Kinetic Models for Magnetized Plasmas

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Outline

Introducing the magnetized Vlasov–Poisson system

The Bernstein–Landau paradox

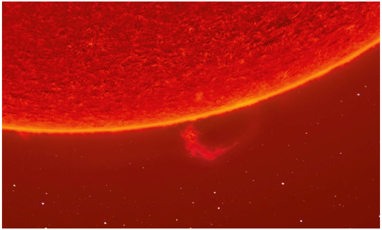
Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system

Plasmas

Definition (Plasma)

Plasma is a state of matter, characterized by an important ionization of particles.

- Term "plasma" first used by Irving Langmuir [1] to describe an ionized gas.
- Plasma represents more than 99% of ordinary matter in the universe.



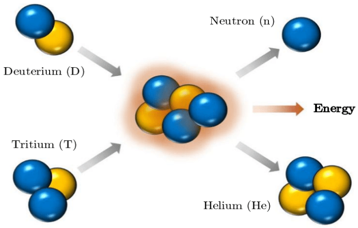
[1] I. Langmuir, Oscillations in Ionized Gases, Proceedings of the National Academy of Science, 14:627–637, 1928.

Nuclear fusion

Definition (Nuclear fusion)

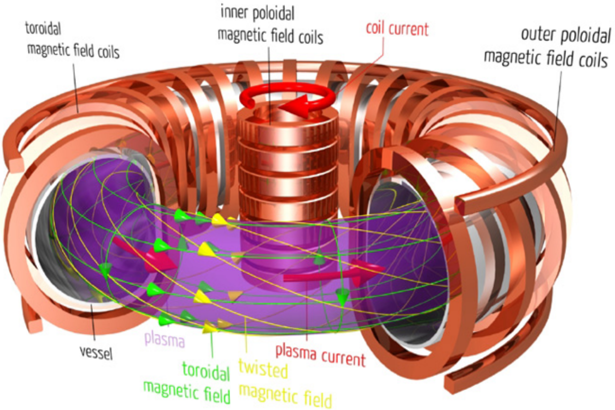
Combination of two atomic nuclei to form one atomic nuclei.

- Fusion of light nuclei releases enormous amounts of energy.
- This reaction takes place in plasmas under certain conditions: very high temperature and high enough particle density.
- One approach to harness nuclear fusion of light nuclei is via magnetic confinement.



Magnetic confinement fusion

- Use an intense external (not self-induced) magnetic field B to confine the hot plasma.
- Feasibility of controlled nuclear fusion: ITER tokamak under construction in Cadarache (south of France).



Kinetic formalism and the Vlasov equation

Trajectory $(X(t), V(t))$ of one particle of mass m subject to a force $F(t)$ given by:

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = \frac{1}{m}F(t). \end{cases} \tag{1}$$

Large number of identical particles allows for a kinetic description of the system. $f(t, x, v)$ is the number density of particles which are located at the position x and have velocity v at time t and satisfies:

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \frac{1}{m}F(t) \cdot \nabla_v f(t,x,v) = 0 \tag{2}$$

with $(t, x, v) \in \mathbb{R}^+ \times X^3 \times \mathbb{R}^3$, $X = \mathbb{R}$ or $X = \mathbb{T}$.

The magnetized Vlasov–Poisson system for electrons

- At the time scale of electrons: ions are static $\Longrightarrow f_{ion}$ is constant.
- At the time scale of ions: electrons are at thermal equilibrium $\Longrightarrow f_{electron} =$ Maxwell–Boltzmann type distribution.

At the time scale of electrons, we have:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q_e}{m_e}(E + v \wedge B) \cdot \nabla_v f = 0, \\ \mathrm{div}_x E(t,x) = \frac{q_{ion}}{\epsilon_0} \int_{\mathbb{R}^3} f_{ion}(x,v) dxdv + \frac{q_e}{\epsilon_0} \int_{\mathbb{R}^3} f(t,x,v) dxdv, \end{cases} \tag{3}$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field and $B \equiv B(t,x)$ the **given** external magnetic field.

Outline

Landau damping

Exponential decrease in time of longitudinal waves in plasma.

- First predicted by Landau with the linearized Vlasov-Poisson system where he showed damping of the electric field E.[2]

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q}{m} E \cdot \nabla_v f_0 = 0, \\ \partial_x E = \frac{q}{\epsilon_0} \int f dv_1 dv_2, \end{cases} \tag{4}$$

with $f = f(t,x,v)$, $(t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$ the distribution of electrons and $f_0 = e^{-\frac{|v|^2}{2}}$ the equilibrium Maxwellian distribution.

- Using the Fourier–Laplace transform, Landau showed the damping of the electric field

$$|E_k(t)| \leq Ce^{-\alpha_k t}. \tag{5}$$

- Irreversible behavior observed in a reversible in time system.
- Recently extended to the nonlinear setting by Mouhot and Villani [3].

[2] L. Landau, On the vibration of the electronic plasma, J. Phys. USSR, 1946.
[3] C. Mouhot and C. Villani, On Landau damping, Acta Math., 2011.

The paradox

The Bernstein-Landau paradox

"In unmagnetized plasmas, waves exhibit Landau Damping, while in magnetized plasmas, waves perpendicular to the magnetic field are exactly undamped, independently of the strength of the magnetic field".

- Magnetized plasmas first studied by Bernstein [4].
- Other physical works [5] have highlighted a discontinuity between the theory of unmagnetized plasmas and the theory of magnetized plasmas, thus speaking of a paradox.
- Very few mathematical papers, recently studied in [6] using Fourier–Laplace in a 3d-3v setting.

[4] I. Bernstein, Waves in a Plasma in a Magnetic Field, Phy. Review, 1958.

[5] A. I. Sukhorukov and P. Stubbe, On the Bernstein-Landau paradox, Phy. of Plasmas, 1997.

[6] J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, J. Stat. Phys., 2020.

Numerical illustration of the influence of B: magnetic recurrence

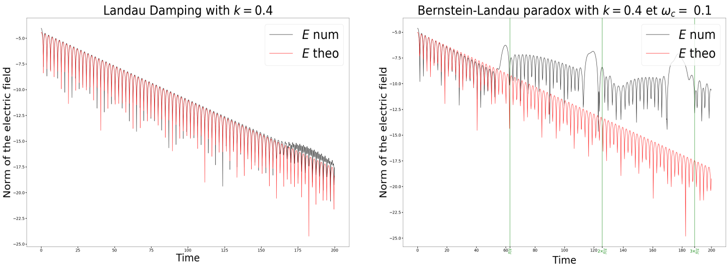


Figure: Damped and undamped electric field

This is a magnetic recurrence [7].

[7] Not a numerical recurrence *Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh*, M. Mehrenberger, L. Navoret, N. Pham, Commun. Comput. Phys., 2020.

The 1d-2v framework

The 1d-2v magnetized Vlasov-Poisson system

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \underbrace{\omega \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) f}_{=-v \wedge B} = 0, \\ \partial_x E = 2\pi - \int f dv_1 dv_2. \end{cases} \quad (6)$$

- The unknowns are the density of electrons $f(t, x, v_1, v_2) \geq 0$ and the electric field $E(t, x)$.
- The domain is $\Omega = \mathbb{T} \times \mathbb{R}^2, \quad \mathbb{T} = [0\,, 2\pi]_{\mathrm{per}}$ is the 1D-torus.
- Here $\omega > 0$ is the constant cyclotron frequency for electrons $(B = (0, 0, \omega))$.
- $\frac{q_e}{m_e}$ normalized to -1.
- Ions are considered as a motionless background of neutralizing positive charge.

Linearized system

We linearize (6) by writing

$$f = f_0 + \varepsilon\sqrt{f_0}u + O(\varepsilon^2) \text{ and } E = \varepsilon F + O(\varepsilon^2).$$

where $(f_0, E_0) = (\exp(-\frac{v_1^2+v_2^2}{2}), 0)$ is a stationary solution of (6).

Linearized Vlasov–Poisson with magnetic field

$$\begin{cases} \partial_t u + v_1\partial_x u + Fv_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right)u = 0, \\ \partial_x F = -\int u e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2. \end{cases} \tag{7}$$

- $\int u e^{-\frac{v_1^2+v_2^2}{4}} dx dv_1 dv_2 = 0$. (total mass equal zero)
- $\int F dx = 0$. (F is derived from a potential)

Reformulation with the Ampère equation

We can rewrite the system with the Ampère equation because both systems are equivalent.

The linear 1d-2v Vlasov–Ampère system

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega(-v_2 \partial_{v_1} + v_1 \partial_{v_2}) u = 0, \\ \boldsymbol{\partial_t F = 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2}. \end{cases} \tag{8}$$

$$\text{with } 1^* g(x) = g(x) - \frac{1}{2\pi} \int_{\mathbb{T}} g(x) dx.$$

A self-adjoint Vlasov–Ampère operator

Final formulation

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = iH \begin{pmatrix} u \\ F \end{pmatrix}, H = i \left(\begin{array}{c|c} \underbrace{v_1\partial_x + \omega(v_2\partial_{v_1} - v_1\partial_{v_2})}_{:=H_0} & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right).$$

$$\mathcal{H} = \underbrace{(L^2(\mathbb{T}\times\mathbb{R}^2) \cap \left\{ \int u\sqrt{f_0}\, dx dv_1 dv_2 = 0 \right\})}_{=L_0^2(\mathbb{T}\times\mathbb{R}^2)} \times \underbrace{(L^2(\mathbb{T}) \cap \left\{ \int F dx = 0 \right\})}_{=L_0^2(\mathbb{T})}$$

and H a **self-adjoint** operator with domain $D(H) := D(H_0) \times L_0^2(\mathbb{T})$.

Spectral properties of Vlasov equations

- For a general self-adjoint operator H with an ambient Hilbert space $\mathcal{H}$: then we have the following decomposition of $\mathcal{H}$ in terms of the spectrum of H: $\mathcal{H} = \mathcal{H}_H^{ac} \oplus \mathcal{H}_H^{sc} \oplus \mathcal{H}_H^{pp}$. [8] [9]
- Spectrum of the Vlasov–Ampère operator with $\omega = 0$ studied in [10] [11]: the operator has only absolutely continuous spectrum and a kernel (as expected) $\mathcal{H} = \ker H \oplus \mathcal{H}_H^{ac}$.

[8] T. Kato, Perturbation theory for linear operators, 1966.

[9] D.R. Yafaev, Scattering theory: Some old and new problems, 2000.

[10] B. Després, Symmetrization of Vlasov–Poisson Equations, SIAM J. Math. Anal., 2014.

[11] B. Després, Trace class properties of the linear Vlasov-Poisson equation, J. of Spectral Theory, 2021.

Discrete spectrum for the Vlasov–Ampère operator

Theorem (Charles, Després, R., Weder [12])

We have completeness of the eigenspaces:

$$\mathcal{H} = \mathcal{H}_H^{pp},$$

and the eigenvalues of H are 0, $m\omega$ and λ_m, $m \neq 0$.

Two different proofs:

- Direct computations.
- Weyl theorem on the invariance of the essential spectrum.

Explanation for the Bernstein–Landau paradox

The magnetic term $(v \wedge B) \cdot \nabla_v f$ can be seen as a perturbation that modifies the domain of the Vlasov–Ampère operator.

[12] F. Charles, B. Després, A. Rege and R. Weder, The magnetized Vlasov–Ampère system and the Bernstein–Landau paradox, J. Stat. Phys., 2021.

Numerical illustration: initialization

Objective: compare the numerical and theoretical solutions of Vlasov-Ampère when initializing with an eigenvector.

- We consider an eigenvector $(w_{n,m}\,, F_n)$ associated to the Fourier mode $n \neq 0$ and the eigenvalue λ_m, so the solution $(u\,, F)$ is given by

$$(u\,, F)(t) = e^{i\lambda_m t}\,(w_{n,m}\,, F_n)\,.$$

- $w_{n,m}$ and F_n are given by

$$w_{n,m} = e^{in(x - \frac{v_2}{\omega})} e^{-\frac{r^2}{4}} \sum_{p\in\mathbb{Z}^*} \frac{p\omega}{p\omega + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega}\right) \text{ and } F_n = -ine^{inx}.$$

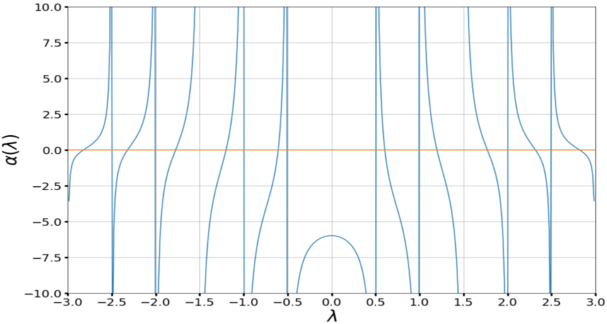
- λ_m is one of the roots of a secular equation given by:

$$\alpha(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m\in\mathbb{Z}^*} \frac{m\omega}{m\omega + \lambda} \int_0^\infty e^{-\frac{r^2}{2}} J_m\left(\frac{nr}{\omega}\right)^2 r dr = 0.$$

Numerical illustration: secular equation

α has a unique root in

- $]m\omega\,,(m+1)\omega[$ for $m \geq 1$,
- $](m-1)\omega\,,m\omega[$ for $m \leq -1$.



For $(n, m) = (1, 2)$, we compute $\lambda_2 \approx 1.19928$.

Numerical illustration: results

Simulations with a classical "backward" semi-lagrangian scheme, $N_x = 33$, $N_{v_1} = N_{v_2} = 63$, $L_x = 2\pi$, $L_{v_1} = L_{v_2} = 10$, $\omega = 0.5$, $n = 1$, and $\mathbf{T_f} = \frac{\pi}{2\lambda_m}$.

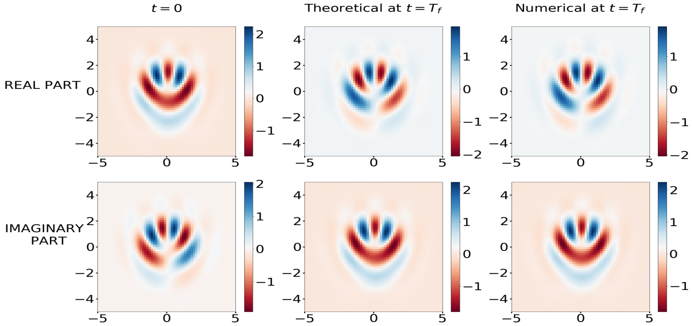


Figure: Real and imaginary parts of u in V1-V2 plane for $x = 0$.

Outline

Back to the nonlinear system in $\mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ E(t,x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|} f(t,y,v) dv dy. \end{cases} \qquad \text{(VPwB)}$$

Energy of the system and macroscopic density ρ

$$\mathcal{E}(t) := \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t,x,v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |E(t,x)|^2 dx,$$

$$\rho(t,x) := \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Results on existence of solutions in the unmagnetized case:

- Existence of weak solutions [Arsenev, 1975]
- Small initial data [Bardos, Degond, 1985]
- Existence of smooth solutions [Pfaffelmoser, 1992]
- Propagation of velocity moments [Lions, Perthame, 1991]

We will first consider a constant B

$$B = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}.$$

Differential inequality on M_k

Let $k \geq 0$, the velocity moment of order k is defined by

$$M_k(t) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dv dx.$$

Propagation of velocity moments:

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x,v) dv dx < \infty \Longrightarrow \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dv dx < \infty$$

$$\begin{aligned} |\frac{d}{dt} M_k(t)| &= |\iint |v|^k (-v \cdot \nabla_x f - (E + v \wedge B) \cdot \nabla_v f) dv dx|, \\ &= |\iint |v|^k \operatorname{div_v} \left((E + v \wedge B) f \right) dv dx|, \\ &= |\iint k |v|^{k-2} v \cdot (E + v \wedge B) f dv dx|, \\ &\leq C \left\| E(t) \right\|_{k+3} M_k(t)^{\frac{k+2}{k+3}}. \end{aligned}$$

Next step: we need to control of $\|E(t)\|_{k+3}$ with $M_k(t)^\alpha$ with $\alpha \leq \frac{1}{k+3}$.

A representation formula for ρ

$$\partial_t f + v\cdot\nabla_x f + (v\wedge B)\cdot\nabla_v f = -E\cdot\nabla_v f$$

$$\begin{cases} \dfrac{d}{ds}(X(s),V(s)) = (V(s), V(s)\wedge B) = (V(s),(\omega V_2(s), -\omega V_1(s), 0)), \\ (X(t),V(t)) = (x,v), \end{cases}$$

$$\begin{cases} V(s) = \begin{pmatrix} v_1\cos(\omega(s-t)) + v_2\sin(\omega(s-t)) \\ -v_1\sin(\omega(s-t)) + v_2\cos(\omega(s-t)) \\ v_3 \end{pmatrix}, \\ X(s) = \begin{pmatrix} x_1 + \frac{v_1}{\omega}\sin(\omega(s-t)) + \frac{v_2}{\omega}(1-\cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t)) - 1) + \frac{v_2}{\omega}\sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix}, \end{cases} \tag{9}$$

$$\rho(t,x) = \underbrace{\int_v f^{in}(X(0),V(0))dv}_{=:\rho_0(t,x)} + \mathrm{div}_x \int_0^t \underbrace{\int_v (fH_t)(s,X(s),V(s))\,dv}_{=:\sigma(s,t,x)} ds.$$

Singularities at multiples of the cyclotron period

$$E(t,x) = -\left(\nabla \frac{1}{|\cdot|} \star \rho\right)(t,x) = E^0(t,x) + \tilde{E}(t,x),$$

$$\|E(t)\|_{k+3} \leq \left\|E^0(t)\right\|_{k+3} + \int_0^t \|\sigma(s,t,x)\|_{k+3}\, ds,$$

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C\frac{\sqrt{2}}{s}\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}. \qquad (10)$$

Proposition (Propagation of moments on a finite interval)

For all $0 \leq t \leq T_\omega := \frac{\pi}{\omega}$ *we have*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)\,dxdv \leq C < +\infty,$$

with $C = C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

Propagation of moments for all time

We have that

- $\left\|f^{in}\right\|_1 = \|f(T_\omega)\|_1$ and $\left\|f^{in}\right\|_\infty = \|f(T_\omega)\|_\infty$,
- $\mathcal{E}(T_\omega) \leq \mathcal{E}_{in}$,
- $M_k(f(T_\omega)) \leq C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

This means $f(T_\omega)$ verifies the same assumptions as $f^{in} \Longrightarrow$ we can show propagation of moments for all time by induction.

Additional results

- Propagation of the regularity of f^{in}.
- Uniqueness if $\rho \in L^\infty([0, T] \times \mathbb{R}^3)$ [13].

[13] G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density, 2006.

Propagation of velocity moments for (VPwB)

Theorem (R. [14])

Let $k_0 > 3$, $T > 0$, $f^{in} = f^{in}(x,v) \geq 0$ *a.e. with* $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f^{in} dxdv < \infty. \tag{11}$$

Then there exists $C > 0$ *and a weak solution* f *to* (VPwB) *with* $B = (0,0,\omega)$ *such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f(t,x,v) dxdv \leq C < +\infty, \quad 0 \leq t \leq T \tag{12}$$

with C *that depends on*

$$T, k_0, \omega, \mathcal{E}_{in}, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}). \tag{13}$$

[14] A. Rege, The Vlasov–Poisson system with a uniform magnetic field: propagation of moments and regularity, SIAM J. Math. Anal., 2021.

Uniqueness in the unmagnetized case

Theorem (Miot [15])

Let $T > 0$. Then there exists at most one solution $f \in L^\infty([0\,,T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to Vlasov–Poisson such that

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{14}$$

New uniqueness criterion which allows for solutions with unbounded ρ.

[15] E. Miot., A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system, Comm. Math. Phys., 2016.

Characteristics

Now we have $B := B(t,x)$ such that

$$B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3). \tag{15}$$

$$\begin{cases} \dfrac{d}{ds} X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds} V(s;t,x,v) = E(s, X(s;t,x,v)) + V(s;t,x,v) \wedge B(s, X(s;t,x,v)). \end{cases}$$

Consider two solutions f_1, f_2 with $(X_1, V_1), (X_2, V_2)$ the corresponding characteristics. We study the distance

$$D(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |X_1(t,x,v) - X_2(t,x,v)| f^{in}(x,v) dxdv. \tag{16}$$

Additional terms in the magnetized case

$$
\begin{aligned}
D(t) \leq &\int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| \\
&+ |V_1(\tau) \wedge B(\tau, X_1(\tau)) - V_2(\tau) \wedge B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
&\leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds \\
&+ \|B\|_\infty \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_1(\tau) - V_2(\tau)| f^{in}(x,v) dx dv d\tau ds \\
&+ \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau)| |B(\tau, X_1(\tau)) - B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
&= I(t) + J(t) + K(t).
\end{aligned}
$$

The additional terms $J(t), K(t)$ are controlled with $\|\rho_1\|_p, \|\rho_2\|_p$ and velocity moments of f^{in}.

Uniqueness criterion with a general magnetic field

Theorem (R.)

Let $T > 0$ and $B := B(t,x)$ such that $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$.
If f^{in} satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x,v)dxdv \leq (C_0 k)^{\frac{k}{3}}, \text{[16]} \tag{17}$$

with C_0 a constant independent of k, then there exists at most one solution $f \in L^\infty([0\,,T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to the Cauchy problem for the magnetized Vlasov–Poisson system. If such a solution exists then it will verify

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{18}$$

[16] E. Miot, A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system, Comm. Math. Phys., 2016.

Propagation of moments with a general magnetic field

$$Q(t) := \sup \left\{ \int_0^t |E(s, X(s; 0, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\}, \tag{19}$$

$$N_T := \sup_{0 \le t \le T} Q(t) \le C, \tag{20}$$

$$\begin{aligned} M_k(t) &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |V(t; 0, x, v)|^k f^{in}(x, v) dv dx, \\ &\le \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v| + N_T)^k \exp(kt \|B\|_\infty) f^{in}(x, v) dv dx. \end{aligned}$$

In the unmagnetized case we have:

Theorem (Pallard [17])

Let $T > 0$, then for all $0 \le t \le T$,

$$Q(t) \le C(T^{\frac{1}{2}} + T^{\frac{7}{5}}). \tag{21}$$

[17] C. Pallard, Moment propagation for weak solutions to the Vlasov-Poisson system, Comm. Partial Differential Equations, 2012.

Estimates on the characteristics

Conjecture

For all time t such that $0 \leq t \leq T_B$,

$$Q(t) \leq C \exp(T_B \|B\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}). \quad (22)$$

- The conjecture is true if we assume $B := B(t)$ independent of x.
- Difficulty when $B := B(t, x)$: control of the difference between velocity characteristics $|V(s) - V_*(s)|$ in terms of $Q(t)$ and $|V(s) - V_*(s)|$:

$$|V(s) - V_*(s)| \leq |v - v_*| + 2Q(t) + \int_s^t |V(s) \wedge B(s, X(s)) - V_*(s) \wedge B(s, X_*(s))| ds.$$

Thank you for your attention!