{1}{2^n}\big)\big\} \\ &\quad\subseteq \big\{|\mathcal{Z}_k^h| \geq 1\,,\, \delta^{|\mathcal{Z}_k^h|} \geq \tfrac{1}{2^{n+1}}\,,\, f(\varphi_{\mathbb{T}_d}(y)) \geq \tfrac{1}{2^{n+1}} \text{ for all } y \in \mathcal{Z}_k^h\big\} \\ &\quad\subseteq \big\{|\mathcal{Z}_k^h| \geq 1\,,\, 2^{n+1} \geq (1/\delta)^{|\mathcal{Z}_k^h|}\,,\, f(\varphi_{\mathbb{T}_d}(y)) \geq \tfrac{1}{2^{n+1}} \text{ for all } y \in \mathcal{Z}_k^h\big\} \\ &\quad\subseteq \big\{1 \leq |\mathcal{Z}_k^h| \leq K_n\,,\, \varphi_{\mathbb{T}_d}(y) \leq \Lambda_n \text{ for all } y \in \mathcal{Z}_k^h\big\} \overset{(3.6)}{=} A_k^{K_n,\Lambda_n} \end{aligned} \tag{3.10}$$

with $K_n := \log_{1/\delta}(2^{n+1})$ and $\Lambda_n := \sup\{a \in \mathbb{R} \,|\, f(a) \geq \frac{1}{2^{n+1}}\}$. We observe that for $n \geq m$ it holds $K_n \geq 1$ since then $1/\delta \leq 2^{m+1} \leq 2^{n+1}$. Moreover, $h \leq \Lambda_n < \infty$ since $f \in \mathcal{S}_h$ (so $f = 1$ on $(-\infty, h)$) and $\lim_{a\to\infty} f(a) = 0$ by Lemma 3.3. As a consequence, Lemma 3.4 and Fatou's lemma imply $\lim_{k\to\infty} \mathbb{P}_a^{\mathbb{T}_d}[A_k^{K_n,\Lambda_n}] = 0$. Therefore, by using the dominated convergence theorem, for $a \in \mathbb{R}$ one has

$$f(a) \underset{(3.10)}{\overset{(3.9)}{\leq}} \lim_{k\to\infty} \Big(q_h^k(a) + \sum_{n\geq m} \frac{1}{2^n} \mathbb{P}_a^{\mathbb{T}_d}\big[A_k^{K_n,\Lambda_n}\big]\Big) = q_h(a).$$

This implies that $f = q_h$, completing the proof. □

As last result of Section 3 we compute the Fréchet derivative of the operators $(R_h)_{h\in\mathbb{R}}$ defined in (3.3). This technical result is one of the main ingredients for proving the existence of exponential moments of $|\mathcal{C}_o^h|$ in the subcritical phase (Section 5). Incidentally, let us mention that its proof is based on the hypercontractivity estimate (1.14) and that the precise relation between p and q in the estimate is vital (for $p = 2$).

Proposition 3.6. *Let $h \in \mathbb{R}$ and consider the operator $R_h : L^2(\nu) \to L^2(\nu)$ from (3.3). Then the Fréchet derivative of R_h at $f \in L^2(\nu)$ is given by $A_h^f : L^2(\nu) \to L^2(\nu)$ with*

$$A_h^f g := \mathbf{1}_{[h,\infty)} \cdot (d-1)\mathbb{E}^Y\big[f(\tfrac{\cdot}{d-1} + Y)\big]^{d-2}\, \mathbb{E}^Y\big[g(\tfrac{\cdot}{d-1} + Y)\big]. \tag{3.11}$$

In particular, if $g \in L^2(\nu)$ vanishes on $(-\infty, h)$, then $A_h^1 g = L_h g$, where A_h^1 is the Fréchet derivative of R_h at the constant function 1 and L_h is given in (1.12). Furthermore, for all $\varepsilon > 0$ there exists $r > 0$ such that $\|A_h^f g\|_{L^2(\nu)} \leq (\lambda_h + \varepsilon)\, \|g\|_{L^2(\nu)}$ if $g \in L^2(\nu)$ vanishes on $(-\infty, h)$ and $\|f - 1\|_{L^2(\nu)} \leq r$.

Proof. We start with some observations. For $u \in L^2(\nu)$ let us abbreviate $\hat{u}(a) \coloneqq \mathbb{E}^Y\big[u(\frac{a}{d-1}+Y)\big]$, $a \in \mathbb{R}$. We further set $p_i \coloneqq \frac{(d-1)^2+1}{i} \geq 2$ for $i = 1, \dots, d-1$. Then for $u \in L^2(\nu)$ one has

$$\|\hat{u}^i\|_{L^{p_i}(\nu)} = \|\hat{u}\|^i_{L^{(d-1)^2+1}(\nu)} \overset{(1.14)}{\leq} \|u\|^i_{L^2(\nu)} < \infty. \tag{3.12}$$

Now if $u, v, w \in L^2(\nu)$ and $i, j, k \in \{0, \dots, d-1\}$ with $i+j+k \leq d-1$, then one has $\hat{u}^i \in L^{p_i}(\nu)$, $\hat{v}^j \in L^{p_j}(\nu)$ and $\hat{w}^k \in L^{p_k}(\nu)$ by (3.12), where we put $p_0 \coloneqq \infty$, and therefore

$$\begin{aligned} \|\hat{u}^i\,\hat{v}^j\,\hat{w}^k\|_{L^2(\nu)} &\overset{2\leq p_{i+j+k}}{\leq} \|\hat{u}^i\,\hat{v}^j\,\hat{w}^k\|_{L^{p_{i+j+k}}(\nu)} \overset{(*)}{\leq} \|\hat{u}^i\|_{L^{p_i}(\nu)}\|\hat{v}^j\|_{L^{p_j}(\nu)}\|\hat{w}^k\|_{L^{p_k}(\nu)} \\ &\overset{(3.12)}{\leq} \|u\|^i_{L^2(\nu)}\|v\|^j_{L^2(\nu)}\|w\|^k_{L^2(\nu)} < \infty, \end{aligned} \tag{3.13}$$

where in $(*)$ we use the generalised Hölder inequality.

To compute the Fréchet derivative of R_h note that for $f, g \in L^2(\nu)$ one has

$$\begin{aligned} R_h(f+g) - R_h f &\overset{(3.3)}{=} \mathbf{1}_{[h,\infty)} \cdot \big((\hat{f}+\hat{g})^{d-1} - \hat{f}^{d-1}\big) \\ &= \mathbf{1}_{[h,\infty)} \cdot \sum_{i=0}^{d-2} \tbinom{d-1}{i} \hat{f}^i \hat{g}^{d-1-i} \;=\; A_h^f g + E_h^f g, \end{aligned} \tag{3.14}$$

where $A_h^f g = \mathbf{1}_{[h,\infty)} \cdot (d-1)\hat{f}^{d-2}\hat{g}$ is the function defined in (3.11) and the operator $E_h^f : L^2(\nu) \to L^2(\nu)$ is given by

$$E_h^f g \coloneqq \mathbf{1}_{[h,\infty)} \cdot \sum_{i=0}^{d-3} \tbinom{d-1}{i} \hat{f}^i \hat{g}^{d-1-i}. \tag{3.15}$$

Note that the map A_h^f is linear and also bounded since $\sup_{\|g\|_{L^2(\nu)}\leq 1} \|A_h^f g\|_{L^2(\nu)} \leq (d-1)\sup_{\|g\|_{L^2(\nu)}\leq 1} \|\hat{f}^{d-2}\hat{g}\|_{L^2(\nu)} < \infty$ by (3.13). To conclude that A_h^f is the Fréchet derivative of R_h at f it remains to show that

$$\frac{\|R_h(f+g) - R_h f - A_h^f g\|_{L^2(\nu)}}{\|g\|_{L^2(\nu)}} \overset{(3.14)}{=} \frac{\|E_h^f g\|_{L^2(\nu)}}{\|g\|_{L^2(\nu)}} \to 0 \quad \text{if } \|g\|_{L^2(\nu)} \to 0. \tag{3.16}$$

This is the case because

$$\|E_h^f g\|_{L^2(\nu)} \underset{(3.13)}{\overset{(3.15)}{\leq}} \sum_{i=0}^{d-3} \tbinom{d-1}{i} \|f\|^i_{L^2(\nu)} \|g\|^{d-1-i}_{L^2(\nu)},$$

implying (3.16). Thus A_h^f is the Fréchet derivative of R_h at f.

From (3.11) and (1.12) we directly see that $A_h^1 g = L_h g$ if $g \in L^2(\nu)$ vanishes on $(-\infty, h)$. It remains to show the second part of the statement. We have $\|A_h^f g\|_{L^2(\nu)} \leq \|A_h^f g - A_h^1 g\|_{L^2(\nu)} + \|A_h^1 g\|_{L^2(\nu)}$. For $g \in L^2(\nu)$ with $g = 0$ on $(-\infty, h)$ one obtains $\|A_h^1 g\|_{L^2(\nu)} = \|L_h g\|_{L^2(\nu)} \leq \lambda_h \|g\|_{L^2(\nu)}$ by (1.13). Moreover, the formula $b^{d-2} - 1 = (b-1)(1 + b + \ldots + b^{d-3})$ and the triangle inequality imply

$$\|A_h^f g - A_f^1 g\|_{L^2(\nu)} \overset{(3.11)}{\leq} (d-1)\,\|\hat{g}\,(\hat{f}^{d-2} - 1)\|_{L^2(\nu)} \leq (d-1) \sum_{i=0}^{d-3} \|\hat{g}\,(\hat{f} - 1)\hat{f}^i\|_{L^2(\nu)}$$

and therefore $\|A_h^f g - A_f^1 g\|_{L^2(\nu)} \leq (d-1)\|g\|_{L^2(\nu)}\|f-1\|_{L^2(\nu)} \sum_{i=0}^{d-3} \|f\|_{L^2(\nu)}^i$ by (3.13). All in all we showed

$$\|A_h^f g\|_{L^2(\nu)} \leq \Big(\lambda_h + (d-1)\|f-1\|_{L^2(\nu)} \sum_{i=0}^{d-3} \|f\|_{L^2(\nu)}^i\Big) \|g\|_{L^2(\nu)}. \tag{3.17}$$

Now let $\varepsilon > 0$ and take $r > 0$ such that $(d-1)((1+r)^{d-2}-1) \leq \varepsilon$. Then if $\|f-1\|_{L^2(\nu)} \leq r$, and hence also $\|f\|_{L^2(\nu)} \leq 1+r$, we have $\|A_h^f g\|_{L^2(\nu)} \leq (\lambda_h + \varepsilon)\,\|g\|_{L^2(\nu)}$ by (3.17). This concludes the proof. □

4 Behaviour of the level sets in the supercritical phase

In this section we study the behaviour of the level sets of the Gaussian free field on $\mathbb{T}_d$ for $h < h_\star$. The main goal is to show that the percolation probabilities η and η^+ are continuous functions of the level h on the interval $(-\infty, h_\star)$ (Theorem 4.1, corresponding to (0.7)) and to prove that $|\mathcal{C}_{\text{o}}^h|$ grows exponentially in the radius with probability bounded away from zero when $h < h_\star$ (Theorem 4.3, corresponding to (0.10)). Along the way we also show the equivalence of the probabilities of forward percolation and of a non-vanishing martingale limit (Proposition 4.2, corresponding to (0.6)). These results essentially come as an application of Theorem 3.1 from Section 3. For this section recall the measure ν defined above (1.12).

4.1 Continuity of the percolation probability

In this section we analyse the continuity properties of the percolation probabilities η and η^+, and show (0.7) in Theorem 4.1. Recall the functions q_h, $h \in \mathbb{R}$, introduced in (3.1) and their relation with η^+ reported in (3.2).

Theorem 4.1. *The functions η and η^+ are left-continuous on $\mathbb{R}$ and continuous on $\mathbb{R} \setminus \{h_\star\}$.*

Proof. Note that

$$\eta^+(h) = \mathbb{P}^{\mathbb{T}_d}\Big[\bigcap_{k\geq 1} \{\mathcal{C}_{\text{o}}^h \cap S_{\mathbb{T}_d}^+(\text{o},k) \neq \emptyset\}\Big] = \lim_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\big[\mathcal{C}_{\text{o}}^h \cap S_{\mathbb{T}_d}^+(\text{o},k) \neq \emptyset\big]. \tag{4.1}$$

Under $\mathbb{P}^{\mathbb{T}_d}$ the vector $(\varphi_{\mathbb{T}_d}(y))_{y\in B_{\mathbb{T}_d}^+(\text{o},k)}$ has a density and thus $h \mapsto \mathbb{P}^{\mathbb{T}_d}[\mathcal{C}_{\text{o}}^h \cap S_{\mathbb{T}_d}^+(\text{o},k) \neq \emptyset]$ is a continuous function. Therefore by (4.1), η^+ is a decreasing limit of continuous functions and hence upper semicontinuous. As η^+ is a non-increasing function, it is thus left-continuous. With the obvious changes in (4.1) one can also show the left-continuity of η.

To show the right-continuity on $\mathbb{R} \setminus \{h_\star\}$ observe first that if $h > h_\star$, then $\eta(h) = \eta^+(h) = 0$ by definition and the comment at the end of Section 1. So it remains to prove the right-continuity on $(-\infty, h_\star)$. Fix $h < h_\star$ and assume $(h_\ell)_{\ell\geq 0}$ is a sequence satisfying $h_\ell \downarrow h$ and $h_\ell < h_\star$ for all $\ell \geq 0$. We will show that $\lim_{\ell\to\infty} \eta^+(h_\ell) = \eta^+(h)$ and $\lim_{\ell\to\infty} \eta(h_\ell) = \eta(h)$. Observe that by (3.2) and the dominated convergence theorem the former follows from the claim

$$\lim_{\ell\to\infty} q_{h_\ell}(a) = q_h(a) \quad \text{for } a \in \mathbb{R} \setminus \{h\}. \tag{4.2}$$

Actually, also $\lim_{\ell\to\infty}\eta(h_\ell) = \eta(h)$ follows from (4.2) by a double application of the dominated convergence theorem since

$$\begin{aligned}\eta(h_\ell) &\overset{(1.10)}{=} \int_{\mathbb{R}} \mathbb{P}_a^{\mathbb{T}_d}[|\mathcal{C}_{\mathrm{o}}^{h_\ell}| = \infty]\, d\nu(a) = \int_{\mathbb{R}} \big(1 - \mathbb{P}_a^{\mathbb{T}_d}[|\mathcal{C}_{\mathrm{o}}^{h_\ell}| < \infty]\big)\mathbf{1}_{[h_\ell,\infty)}(a)\, d\nu(a)\\ &= \int_{\mathbb{R}} \big(1 - \mathbb{P}_a^{\mathbb{T}_d}[|\mathcal{C}_{\mathrm{o}}^{h_\ell} \cap U_{x_i}| < \infty \text{ for all } i = 1,\dots,d]\big)\mathbf{1}_{[h_\ell,\infty)}(a)\, d\nu(a)\\ &\underset{(3.1)}{\overset{(1.11)}{=}} \int_{\mathbb{R}} \big(1 - \mathbb{E}^Y[q_{h_\ell}(\tfrac{a}{d-1} + Y)]^d\big)\mathbf{1}_{[h_\ell,\infty)}(a)\, d\nu(a).\end{aligned}$$

Hence it remains to show (4.2).

Define the two auxiliary functions $\widetilde{q}_h$ and q'_h on $\mathbb{R}$ by

$$\widetilde{q}_h(a) \coloneqq \lim_{\ell\to\infty} q_{h_\ell}(a) = \inf_{\ell\geq 0} q_{h_\ell}(a) \quad \text{for } a \in \mathbb{R} \tag{4.3}$$

and

$$q'_h(a) \coloneqq \begin{cases} \widetilde{q}_h(a), & \text{if } a \in \mathbb{R}\setminus\{h\}\\ (R_h\widetilde{q}_h)(h), & \text{if } a = h.\end{cases} \tag{4.4}$$

We will now apply Theorem 3.1 to show $q'_h = q_h$. From this the claim (4.2) follows by (4.4) and (4.3).

Since $h_\ell < h_\star$, one has $q_{h_\ell} \not\equiv 1$ for all $\ell \geq 0$ (see below (3.2)). This implies $\widetilde{q}_h \not\equiv 1$ by (4.3) (being a decreasing limit) and hence also $q'_h \not\equiv 1$ by (4.4). Moreover if $a < h$, then $a < h_\ell$ for all $\ell \geq 0$, which yields $q_{h_\ell}(a) = 1$ for all $\ell \geq 0$. This implies $q'_h(a) = 1$ for $a < h$ by (4.3) and (4.4). Thus $q'_h \in \mathcal{S}_h$. Finally, for $a > h$ and $\ell \geq 0$ such that $h_\ell \leq a$, one finds by Lemma 3.2 and (3.3) that $q_{h_\ell}(a) = \mathbb{E}^Y\big[q_{h_\ell}(\frac{a}{d-1} + Y)\big]^{d-1}$. If we let ℓ tend to infinity on both sides, then (4.3) and the dominated convergence theorem give $\widetilde{q}_h(a) = \mathbb{E}^Y\big[\widetilde{q}_h(\frac{a}{d-1} + Y)\big]^{d-1}$ for all $a > h$. This together with (4.4) shows $q'_h = R_h q'_h$. By Theorem 3.1 we conclude that $q'_h = q_h$. The proof is complete. □

4.2 Percolation probability and non-triviality of the martingale limit

Recall the martingale $(M_k^{\geq h})_{k\geq 0}$ from (1.16). We now apply Theorem 3.1 from Section 3 to show in Proposition 4.2 the equivalence (0.6) between the probability of non-vanishing of the martingale limit and $\eta^+(h)$. From the discussion at the end of Section 1.2 we already know that $\mathbb{P}^{\mathbb{T}_d}\big[M_\infty^{\geq h} > 0\big] = \eta^+(h) = 0$ for $h > h_\star$. We now prove that the first equality remains true also if $h < h_\star$.

Proposition 4.2. *One has*

$$\eta^+(h) = \mathbb{P}^{\mathbb{T}_d}\big[M_\infty^{\geq h} > 0\big] \quad \textit{for all } h \in \mathbb{R}\setminus\{h_\star\}. \tag{4.5}$$

Proof. For every $h \in \mathbb{R}$ we introduce the function $m_h(a) \coloneqq \mathbb{P}_a^{\mathbb{T}_d}\big[M_\infty^{\geq h} = 0\big]$ for $a \in \mathbb{R}$, where $\mathbb{P}_a^{\mathbb{T}_d}$ is the conditional probability defined in (1.10). We note that

$$\int_{\mathbb{R}} m_h(a)\, d\nu(a) = \mathbb{E}^{\mathbb{T}_d}[m_h(\varphi_{\mathbb{T}_d}(\mathrm{o}))] \overset{(1.10)}{=} \mathbb{P}^{\mathbb{T}_d}\big[M_\infty^{\geq h} = 0\big]. \tag{4.6}$$

By (3.2) and (4.6) it is enough to show that for $h \neq h_\star$ one has $q_h = m_h$. This will follow from Theorem 3.1. Note that $m_h \in \mathcal{S}_h$ since $m_h(a) \geq \mathbb{P}_a^{\mathbb{T}_d}[\mathcal{Z}_0^h = \emptyset] = \mathbf{1}_{(-\infty,h)}(a)$

by (1.16) and (1.10). We also have that $R_h m_h = m_h$. Indeed, recall (1.1) and denote $S^+_{\mathbb{T}_d}(\mathrm{o},1) =: \{x_1,\dots,x_{d-1}\}$. Let us write $M^{\geq h}_{k,i} := \lambda_h^{-k}\sum_{y\in\mathcal{Z}^h_k\cap U_{x_i}} \chi_h(\varphi_{\mathbb{T}_d}(y))$ for $k\geq 1$ and $i=1,\dots,d-1$, so that $M^{\geq h}_k = \sum_{i=1}^{d-1} M^{\geq h}_{k,i}$ for $k\geq 1$. Then for $a\in\mathbb{R}$

$$
\begin{aligned}
m_h(a) &= \mathbb{P}^{\mathbb{T}_d}_a\big[\varphi_{\mathbb{T}_d}(\mathrm{o}) < h\,,\, M^{\geq h}_\infty = 0\big] + \mathbb{P}^{\mathbb{T}_d}_a\big[\varphi_{\mathbb{T}_d}(\mathrm{o}) \geq h\,,\, M^{\geq h}_\infty = 0\big] \\
&\overset{(1.10)}{=} \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a)\,\mathbb{P}^{\mathbb{T}_d}_a\big[\lim_{k\to\infty} M^{\geq h}_{k,i} = 0 \text{ for } i=1,\dots,d-1\big] \\
&\overset{(1.11)}{=} \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a)\,\mathbb{E}^Y\Big[\mathbb{P}^{\mathbb{T}_d}_{\frac{a}{d-1}+Y}\big[M^{\geq h}_\infty = 0\big]\Big]^{d-1} \overset{(3.3)}{=} (R_h m_h)(a).
\end{aligned}
$$

Now if $h > h_\star$, then by Theorem 3.1 we find $m_h = q_h \equiv 1$. On the other hand, if $h < h_\star$, then (4.6) and (1.18) imply that m_h is not the constant 1 function and so $m_h = q_h$ by Theorem 3.1 again. The proof is complete. □

4.3 Geometrical growth of $|\mathcal{C}^h_\mathrm{o}|$ in the supercritical phase

We come to the proof of (0.10), essentially that for $h < h_\star$ the number of vertices in $\mathbb{T}^+_d$ connected over distance k above level h to the root $\mathrm{o}\in\mathbb{T}_d$ grows exponentially in k with positive probability. Recall the notation from (1.15).

Theorem 4.3. *Let $h < h_\star$ (so that $\lambda_h > 1$, see Proposition 1.1). Then*

$$
\lim_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\Big[\big|\mathcal{Z}^h_k\big| \geq \frac{\lambda_h^k}{k^2}\Big] = \eta^+(h) > 0.
$$

Proof. Note that one directly has

$$
\limsup_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\Big[\big|\mathcal{Z}^h_k\big| \geq \frac{\lambda_h^k}{k^2}\Big] \leq \limsup_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\big[\mathcal{C}^h_\mathrm{o} \cap S^+_{\mathbb{T}_d}(\mathrm{o},k) \neq \emptyset\big] \overset{(1.19)}{=} \eta^+(h).
$$

Thus we only have to find a corresponding lower bound. By Fatou's lemma

$$
\begin{aligned}
\eta^+(h) &\overset{(4.5)}{=} \mathbb{P}^{\mathbb{T}_d}\big[M^{\geq h}_\infty > 0\big] \leq \mathbb{P}^{\mathbb{T}_d}\big[M^{\geq h}_k \geq \tfrac{1}{k} \text{ for all } k \text{ large enough}\big] \\
&\leq \liminf_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\big[M^{\geq h}_k \geq \tfrac{1}{k}\big] \\
&\leq \liminf_{k\to\infty} \Big(\mathbb{P}^{\mathbb{T}_d}\big[M^{\geq h}_k \geq \tfrac{1}{k}\,,\, A^h_k\big] + \mathbb{P}^{\mathbb{T}_d}\big[(A^h_k)^\mathsf{c}\big]\Big),
\end{aligned} \tag{4.7}
$$

where we introduced the event

$$
A^h_k := \Big\{ \sup_{y\in S^+_{\mathbb{T}_d}(\mathrm{o},k)} \chi_h(\varphi_{\mathbb{T}_d}(y)) \leq k \Big\} \quad \text{for } k\geq 0.
$$

On the event A^h_k the inequality $M^{\geq h}_k \geq \frac{1}{k}$ implies $|\mathcal{Z}^h_k| \geq \frac{\lambda_h^k}{k^2}$ by (1.16). Hence

$$
\mathbb{P}^{\mathbb{T}_d}\big[M^{\geq h}_k \geq \tfrac{1}{k}\,,\, A^h_k\big] \leq \mathbb{P}^{\mathbb{T}_d}\Big[\big|\mathcal{Z}^h_k\big| \geq \frac{\lambda_h^k}{k^2}\Big]. \tag{4.8}
$$

To deal with the event $(A^h_k)^\mathsf{c}$ note that by Proposition 2.1 and Remark 2.2 (here $h < h_\star$) one has $\chi_h(a) \leq c_h a$ for $a\geq h$ and $\chi_h(a) = 0$ for $a < h$. Thus, for $y\in\mathbb{T}_d$ and for $k\geq 0$

$$
\begin{aligned}
&\mathbb{P}^{\mathbb{T}_d}\big[\chi_h(\varphi_{\mathbb{T}_d}(y)) > k\big] = \mathbb{P}^{\mathbb{T}_d}\big[\chi_h(\varphi_{\mathbb{T}_d}(y)) > k\,,\, \varphi_{\mathbb{T}_d}(y) \geq h\big] \\
&\quad\leq \mathbb{P}^{\mathbb{T}_d}\big[c_h\varphi_{\mathbb{T}_d}(y) > k\,,\, \varphi_{\mathbb{T}_d}(y) \geq h\big] \overset{(0.1)}{\leq} \exp\Big(-\frac{k^2}{2c_h^2 g_{\mathbb{T}_d}(\mathrm{o},\mathrm{o})}\Big),
\end{aligned}
$$

where in the last step we use the exponential Markov inequality. Hence, by a union bound, for $k \geq 0$

$$\mathbb{P}^{\mathbb{T}_d}\big[(A_k^h)^{\mathsf{c}}\big] \leq \underbrace{\big|S_{\mathbb{T}_d}^+(\mathrm{o},k)\big|}_{=(d-1)^k} \exp\Big(-\frac{k^2}{2c_h^2 g_{\mathbb{T}_d}(\mathrm{o},\mathrm{o})}\Big) \xrightarrow{k\to\infty} 0. \tag{4.9}$$

From (4.7), (4.8) and (4.9) we have that $\liminf_{k\to\infty} \mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{Z}_k^h| \geq \frac{\lambda_h^k}{k^2}\big] \geq \eta^+(h)$ and the proof of Theorem 4.3 follows. □

5 Exponential moments of $|\mathcal{C}_\mathrm{o}^h|$ in the subcritical phase

This section proves that for every $h > h_\star$ the cardinality of the connected component of the level set of $\varphi_{\mathbb{T}_d}$ above level h in $\mathbb{T}_d^+$ containing the root $\mathrm{o} \in \mathbb{T}_d$ has exponential moments and actually, as a function of the value of $\varphi_{\mathbb{T}_d}(\mathrm{o})$, these exponential moments do not grow too fast. This is the content of Theorem 5.1 below (corresponding to (0.8)). In its proof we will use Proposition 3.6 from Section 3.

To state the result, we define for every $h \in \mathbb{R}$ and $\delta > 0$ the (potentially infinite) function

$$g_{h,\delta}(a) := \mathbb{E}_a^{\mathbb{T}_d}\Big[(1+\delta)^{|\mathcal{C}_\mathrm{o}^h \cap \mathbb{T}_d^+|}\Big] \quad \text{for } a \in \mathbb{R}, \tag{5.1}$$

where we use the notation for the conditional distribution of $\varphi_{\mathbb{T}_d}$ given $\varphi_{\mathbb{T}_d}(\mathrm{o}) = a$ defined in (1.10). Observe that (recall ν from above (1.12))

$$\int_{\mathbb{R}} g_{h,\delta}(a)\, d\nu(a) = \mathbb{E}^{\mathbb{T}_d}[g_{h,\delta}(\varphi_{\mathbb{T}_d}(\mathrm{o}))] \overset{(1.10)}{=} \mathbb{E}^{\mathbb{T}_d}\Big[(1+\delta)^{|\mathcal{C}_\mathrm{o}^{\mathbb{T}_d,h} \cap \mathbb{T}_d^+|}\Big]. \tag{5.2}$$

Note that if $q_h(a) < 1$ for q_h from (3.1) (in particular this is the case in the supercritical phase $h < h_\star$ for $a \geq h$), then $g_{h,\delta}(a)$ is infinite. The main goal of this section is to show that in the subcritical phase $h > h_\star$ there exists $\delta > 0$ such that the right hand side of (5.2) is finite and such that $g_{h,\delta}(a)$ does not grow too fast as a tends to infinity. Recall the space $L^2(\nu)$ defined in (1.12).

Theorem 5.1. *Let $h > h_\star$. Then there exists $\delta_h > 0$ such that*

$$g_{h,\delta_h} \in L^2(\nu). \tag{5.3}$$

Moreover, g_{h,δ_h} equals 1 on $(-\infty, h)$ and $g_{h,\delta_h}(a)$ is finite for all $a \in \mathbb{R}$. Finally, g_{h,δ_h} is continuous on $[h,\infty)$ and for all $\gamma > 0$ there exist $c_{h,\gamma} > 0$ and $c'_{h,\gamma} > 0$ such that

$$g_{h,\delta_h}(a) \leq c_{h,\gamma} \exp(c'_{h,\gamma} a^{1+\gamma}) \quad \text{for all } a \geq h. \tag{5.4}$$

In particular, (5.2) and (5.3) imply $\mathbb{E}^{\mathbb{T}_d}\big[(1+\delta_h)^{|\mathcal{C}_\mathrm{o}^h \cap \mathbb{T}_d^+|}\big] < \infty$.

Remark 5.2. Note that (0.9) follows from Theorem 5.1 by the exponential Markov inequality. More precisely, for $h < h_\star$ and $a \in \mathbb{R}$ take say $\gamma = 1$ in (5.4). Then $\mathbb{P}^{\mathbb{T}_d}\big[|\mathcal{C}_\mathrm{o}^h| \geq k \,\big|\, \varphi_{\mathbb{T}_d}(\mathrm{o}) = a\big] \leq \mathbb{P}_a^{\mathbb{T}_d}\big[|\mathcal{C}_\mathrm{o}^h \cap \mathbb{T}_d^+| \geq k\big] \leq (1+\delta_h)^{-k} c_h \exp(c'_h a^2)$, thus (0.9). □

The proof of Theorem 5.1 is split into various lemmas. The first one characterises $g_{h,\delta}$ as a monotone limit of functions in $L^2(\nu)$ which are obtained via iterated applications of a certain operator $R_{h,\delta}$ (see (5.5)) to the constant 1 function (Lemma 5.3). The second lemma shows that for $h > h_\star$ we can choose $\delta > 0$ such that the operator $R_{h,\delta}$ is a strict contraction on a closed subset of $L^2(\nu)$ including the constant 1 function (Lemma 5.4). This is an application of the technical Proposition 3.6 from Section 3. The combination of these two results will quickly lead to (5.3) via the Banach-Caccioppoli fixed-point theorem and to the other properties of $g_{h,\delta}$ stated in Theorem 5.1 except for (5.4). This is the content of Corollary 5.5. It then remains to prove (5.4). We first show a weaker statement in which $\gamma = 1$ on the right hand side (Lemma 5.6). It implies a recursive bound on $g_{h,\delta}$ (Lemma 5.7) which subsequently can be used to show the stronger statement (Lemma 5.8).

Let us introduce for every $h \in \mathbb{R}$ and $\delta > 0$ the operator $R_{h,\delta}$ on $L^2(\nu)$ through:

$$(R_{h,\delta} f)(a) \coloneqq \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a) \cdot (1+\delta)\mathbb{E}^Y\left[f(\tfrac{a}{d-1} + Y)\right]^{d-1} \quad \text{for } f \in L^2(\nu) \text{ and } a \in \mathbb{R}, \tag{5.5}$$

where, as usual, $Y \sim \mathcal{N}(0, \frac{d}{d-1})$. By the same observations as below (3.3) one can check that indeed $R_{h,\delta} f \in L^2(\nu)$ for $f \in L^2(\nu)$. Note also that $R_{h,\delta}$ for $\delta > 0$ can be expressed in terms of the operator R_h from (3.3) via $R_{h,\delta} = (\mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,\infty)} \cdot (1+\delta))R_h$. The role of $R_{h,\delta}$ can be seen from the following lemma.

Lemma 5.3. *Let $h \in \mathbb{R}$, $\delta > 0$ and define the (bounded) functions*

$$g^k_{h,\delta}(a) \coloneqq \mathbb{E}^{\mathbb{T}_d}_a\left[(1+\delta)^{|\mathcal{C}^h_o \cap B^+_{\mathbb{T}_d}(o,k)|}\right] \quad \text{for } a \in \mathbb{R},\ k \geq 0. \tag{5.6}$$

Then one has

$$1 \leq g^0_{h,\delta} \leq g^1_{h,\delta} \leq g^2_{h,\delta} \leq \ldots \leq g_{h,\delta} \qquad \text{and} \qquad \lim_{k\to\infty} g^k_{h,\delta} = g_{h,\delta}. \tag{5.7}$$

Moreover, for every $k \geq 0$ one has

$$g^k_{h,\delta} = R^{k+1}_{h,\delta} 1 \qquad \text{and} \qquad g^{k+1}_{h,\delta} = R_{h,\delta} g^k_{h,\delta}. \tag{5.8}$$

Proof. The first part of (5.7) is clear by definition and the second part follows by the monotone convergence theorem. Claim (5.8) can be seen via induction on $k \geq 0$. Indeed, for $k = 0$ it holds $R_{h,\delta} 1 = \mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,\infty)} \cdot (1+\delta) = g^0_{h,\delta}$ by (5.5) and (1.10). Furthermore, for $k \geq 0$ and $a \in \mathbb{R}$ one has (recall (1.1) and denote $S^+_{\mathbb{T}_d}(o,1) =: \{x_1, \ldots, x_{d-1}\}$)

$$\begin{aligned}
g^{k+1}_{h,\delta}(a) &= \mathbb{E}^{\mathbb{T}_d}_a\left[\left(\mathbf{1}_{\{\varphi_{\mathbb{T}_d}(o)<h\}} + \mathbf{1}_{\{\varphi_{\mathbb{T}_d}(o)\geq h\}}\right)(1+\delta)^{|\mathcal{C}^h_o \cap B^+_{\mathbb{T}_d}(o,k+1)|}\right] \\
&\overset{(1.10)}{=} \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a) \cdot (1+\delta)\mathbb{E}^{\mathbb{T}_d}_a\left[\prod_{i=1}^{d-1}(1+\delta)^{|\mathcal{C}^h_o \cap B^+_{\mathbb{T}_d}(o,k+1)\cap U_{x_i}|}\right] \\
&\overset{(1.11)}{=} \mathbf{1}_{(-\infty,h)}(a) + \mathbf{1}_{[h,\infty)}(a) \cdot (1+\delta)\mathbb{E}^Y\left[\mathbb{E}^{\mathbb{T}_d}_{\frac{a}{d-1}+Y}\left[(1+\delta)^{|\mathcal{C}^{\mathbb{T}_d,h}_o \cap B^+_{\mathbb{T}_d}(o,k)|}\right]\right]^{d-1} \\
&\underset{(5.5)}{\overset{(5.6)}{=}} (R_{h,\delta}\, g^k_{h,\delta})(a) \overset{(*)}{=} (R^{k+2}_{h,\delta} 1)(a),
\end{aligned}$$

where in $(*)$ we use the induction hypothesis. This shows the first half of (5.8), which implies the second half. ☐

For the next lemma we define for $h \in \mathbb{R}$ and $r > 0$

$$B_{h,r} \coloneqq \{f \in L^2(\nu) \,|\, f \geq 1,\, f \text{ equals } 1 \text{ on } (-\infty, h) \text{ and } \|f - 1\|_{L^2(\nu)} \leq r\}.$$

Since $B_{h,r}$ is a closed subset of $L^2(\nu)$, it is a complete metric space.

Lemma 5.4. *Let $h > h_\star$. Then there exists $\delta_h > 0$ and $r_h > 0$ such that R_{h,δ_h} is a (strict) contraction on the complete metric space B_{h,r_h}. In particular, by the Banach-Caccioppoli fixed-point theorem there exists a unique $f^\star \in B_{h,r_h}$ with $R_{h,\delta_h} f^\star = f^\star$ and for all $f \in B_{h,r_h}$ one has $\|R^k_{h,\delta_h} f - f^\star\|_{L^2(\nu)} \to 0$ as $k \to \infty$.*

Proof. Let $\delta > 0$ and consider $f, g \in L^2(\nu)$. By the relationship between $R_{h,\delta}$ and R_h explained below (5.5) one has

$$\begin{aligned} \|R_{h,\delta} g - R_{h,\delta} f\|_{L^2(\nu)} &= \|(\mathbf{1}_{(-\infty,h)} + \mathbf{1}_{[h,\infty)} \cdot (1+\delta))(R_h g - R_h f)\|_{L^2(\nu)} \\ &\leq (1+\delta) \cdot \|R_h g - R_h f\|_{L^2(\nu)} = (1+\delta) \cdot \|R_h(f + g - f) - R_h f\|_{L^2(\nu)} \\ &\overset{(3.14)}{\leq} (1+\delta)\Big(\|A^f_h(g-f)\|_{L^2(\nu)} + \|E^f_h(g-f)\|_{L^2(\nu)}\Big). \end{aligned} \tag{5.9}$$

Since $h > h_\star$ (and thus $\lambda_h < 1$ by Proposition 1.1), we can choose $\varepsilon_h > 0$ such that $\lambda_h + 2\varepsilon_h < 1$. Now on the one hand, by Proposition 3.6 there exists $s_h > 0$ such that $\|A^f_h(g-f)\|_{L^2(\nu)} \leq (\lambda_h + \varepsilon_h)\,\|g - f\|_{L^2(\nu)}$ for $f, g \in B_{h,s_h}$, because then $f - g$ vanishes on $(-\infty, h)$ and $\|f - 1\|_{L^2(\nu)} \leq s_h$. On the other hand, by (3.16) there exists $s'_h > 0$ such that $\|E^f_h(g-f)\|_{L^2(\nu)} \leq \varepsilon_h \|g - f\|_{L^2(\nu)}$ if $\|g - f\|_{L^2(\nu)} \leq s'_h$. Hence if $f, g \in B_{h,r_h}$ with $r_h \coloneqq \frac{1}{2}\min\{s_h, s'_h\}$, then both conditions are simultaneously satisfied and one has

$$\|R_{h,\delta} g - R_{h,\delta} f\|_{L^2(\nu)} \overset{(5.9)}{\leq} (1+\delta)(\lambda_h + 2\varepsilon_h)\,\|g - f\|_{L^2(\nu)}. \tag{5.10}$$

Moreover, since $R_{h,\delta} 1 = 1 + \mathbf{1}_{[h,\infty)}\delta$ by (5.5), one also has for $f \in B_{h,r_h}$ that

$$\begin{aligned} \|R_{h,\delta} f - 1\|_{L^2(\nu)} &\leq \|R_{h,\delta} f - R_{h,\delta} 1\|_{L^2(\nu)} + \|R_{h,\delta} 1 - 1\|_{L^2(\nu)} \\ &\overset{(5.10)}{\leq} (1+\delta)(\lambda_h + 2\varepsilon_h) r_h + \delta. \end{aligned} \tag{5.11}$$

Due to $\lambda_h + 2\varepsilon_h < 1$, we can choose $\delta = \delta_h > 0$ such that $(1+\delta_h)(\lambda_h + 2\varepsilon_h) r_h + \delta_h \leq r_h$. This also implies $(1+\delta_h)(\lambda_h + 2\varepsilon_h) \eqqcolon \Delta_h < 1$. Then R_{h,δ_h} maps the space B_{h,r_h} to itself. Indeed, for $f \in B_{h,r_h}$ one has $R_{h,\delta_h} f \geq 1$ and $R_{h,\delta_h} f = 1$ on $(-\infty, h)$ by definition of R_{h,δ_h}, and furthermore $\|R_{h,\delta_h} f - 1\|_{L^2(\nu)} \leq r_h$ by (5.11). Finally, (5.10) shows $\|R_{h,\delta_h} g - R_{h,\delta_h} f\|_{L^2(\nu)} \leq \Delta_h \|g - f\|_{L^2(\nu)}$ for $f, g \in B_{h,r_h}$, i.e. that R_{h,δ_h} is a strict contraction. □

With Lemma 5.3 and Lemma 5.4 at hand, we can readily show the first half of Theorem 5.1.

Corollary 5.5. *Let $h > h_\star$. Then there exists $\delta_h > 0$ such that $g_{h,\delta_h} \in L^2(\nu)$. Moreover, g_{h,δ_h} equals 1 on $(-\infty, h)$, satisfies $R_{h,\delta_h} g_{h,\delta_h} = g_{h,\delta_h}$ and is continuous. Finally, $g_{h,\delta_h}(a)$ is finite for all $a \in \mathbb{R}$.*

Proof. Consider $\delta_h > 0$, $r_h > 0$ and $f^\star \in L^2(\nu)$ from Lemma 5.4. We start by showing that ν-almost everywhere $f^\star = g_{h,\delta_h}$ and hence $g_{h,\delta_h} \in L^2(\nu)$. Note that by Lemma 5.4 one has $1 \le f^\star$ and thus by (5.8) also $g^k_{h,\delta_h} = R^{k+1}_{h,\delta_h} 1 \le R^{k+1}_{h,\delta_h} f^\star = f^\star$ for all $k \ge 0$. By (5.7) and the monotone convergence theorem this shows

$$\| \lim_{k\to\infty} g^k_{h,\delta_h} - f^\star \|_{L^2(\nu)} = \lim_{k\to\infty} \| g^k_{h,\delta_h} - f^\star \|_{L^2(\nu)} \overset{(5.8)}{=} \lim_{k\to\infty} \| R^k_{h,\delta_h} 1 - f^\star \|_{L^2(\nu)}. \tag{5.12}$$

Now since $1 \in B_{h,r_h}$, we know from Lemma 5.4 that $\lim_{k\to\infty} \| R^k_{h,\delta_h} 1 - f^\star \|_{L^2(\nu)} = 0$. Hence $\| \lim_{k\to\infty} g^k_{h,\delta_h} - f^\star \|_{L^2(\nu)} = 0$ by (5.12) and so ν-almost everywhere $f^\star = g_{h,\delta_h}$ by (5.7). By (5.1) and (1.10) it is obvious that g_{h,δ_h} equals 1 on $(-\infty, h)$. Now if we take k to infinity on both sides of the equation $g^{k+1}_{h,\delta_h} = R_{\delta_h,h} g^k_{h,\delta_h}$ from (5.8), we obtain

$$g_{h,\delta_h} = R_{h,\delta_h} g_{h,\delta_h} \tag{5.13}$$

by (5.7) and the monotone convergence theorem. The right hand side of (5.13) satisfies $(R_{h,\delta_h} g_{h,\delta_h})(a) < \infty$ for all $a \in \mathbb{R}$ by (5.5) and (5.3). Hence $g_{h,\delta_h}(a) < \infty$ for all $a \in \mathbb{R}$. With (5.13) established, we can show the continuity of g_{h,δ_h} on $[h, \infty)$ in the same way as the continuity of f in Lemma 3.3. Hence the proof of Corollary 5.5 is complete. □

It remains to prove (5.4). In the next lemma we show a weaker statement by applying results from Corollary 5.5.

Lemma 5.6. *Let $h > h_\star$ and consider the function $g_{h,\delta_h} \in L^2(\nu)$ from Corollary 5.5. For all $\zeta > 0$ there exists $c_{h,\zeta} > 0$ such that*

$$g_{h,\delta_h}(a) \le c_{h,\zeta} \exp(\zeta a^2) \quad \textit{for } a \ge h. \tag{5.14}$$

Proof. We will first show that $g_{h,\delta_h} \in L^q(\nu)$ for all $q \ge 1$, which will then imply (5.14). Let us define

$$\begin{aligned} &p_0 := 2 \quad \text{and} \quad p_{i+1} := (p_i - 1)(d-1) + \tfrac{1}{d-1} \quad \text{for } i \ge 0, \\ &q_i := (p_i - 1)(d-1)^2 + 1 \quad \text{for } i \ge 0. \end{aligned} \tag{5.15}$$

We prove by induction that $g_{h,\delta_h} \in L^{p_i}(\nu)$ for all $i \ge 0$ by using the hypercontractivity estimate (1.14). For $i = 0$ we have $p_0 = 2$ and hence $g_{h,\delta_h} \in L^{p_0}(\nu)$ as seen in Corollary 5.5. Now assume $g_{h,\delta_h} \in L^{p_i}(\nu)$ for $i \ge 0$. Since $g_{h,\delta_h} = R_{h,\delta_h} g_{h,\delta_h}$ by Corollary 5.5, it follows that to prove $g_{h,\delta_h} \in L^{p_{i+1}}(\nu)$ it is enough to show $\hat{g}^{d-1}_{h,\delta_h} \in L^{p_{i+1}}(\nu)$, where we abbreviated $\hat{g}_{h,\delta_h}(a) := \mathbb{E}^Y\big[g_{h,\delta_h}(\frac{a}{d-1} + Y)\big]$. And indeed we have, using (5.15), (1.14) and the induction hypothesis, that $\|\hat{g}^{d-1}_{h,\delta_h}\|_{L^{p_{i+1}}(\nu)} = \|\hat{g}_{h,\delta_h}\|^{d-1}_{L^{q_i}(\nu)} \le \|g_{h,\delta_h}\|^{d-1}_{L^{p_i}(\nu)} < \infty$. Next we show that the sequence $(p_i)_{i\ge0}$ diverges to infinity as i tends to infinity. To see this, note that $r_i := \frac{d-2}{d-1}(d-1)^i + \frac{1}{d-1} + 1$, $i \ge 0$, solves the recursion for $(p_i)_{i\ge0}$ given in (5.15) and clearly $r_i \xrightarrow{i\to\infty} \infty$. This implies that $g_{h,\delta_h} \in L^q(\nu)$ for all $q \ge 1$ since we can take $i \ge 0$ such that $q < p_i$. Then $g_{h,\delta_h} \in L^{p_i}(\nu) \subseteq L^q(\nu)$.

We turn to show (5.14). Let $\zeta > 0$ and take $q \ge 1$ large enough such that $2q \ge \frac{1}{\zeta}$. Since $g_{h,\delta_h} \in L^q(\nu)$ as just shown and $\nu = \mathcal{N}(0, \frac{d-1}{d-2})$ from above (1.12), one has $\lim_{a\to\infty} g_{h,\delta_h}(a)^q \exp(-\frac{(d-2)a^2}{(d-1)2}) = 0$, which implies $\lim_{a\to\infty} g_{h,\delta_h}(a) \exp(-\frac{a^2}{2q}) = 0$. But this shows the statement of the lemma by the choice of q (use that g_{h,δ_h} is continuous on $[h, \infty)$ by Corollary 5.5). □

The estimate obtained in the previous lemma can be used to derive the following recursive bound on g_{h,δ_h} which is the final ingredient for the proof of (5.4)

Lemma 5.7. *Let $h > h_\star$ and consider the function $g_{h,\delta_h} \in L^2(\nu)$ from Corollary 5.5. For all $\eta > 0$ there exists $c_{h,\eta} > 0$ such that*

$$g_{h,\delta_h}(a) \le (1+2\delta_h) g_{h,\delta_h}\big(\tfrac{a}{d-1}(1+\eta)\big)^{d-1} \qquad \text{for all } a \ge c_{h,\eta}. \tag{5.16}$$

Proof. Let $\eta > 0$. Because $g_{h,\delta_h} = R_{h,\delta_h}\, g_{h,\delta_h}$ as obtained in (5.13), one has for $a \ge h$ and with $Y \sim \mathcal{N}(0, \frac{d}{d-1})$ that

$$\begin{aligned} g_{h,\delta_h}(a) &= (1+\delta_h)\Big(\mathbb{E}^Y\Big[g_{h,\delta_h}(\tfrac{a}{d-1}+Y)\mathbf{1}_{\{Y<\frac{\eta a}{d-1}\}}\Big] + \mathbb{E}^Y\Big[g_{h,\delta_h}(\tfrac{a}{d-1}+Y)\mathbf{1}_{\{Y\ge\frac{\eta a}{d-1}\}}\Big]\Big)^{d-1} \\ &\le (1+\delta_h)\Big(g_{h,\delta_h}\big(\tfrac{a}{d-1}(1+\eta)\big) + \mathbb{E}^Y\Big[g_{h,\delta_h}(\tfrac{a}{d-1}+Y)\mathbf{1}_{\{Y\ge\frac{\eta a}{d-1}\}}\Big]\Big)^{d-1}, \end{aligned} \tag{5.17}$$

where in the last step we use that g_{h,δ_h} is a non-decreasing function (see (5.1) and (1.10)). Because $g_{h,\delta_h} \ge 1$, we further obtain from (5.17) that for $a \ge h$

$$g_{h,\delta_h}(a) \le (1+\delta_h)\Big(1+\mathbb{E}^Y\Big[g_{h,\delta_h}(\tfrac{a}{d-1}+Y)\mathbf{1}_{\{Y\ge\eta\frac{a}{d-1}\}}\Big]\Big)^{d-1} g_{h,\delta_h}\big(\tfrac{a}{d-1}(1+\eta)\big)^{d-1}. \tag{5.18}$$

To bound the expectation on the right hand side of (5.18) we will apply (5.14) for some $\zeta > 0$ depending on η. Choose $0 < \zeta < \frac{d-1}{2d}\frac{\eta^2}{(1+\eta)^2}$, so that in particular $\zeta < \frac{d-1}{2d}$ and $\zeta(1+\eta)^2 - \frac{d-1}{2d}\eta^2 < 0$. Then $z \mapsto \zeta(1+z)^2 - \frac{d-1}{2d}z^2 = (\zeta - \frac{d-1}{2d})z^2 + 2\zeta z + \zeta$ is a parabola with negative leading coefficient and two zeros of opposite sign. As the parabola is negative at $z = \eta > 0$, this shows that $\zeta(1+z)^2 - \frac{d-1}{2d}z^2 < 0$ for all $z \ge \eta$. Hence there exists $c_{\zeta,\eta} = c'_\eta > 0$ such that

$$\zeta(1+z)^2 - \tfrac{d-1}{2d}z^2 < -c'_\eta z \quad \text{for all } z \ge \eta. \tag{5.19}$$

Since $Z := \frac{d-1}{a}Y$ satisfies $Z \sim \mathcal{N}(0,\sigma_a^2)$ for $\sigma_a^2 := \frac{(d-1)^2}{a^2}\frac{d}{d-1}$, we get the following bound for $a \ge (d-1)h$

$$\begin{aligned} \mathbb{E}^Y\Big[g_{h,\delta_h}(\tfrac{a}{d-1}+Y)\mathbf{1}_{\{Y\ge\eta\frac{a}{d-1}\}}\Big] &= \frac{1}{\sqrt{2\pi\sigma_a^2}}\int_\eta^\infty g_{h,\delta_h}\big(\tfrac{a}{d-1}(1+z)\big)\exp\big(-\tfrac{z^2}{2\sigma_a^2}\big)\,dz \\ &\overset{(5.14)}{\le} \frac{c_{h,\eta}}{\sqrt{2\pi\sigma_a^2}}\int_\eta^\infty \exp\Big((\tfrac{a}{d-1})^2\big(\zeta(1+z)^2 - \tfrac{d-1}{2d}z^2\big)\Big)\,dz \\ &\overset{(5.19)}{\le} \frac{c_{h,\eta}}{\sqrt{2\pi\sigma_a^2}}\int_\eta^\infty \exp\big(-(\tfrac{a}{d-1})^2 c'_\eta z\big)\,dz = \frac{c_{h,\eta}}{\sqrt{2\pi\sigma_a^2}}\frac{(d-1)^2}{a^2 c'_\eta}\exp\big(-(\tfrac{a}{d-1})^2 c'_\eta \eta\big), \end{aligned}$$

which tends to zero as a tends to infinity. Hence there is $c_{h,\eta} > 0$ such that $(1+\delta_h)\big(1+\mathbb{E}^Y\big[g_{h,\delta_h}(\frac{a}{d-1}+Y)\mathbf{1}_{\{Y\ge\eta\frac{a}{d-1}\}}\big]\big)^{d-1} \le (1+2\delta_h)$ for all $a \ge c_{h,\eta}$. This, together with (5.18), concludes the proof of Lemma 5.7. □

The next and final lemma shows (5.4) and hence concludes the proof of Theorem 5.1.

Lemma 5.8. *Let $h > h_\star$ and consider the function $g_{h,\delta_h} \in L^2(\nu)$ from Corollary 5.5. For all $\gamma > 0$ there exist $c_{h,\gamma} > 0$ and $c'_{h,\gamma} > 0$ such that $g_{h,\delta_h}(a) \le c_{h,\gamma}\exp(c'_{h,\gamma}a^{1+\gamma})$ for all $a \ge h$.*

Proof. Let $\gamma > 0$ and take $\eta > 0$ such that $1 + \gamma = \log_{\frac{d-1}{1+\eta}}(d-1)$, in particular $\frac{d-1}{1+\eta} > 1$. We abbreviate $K \coloneqq c_{h,\eta}$ for the constant from (5.16). Since g_{h,δ_h} is continuous on $[h, \infty)$ by Corollary 5.5, it is enough to find the requested bound on g_{h,δ_h} for all $a \geq K$. Define the intervals

$$J_k \coloneqq \Big[K\big(\tfrac{d-1}{1+\eta}\big)^k, K\big(\tfrac{d-1}{1+\eta}\big)^{k+1}\Big) \quad \text{for all } k \geq 0,$$

which form a disjoint decomposition of $[K, \infty)$. For $a \geq K$ let $k(a) \geq 0$ be the unique $k \geq 0$ with $a \in J_k$, that is, $k(a) \coloneqq \lfloor \log_{\frac{d-1}{1+\eta}}(\frac{a}{K}) \rfloor$. For such a one can apply (5.16) iteratively $k(a)$ times to obtain

$$\begin{aligned} g_{h,\delta_h}(a) &\leq (1+2\delta_h) g_{h,\delta_h}\big(\tfrac{a}{d-1}(1+\eta)\big)^{d-1} \\ &\leq (1+2\delta_h)^{1+(d-1)} g_{h,\delta_h}\Big(\tfrac{a}{(d-1)^2}(1+\eta)^2\Big)^{(d-1)^2} \\ &\leq \ldots \leq (1+2\delta_h)^{\sum_{i=0}^{k(a)-1}(d-1)^i} g_{h,\delta_h}\Big(\underbrace{\tfrac{a}{(d-1)^{k(a)}}(1+\eta)^{k(a)}}_{\in J_0}\Big)^{(d-1)^{k(a)}} \\ &\leq \Big((1+2\delta_h) \sup_{b \in J_0} g_{h,\delta_h}(b)\Big)^{(d-1)^{k(a)}}. \end{aligned} \tag{5.20}$$

Note that $(d-1)^{k(a)} \leq (d-1)^{\log_{\frac{d-1}{1+\eta}}\left(\frac{a}{K}\right)} = \big(\frac{a}{K}\big)^{\log_{\frac{d-1}{1+\eta}}(d-1)} = \big(\frac{a}{K}\big)^{1+\gamma}$ and therefore (5.20) implies that

$$g_{h,\delta_h}(a) \leq \exp\Big(\big(\tfrac{a}{K}\big)^{1+\gamma} \ln\Big((1+2\delta_h) \sup_{b \in J_0} g_{h,\delta_h}(b)\Big)\Big) \leq c_{h,\gamma} \exp(c'_{h,\gamma} a^{1+\gamma}) \quad \text{for } a \geq K.$$

As explained above, this proves the lemma. □

We end with some concluding remarks. One might naturally wonder what can be said about level-set percolation of the Gaussian free field on $\mathbb{T}_d$ near criticality. For example: can the result from Theorem 4.1 be extended to $h_\star$, i.e. are the functions η and η^+ continuous or not at $h_\star$? Or also: does the equality (4.5) hold for $h = h_\star$, too?

Independently from that, and as remarked in the introduction, we apply a number of the results obtained here in the accompanying paper [AČ19] to establish a phase transition for level-set percolation of the zero-average Gaussian free field on a class of finite regular expanders.

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