# Tunable Localisation in Parity-Time-Symmetric Resonator Arrays with Imaginary Gauge Potentials

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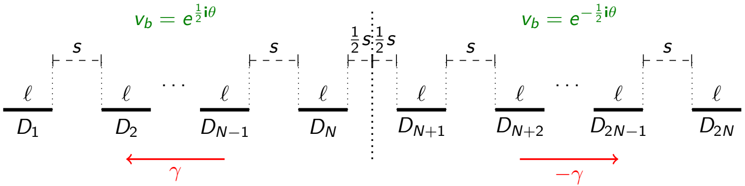
# High-contrast resonators with balanced non-hermiticity




- Array of $2N$ resonators in 1D, symmetric about the origin
- Introduce two types of non-Hermiticity: gain and loss and non-reciprocal gauge potential[1]
- Modal decomposition yields modified Helmholtz equation for resonant modes

$$\begin{cases} \frac{d^2}{dx^2}u + \frac{\omega^2}{v^2}u = 0, & \text{in } \mathbb{R} \setminus \cup D_i \\ \frac{d^2}{dx^2}u + \gamma(x)\frac{d}{dx}u + \frac{\omega^2}{v_b(x)^2}u = 0, & \text{in } D_i \\ u|_+ - u|_- = 0, & \text{on } \partial D_i \\ \delta \frac{du}{dx}|_+ - \frac{du}{dx}|_- = 0, & \text{on } \partial D_i \\ u \text{ satisfies Sommerfeld radiation condition} \end{cases} \tag{1}$$

$\delta := \rho_b/\rho \ll 1$, contrast ratio

[1] Jana and Sirota, "Emerging Exceptional Point with Breakdown of Skin Effect in Non-Hermitian Systems".

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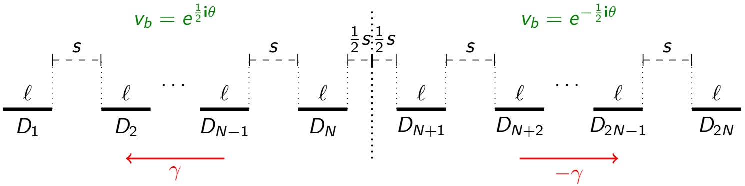


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**Goal:** Understand system as we tune $\theta$ from 0 to $\pi/2$

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# Subwavelength high-contrast regime

- We are looking for Subwavelength resonant frequencies in the high-contrast regime, i.e. resonant frequencies $\omega$ with

$$\omega \to 0 \quad \text{as} \quad \delta \to 0$$

  for which there exist non-trivial solutions to the Helmholtz equation.
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## Theorem

*There exist exactly $2 * 2N$ subwavelength resonant frequencies which are approximated by eigenvalues and eigenvectors of a capacitance matrix*[2] $\mathcal{C}^{\theta,\gamma} \in \mathbb{C}^{2N\times 2N}$, *i.e. for an eigenpair* $(\lambda_i, \boldsymbol{a}_i)$ *of* $\mathcal{C}^{\theta,\gamma}$ *we have*

$$\omega_i = \pm\sqrt{\delta\lambda_i} + \mathcal{O}(\delta) \quad \textit{and} \quad u_i(x) = \boldsymbol{a}_i^{(j)} + \mathcal{O}(\delta) \quad x \in D_j.$$

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# Capaticance matrix

Goal: Solve eigenproblem for capacitance matrix

$$\mathcal{C}^{\theta,\gamma} = V^{\theta} C^{\gamma} = \left(\begin{array}{c|c} e^{\mathrm{i}\theta} I_N & \mathbf{0} \\ \hline \mathbf{0} & e^{-\mathrm{i}\theta} I_N \end{array}\right) \left(\begin{array}{cccc|cccc} \alpha+\beta & \eta & & & & & & \\ \beta & \alpha & \ddots & & & & & \\ & \ddots & \ddots & \eta & & & & \\ & & \beta & \alpha & \eta & & & \\ \hline & & & \eta & \alpha & \beta & & \\ & & & & \eta & \ddots & \ddots & \\ & & & & & \ddots & \alpha & \beta \\ & & & & & & \eta & \alpha+\beta \end{array}\right) \in \mathbb{R}^{2N\times 2N}$$

with $\alpha = \frac{\gamma}{1-e^{-\gamma}} - \frac{\gamma}{1-e^{\gamma}} = \gamma \coth(\gamma/2), \eta = \frac{-\gamma}{1-e^{-\gamma}}, \beta = \frac{\gamma}{1-e^{\gamma}}.$

*Here we assumed $s = \ell = 1$ for the sake of simplicity. The general case functions analogously.*

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- $\mathcal{C}^{\theta,\gamma}$ is tridiagonal and almost Toeplitz in upper / lower part
- $P\mathcal{C}^{\theta,\gamma}P = \overline{\mathcal{C}^{\theta,\gamma}}$

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# PT-symmetry and Exceptional Points

- $P\mathcal{C}^{\theta,\gamma}P = \overline{\mathcal{C}^{\theta,\gamma}}$ is a PT-symmetry
- Thus, the eigenvalues of $\mathcal{C}^{\theta,\gamma}$ are real or come in complex conjugate pairs, i.e. $\sigma(\mathcal{C}^{\theta,\gamma}) = \overline{\sigma(\mathcal{C}^{\theta,\gamma})}$
- Because $\mathcal{C}^{\theta,\gamma}$ is tridiagonal: Eigenspaces are always one-dimensional
- Because $\mathcal{C}^{\theta,\gamma}$ is PT-Symmetric: Real eigenvalues must meet pairwise to become complex
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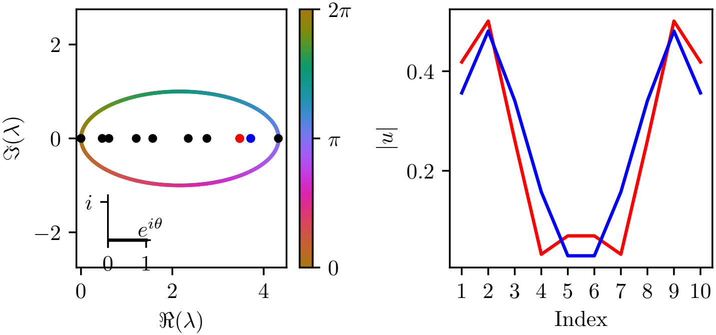
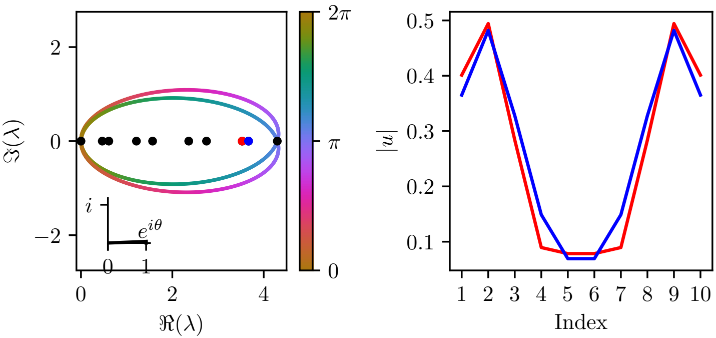
## Definition

We call $\theta \in [0, \pi/2]$ an *exceptional point* if $\mathcal{C}^{\theta,\gamma}$ is not diagonalisable.

# Decoupling ($\theta = 0$)



# Decoupling ($\theta = 0.04$)



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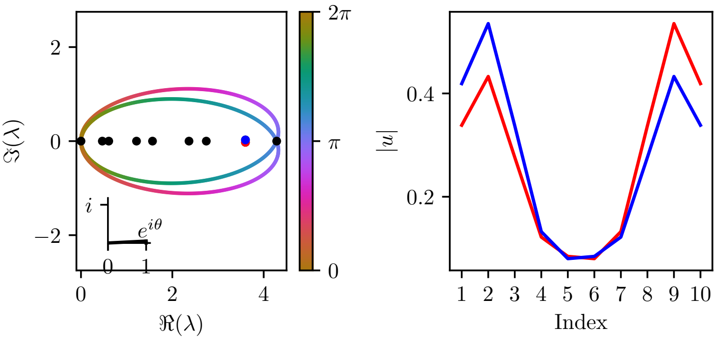
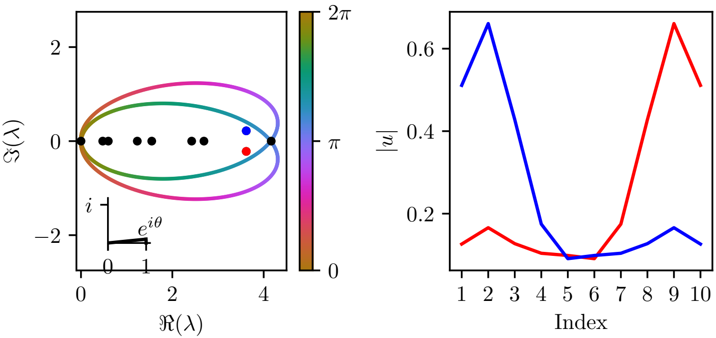
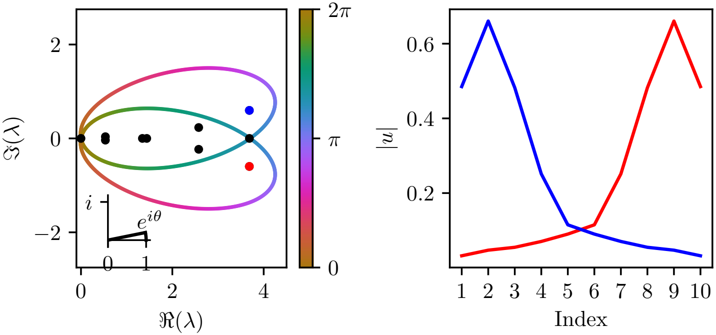


# Decoupling ($\theta = 0.1$)



# Decoupling ($\theta = 0.2$)



# Chebyshev formalism

Exploit the tridiagonal Toeplitz structure of $\mathcal{C}^{\theta,\gamma}$ to recursively determine its eigenvectors:

## Theorem

*For $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{C}^{\theta,\gamma}$, the corresponding eigenvector is given by $\boldsymbol{u} = (\boldsymbol{x}, \boldsymbol{y})^\top$ where*

$$\begin{aligned} \boldsymbol{x} &= \left( P_0(\mu^\theta(\lambda)), \left(e^{-\frac{\gamma}{2}}\right) P_1(\mu^\theta(\lambda)), \cdots, \left(e^{-\frac{\gamma}{2}}\right)^{N-1} P_{N-1}(\mu^\theta(\lambda)) \right), \\ \boldsymbol{y} &= C \left( \left(e^{-\frac{\gamma}{2}}\right)^{N-1} P_{N-1}(\mu^{-\theta}(\lambda)), \cdots, \left(e^{-\frac{\gamma}{2}}\right) P_1(\mu^{-\theta}(\lambda)), P_0(\mu^{-\theta}(\lambda)) \right). \end{aligned} \tag{2}$$

*With affine transformation* $\mu^\theta(\lambda) := e^{-\mathrm{i}\theta}\lambda \frac{1}{\gamma} \sinh \frac{\gamma}{2} - \cosh \frac{\gamma}{2}$ *and*
$P_n(x) := U_n(x) + e^{-\frac{\gamma}{2}} U_{n-1}(x)$, *the sum of two Chebyshev polynomials of the second kind.*

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Continuity across the interface:

$$C = e^{-\frac{\gamma}{2}} \frac{P_N(\mu^\theta(\lambda))}{P_{N-1}(\mu^{-\theta}(\lambda))} = e^{\frac{\gamma}{2}} \frac{P_{N-1}(\mu^\theta(\lambda))}{P_N(\mu^{-\theta}(\lambda))}$$

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Characteristic equation:

$$\frac{P_N(\mu^\theta(\lambda))P_N(\mu^{-\theta}(\lambda))}{P_{N-1}(\mu^\theta(\lambda))P_{N-1}(\mu^{-\theta}(\lambda))} = e^\gamma$$

# Understanding the characteristic equation asymptotically

Goal: Understand $\frac{P_N(\mu^{\theta}(\lambda))P_N(\mu^{-\theta}(\lambda))}{P_{N-1}(\mu^{\theta}(\lambda))P_{N-1}(\mu^{-\theta}(\lambda))} = e^{\gamma}$ as $N \to \infty$.

Idea: Write Chebyshev polynomials of second kind as

$$U_n(\mu) = \frac{a(\mu)^{n+1} - a(\mu)^{-(n+1)}}{2\sqrt{\mu+1}\sqrt{\mu-1}},$$

where $a(\mu) = \mu + \sqrt{\mu+1}\sqrt{\mu-1}$ for $\mu \in \mathbb{C}$ and find

$$\frac{P_n(\mu)}{P_{n-1}(\mu)} = \frac{U_n(\mu) + e^{-\frac{\gamma}{2}}U_{n-1}(\mu)}{U_{n-1}(\mu) + e^{-\frac{\gamma}{2}}U_{n-2}(\mu)} \xrightarrow{unif.} a(\mu) \quad \text{as } n \to \infty$$

outside of any $\varepsilon$-neighbourhood of $[-1, 1]$.

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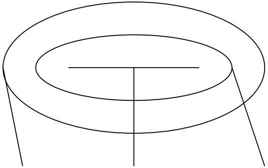


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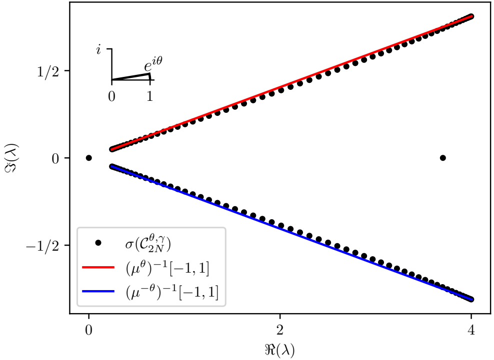
$\implies$ $|a(\mu)|$ controls the asymptotic growth behaviour of $P_n(\mu)$ as $n \to \infty$

We can understand $a$:

- $|a| > 1$ and level sets of $|a| = c$ are ellipses for $c > 1$ and $[-1, 1]$ for $c = 1$
- For any $\gamma > 0, \theta \in [0, \pi/2]$, $a(\mu^\theta(\lambda))a(\mu^{-\theta}(\lambda)) = e^\gamma$ has exactly two solutions, both on the real line

# Location of eigenvalues



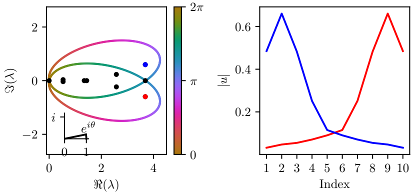

## Lemma

*For any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that all but two eigenvalues of $\mathcal{C}^{\theta,\gamma}$ lie in $\varepsilon$-neighbourhood of red/blue lines.*

Proof Idea:

- $a(\mu^{\theta}(\lambda))a(\mu^{-\theta}(\lambda)) = e^{\gamma}$ has exactly two solutions, both real
- $\frac{P_N(\mu^{\theta}(\lambda))P_N(\mu^{-\theta}(\lambda))}{P_{N-1}(\mu^{\theta}(\lambda))P_{N-1}(\mu^{-\theta}(\lambda))} \xrightarrow{unif.} a(\mu^{\theta}(\lambda))a(\mu^{-\theta}(\lambda))$ outside $\varepsilon$-neighbourhood of red / blue lines
- Thus for $N$ large enough charateristic equation has exactly two solutions outside these neighbourhoods
- But as charateristic equation is equivalent to a degree $2N$ polynomial, the $2N-2$ remaining solutions must lie in the $\varepsilon$-neighbourhoods

# Topological origin of eigenvector decoupling



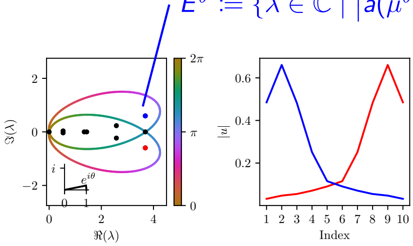

- Recall that

$$\boldsymbol{x} = \left(P_0(\mu^\theta(\lambda)), \left(e^{-\frac{\gamma}{2}}\right) P_1(\mu^\theta(\lambda)), \cdots, \left(e^{-\frac{\gamma}{2}}\right)^{N-1} P_{N-1}(\mu^\theta(\lambda))\right)$$

- Thus $\frac{\boldsymbol{x}^{(j+1)}}{\boldsymbol{x}^{(j)}} = e^{-\frac{\gamma}{2}} \frac{P_j(\mu^\theta(\lambda))}{P_{j-1}(\mu^\theta(\lambda))} \implies$ $\boldsymbol{x}$ decays iff $\left|a(\mu^\theta(\lambda))\right| < e^{\frac{\gamma}{2}}$
- $\boldsymbol{y}$ grows iff $\left|a(\mu^{-\theta}(\lambda))\right| < e^{\frac{\gamma}{2}}$

# Topological origin of eigenvector decoupling

$E^{\theta} := \{\lambda \in \mathbb{C} \mid |a(\mu^{\theta}(\lambda))| < e^{\frac{\gamma}{2}}\}$, location in ellipse determines growth behaviour

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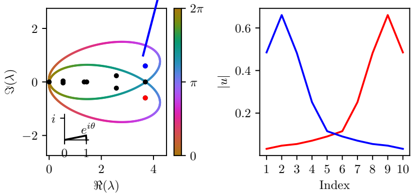


$E^\theta$ turns out to be exactly the interior of the ellipse drawn out by the Toeplitz symbol $z \in \mathbb{T} \mapsto e^{\mathbf{i}\theta}(\beta z + \alpha + \eta z^{-1})$

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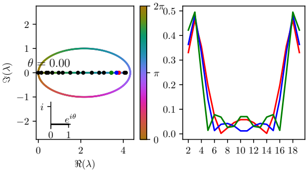
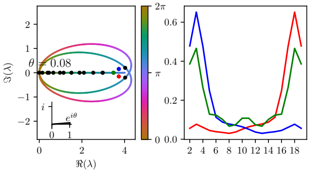
# Recap

1. Introduced two kinds of non-Hermiticity to 1D resonator array via energy gain/loss and non-reciprocal gauge potential in a balanced and thus PT-symmetric way
2. Used capacitance matrix approximation to reduce the subwavelength resonance problem to a finite eigenproblem on tridiagonal Toeplitz matrix with interface
3. Used Chebyshev polynomials to recursively construct eigenvectors and got characteristic equation for eigenvalues
4. Found limit $a(\mu)$ of Chebyshev polynomial ratios and used it to understand characteristic equation and decoupling asymptotically
   $\implies$ Eigenvalues go through exceptional points and corresponding eigenmodes begin to decouple as gain-to-loss ratio $\theta$ is increased
5. Found the topological origin of the decoupling by relating it to the winding of Toeplitz symbols

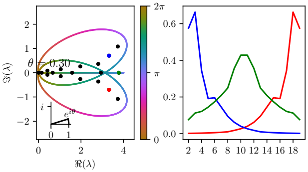
# Outlook



(a) $\theta = 0$



(b) $\theta = 0.08$



(c) $\theta = 0.3$

- Findings could be extended and embedded into larger framework for tridiagonal interfaced Toeplitz matrices
  - Decoupling into delocalized modes for three-part resonator arrays
  - Findings are also be applicable to quantum mechanical setting
- Use exceptional points to get sensor arrays with higher order sensistivity

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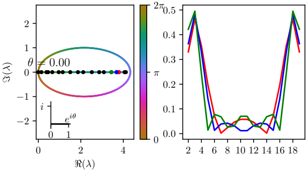


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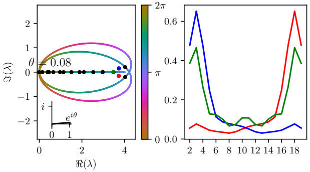


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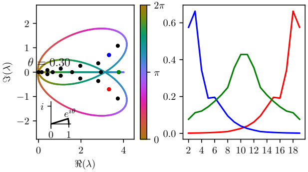


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# Questions?

# Capaticance matrix approximation

The subwavelength resonant frequencies can be approximated by an eigenvalue problem:

## Theorem (From[3])

*The $N$ subwavelength eigenfrequencies $\omega_i$, as $\delta \to 0$, are*

$$\omega_i = \sqrt{\delta \lambda_i} + \mathcal{O}(\delta),$$

*where $(\lambda_i)_{1 \leq i \leq N}$ are the eigenvalues of the eigenvalue problem*

$$VL^{-1}\mathcal{C}^\gamma \boldsymbol{a}_i = \lambda_i \boldsymbol{a}_i$$

$C^\gamma$ is the *capacitance matrix*. We can explicitly find its entries.

*with $V = v_b^2 I_N$ and $L_{ij} = \ell_i \delta_{ij}$. Furthermore, let $u_i$ be a subwavelength eigenmode corresponding to $\omega_i$ and let $\boldsymbol{a}_i$ be the corresponding eigenvector of $VL^{-1}\mathcal{C}^\gamma$. Then,*

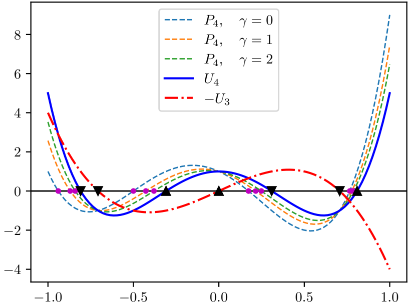
$$u_i(x) = \boldsymbol{a}_i^{(j)} + \mathcal{O}(\delta) \quad \text{for } x \in D_j,$$

*where $\boldsymbol{a}^{(j)}$ denotes the $j$-th entry of the eigenvector.*

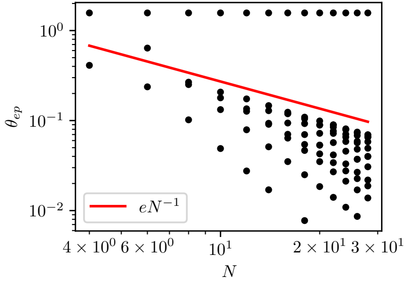
[3] Feppon, Cheng, and Ammari, *Subwavelength Resonances in 1D High-Contrast Acoustic Media*.

# Chebyshev Corollaries

- For $0 \le \theta < \varepsilon$ all eigenvalues of $\mathcal{C}^{\theta,\gamma}$ are real. In this regime the eigenvectors are symmetric about their middle $\implies$ use Equation (2) and $\mathcal{C}^{\theta,\gamma}$ diagonalisable for $\theta = 0$.
- For $\theta = \pi/2$ all eigenvalues of $\mathcal{C}^{\theta,\gamma}$ lie on the imaginary axis. Thus they all must have passed through an exceptional point $\implies$ use charateristic equation and the fact that $U_N$ and $U_{N-1}$ are fully interlaced.



# Density of exceptional points




## Theorem

*For any $\theta > 0$ we can find a $N$ such that all but two eigenvalues of $\mathcal{C}^{\theta,\gamma}$ are away from the real line and must thus have passed through an exceptional point.*

Proof Idea:

- Red / blue lines are away from the real line
- Pick $\varepsilon$-neighbourhoods of red / blue lines that don't touch the real line
- Use uniform convergence from last slide to find $N$ such that all but two eigenvalues are in these $\varepsilon$-neighbourhoods