Wave localisation and density of states in aperiodic block disordered systems

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Motivation

Can we predict the spectral properties of aperiodic block disordered systems from their building blocks?

Governing Equation (Helmholtz)

- Time-harmonic wave resonance in a 1D system of N resonators $\mathcal{D} = \bigsqcup_{i=1}^{N}(x_i^{\mathsf{L}}, x_i^{\mathsf{R}})$ is described by:

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2}u + \omega^2 u = 0, & \text{in } \mathbb{R}\setminus\{x_i^{\mathsf{L,R}}\}, \\ u|_{\mathsf{R}}(x_i^{\mathsf{L,R}}) - u|_{\mathsf{L}}(x_i^{\mathsf{L,R}}) = 0, \\ \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{in}}(x_i^{\mathsf{L,R}}) - \frac{\rho_{\mathsf{in}}}{\rho_{\mathsf{out}}}\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{\mathsf{out}}(x_i^{\mathsf{L,R}}) = 0 \end{cases}$$

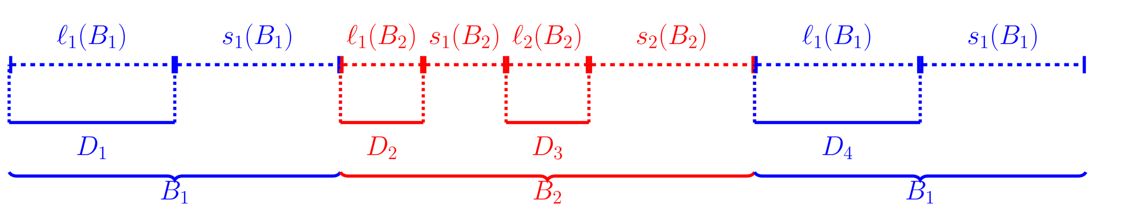
- $\rho_{\mathsf{in,out}}$ denote the density inside and outside the resonators, respectively.
- Resonator array is described by the *resonator lengths* $\ell_i = x_i^{\mathsf{R}} - x_i^{\mathsf{L}}$ and *resonator spacings* $s_i = x_{i+1}^{\mathsf{L}} - x_i^{\mathsf{R}}$.

Subwavelength Asymptotics

- *High-contrast regime*, where the density ratio $\delta = \rho_{\mathsf{in}}/\rho_{\mathsf{out}} \to 0$.
- Look for *subwavelength resonant frequencies*, which $\omega(\delta) \to 0$ as $\delta \to 0$.

Theorem 1 (Capacitance Matrix Approximation [3])**.** *Eigenpairs of* tridiagonal capacitance matrix $VC \in \mathbb{R}^{N\times N}$ *give subwavelength resonant modes to leading order:*

$$\omega_i(\delta) \approx \sqrt{\delta\lambda_i}, \quad u(x) \approx \sum_{i=1}^{N} \boldsymbol{u}_i V_i(x).$$



Schematic of a block disordered system built from B_1 (single resonator) and B_2 (dimer) blocks, arranged according to the sequence $\chi = (1, 2, 1)$.

Block Disordered Systems

- Construct resonator arrays by concatenating M *building blocks* $B_{\chi^{(j)}}$ selected from a finite set of D distinct blocks $\{B_1, \dots, B_D\}$.
- Sequence $\chi = (\chi^{(1)}, \dots, \chi^{(M)}) \in \{1, \dots, D\}^M$ dictates the block arrangement along the line.
- Each block B_d contains ≥ 1 resonators described by lengths $\ell_k(B_d)$, spacings $s_k(B_d)$.
- χ is sampled *independently and identically (IID)* with block probabilities $p_1, \dots, p_D$.

Propagation Matrix

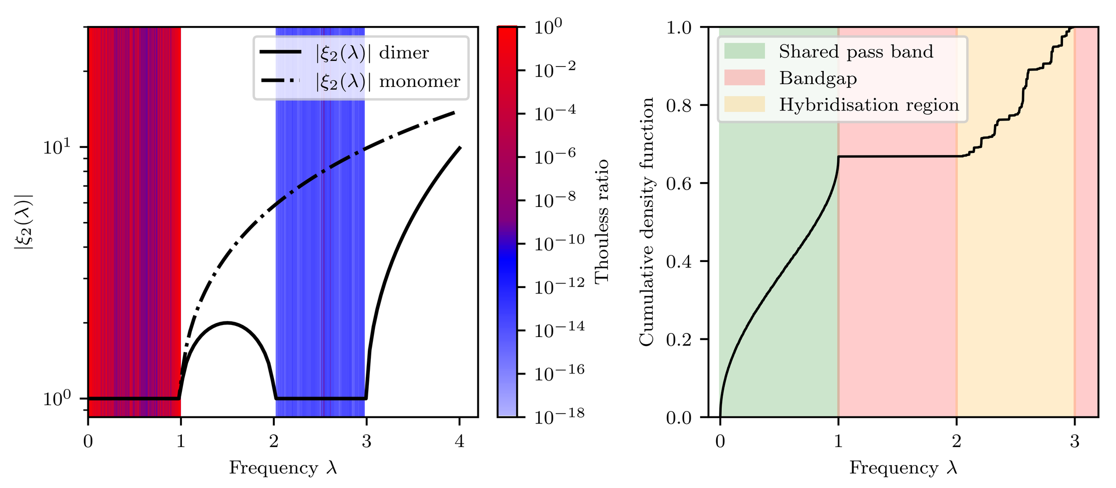
- For a single resonator, its *subwavelength propagation matrix* at frequency λ is:

$$P_{\ell_i, s_i}(\lambda) = \begin{pmatrix} 1 - s_i\ell_i\lambda & s_i \\ -\ell_i\lambda & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

- $P_{\ell_i,s_i}(\lambda)$ propagates $(u, u')^\top$ from the left of x_i^{L} to the left of x_{i+1}^{L}.
- **Block Propagation:** For a block B_d, $P_{B_d}(\lambda) = \prod_k P_{\ell_k(B_d), s_k(B_d)}(\lambda)$.
- **Block Bandgap:** λ is in the *bandgap* of B_d if $|\mathrm{tr}\, P_{B_d}(\lambda)| > 2$ (exponential decay, DoS vanishes).

Saxon-Hutner Type Result

Theorem 2. *If* $|\mathrm{tr}\, P_{B_d}(\lambda)| > 2$ *for every block* B_d*, then the array of blocks* $B_{\chi^{(j)}}$ *has a* bandgap *at frequency* λ.

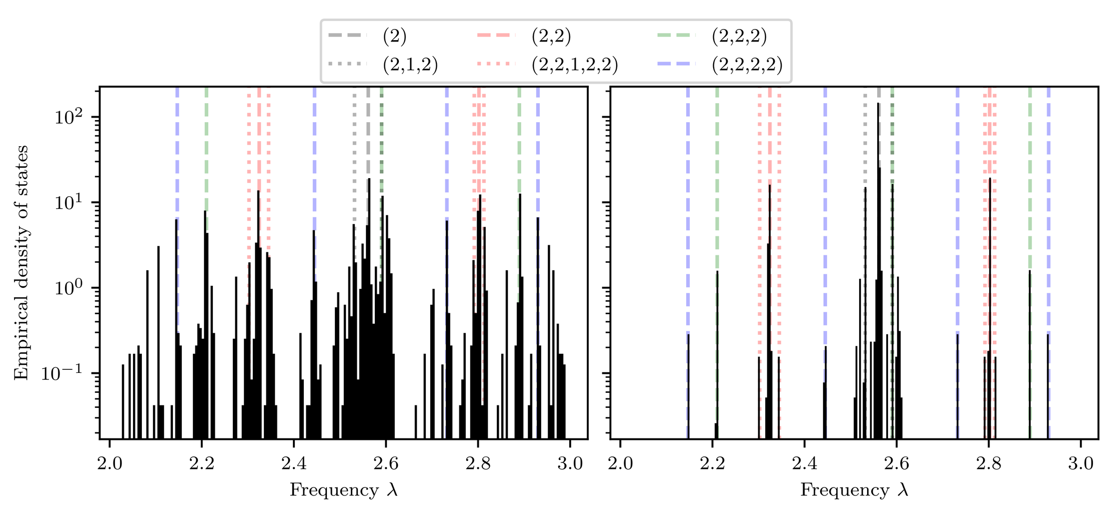


(Left) Maximal propagation matrix eigenvalue $|\xi_2(\lambda)|$ for single resonator (B_1) and dimer (B_2) blocks. Vertical lines are eigenvalues of a random system made from these blocks with color indicating localisation. **(Right)** Cumulative Density of States (CDF) for the random system, showing distinct spectral regions.

Spectral Regions

Spectrum of a block disordered system can be divided into *three distinct regions*:

- **Shared Pass Band:** Intersection of the pass bands of *all* block types. Modes here are weakly localised.
- **Bandgap:** Intersection of the bandgaps of *all* block types. The DoS here is zero.
- **Hybridisation Region:** In the pass band of *some* blocks but in the gap of *others*. Supports highly localised *hybridised bound states*. Self-similar DoS.



DoS in the hybridisation region $(2, 3)$ for random systems of single resonator (B_1) and dimer (B_2) blocks. Peaks correspond to modes of local dimer "meta-atom" arrangements like (2), $(2, 2)$, $(2, 1, 2)$. Self-similarity is especially clear for low dimer density p_2 **(Right)**

Self-similar DoS

- *Hybridisation regions*: In the pass band of *some* blocks but the gap of others.
- Local arrangements of *pass band* blocks (*Meta-atoms*) act as *isolated defects* and contribute resonant modes in the hybridisation region, leading to a *self-similar* DoS.
- DoS slightly smoothed by weak interactions between these modes (*hybridisation*)

Meta-atom DoS Estimate

- **Goal:** Estimate the DoS in hybridisation regions.
- **Approach:** Leverage the fractal-like concentration of DoS around meta-atom modes.
- **Key Steps:**
  1. *Pre-compute meta-atom spectra* when embedded in bandgap blocks.
  2. Scan the long disordered sequence χ: *iteratively match the largest possible meta-atom*, add its pre-computed spectrum to the total, and advance.

$\implies$ Amortised $\mathcal{O}(N)$ complexity.

Non-IID Sampling

The meta-atom estimate is *robust beyond IID block sampling*. It effectively handles:

- **Bounded-length Sampling:** Limits consecutive identical blocks.
- **Hyperuniform Sampling:** Suppresses large-scale density fluctuations (e.g., via chunking or softmax sampling).
- **Quasiperiodic Sampling:** Deterministic, non-periodic sequences (e.g., Fibonacci).

Conclusion and Outlook

- Constituent block band properties predict spectral regions.
- Meta-atom approach enables rapid DoS estimation in hybridisation regions.
- **Outlook:** Higher dimensions (2D/3D), non-Hermitian block disordered systems

References

[1] Habib Ammari, Silvio Barandun, Bryn Davies, Erik Orvehed Hiltunen, and Alexander Uhlmann. Universal estimates for the density of states for aperiodic block subwavelength resonator systems, May 2025.

[2] Habib Ammari, Silvio Barandun, and Alexander Uhlmann. Subwavelength Localisation in Disordered Systems, May 2025.

[3] Florian Feppon, Zijian Cheng, and Habib Ammari. Subwavelength Resonances in One-Dimensional High-Contrast Acoustic Media. *SIAM Journal on Applied Mathematics*, 83(2):625–665, April 2023.