Character formulas for limits of discrete series of p-adic SL_2

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Abstract

Let F be a non-archimedean field of characteristic greater than 2. In this expository note, we compute trace characters for the two irreducible summands of the parabolic induction to $\mathrm{SL}_2(F)$ of a non-trivial order 2 character $\chi : F^\times \to \{\pm\}$, using endoscopic character identities. We then compare this to the trace characters of regular supercuspidal representations, inspired by the limits of discrete series for $\mathrm{SL}_2(\mathbb{R})$. [1]

Archimedean motivation:

In this section, let $G \coloneqq \mathrm{SL}_2(\mathbb{R})$. There are two (stable) conjugacy classes of maximal tori in G, represented by the split torus $T \simeq \mathbb{R}^\times$ consisting of diagonal matrices, and the compact torus $S = \mathrm{SO}_2(\mathbb{R})$.

The discrete series representations of G are parametrized by non-trivial characters $\theta : S \to \mathbb{C}^\times$, corresponding to non-zero integers via

$$\theta = \theta_k : \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \mapsto e^{ikx}.$$

To each integer $k \geq 1$, there are two discrete series representations $\pi^\pm(S, \theta_k)$ whose trace characters are given by

$$\Theta_{\pi^+(S,\theta)}(g) = \begin{cases} \dfrac{-\theta(x)}{e^{ix} - e^{-ix}}, & g \sim \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \in S, \\[2ex] \mathrm{sgn}(t)^{k+1} \dfrac{\theta(-it)(1 - \mathrm{sgn}(t)) + \theta(it)(1 + \mathrm{sgn}(t))}{2|e^t - e^{-t}|}, & g \sim \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T \end{cases} \tag{1}$$

[1] This was one of Tasho Kaletha's projects at the 2025 Arizona Winter School, with project assistant Jialiang Zou.

and

$$\Theta_{\pi^-(S,\theta)}(g) = \begin{cases} \dfrac{\theta(x)^{-1}}{e^{ix} - e^{-ix}}, & g \sim \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \in S, \\ \operatorname{sgn}(t)^{k+1} \dfrac{\operatorname{sgn}(-it)(1 - \operatorname{sgn}(t)) + \theta(it)(1 + \operatorname{sgn}(t))}{2|e^t - e^{-t}|}, & g \sim \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T. \end{cases} \tag{2}$$

Let $B \subset G$ denote the Borel subgroup, consisting of upper triangular matrices. Inflating the sign character $\operatorname{sgn} : T \to \{\pm\}$ to B and inducing to G produces a representation $I_B^G(\operatorname{sgn})$ which decomposes as a direct sum

$$I_B^G(\operatorname{sgn}) = \pi^+ \oplus \pi^-.$$

The subrepresentations $\pi^\pm$ are irreducible and tempered, but not discrete series. However, their trace characters are given by

$$\Theta_{\pi^\pm}(g) = \begin{cases} \mp \dfrac{1}{e^{ix} - e^{-ix}}, & g \sim \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \in S, \\ \dfrac{\operatorname{sgn}(t)}{|e^t - e^{-t}|}, & g \sim \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T. \end{cases} \tag{3}$$

which match exactly with the trace character of the discrete series $\pi^\pm(S, \theta)$, if we formally inserted the trivial character as θ.

These representations $\pi^\pm$ are called "limits of discrete series" and can be denoted $\pi^\pm(S, 1)$. For more on the archimedean case, see [Kna86].

Non-archimedean case:

Let F be non-archimedean field of odd residue characteristic. A non-trivial character of order two $\chi : F^\times \to \{\pm\}$ corresponds via class field theory to a quadratic extension E/F. We can view χ as a character of the diagonal torus T in $G \coloneqq \mathrm{SL}_2(F)$. The parabolic induction $I_B^G(\chi)$ splits into a direct sum of irreducibles

$$I_B^G(\chi) = \pi^+ \oplus \pi^-.$$

The representations $\pi^\pm$ constitute an L-packet, whose parameter $\varphi : W_F \to {}^L G$ factors

$$\begin{array}{ccc} \varphi\colon W_F & \longrightarrow & {}^L G. \\ & \searrow^{\varphi_T} \quad \nearrow^{{}^L j_T} & \\ & {}^L T & \end{array}$$

Here ${}^L G = \mathrm{PGL}_2(\mathbb{C}) \times \Gamma$ and ${}^L T = \mathbb{C}^\times \times \Gamma$, where $\Gamma = \mathrm{Gal}(\overline{F}/F)$. The slanted maps are

$$\varphi_T : \tau \mapsto (\chi(\tau), \tau)$$
$${}^L j_T : (z, \tau) \mapsto \left(\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, \tau \right).$$

where in the first line we have identified $\chi : F^\times \to \{\pm 1\}$ with a character of W_F by local class field theory.

Remark. The labeling π^+, π^- is not unique, but upon fixing a Whittaker datum $\mathfrak{w}$ we can choose π^+ to be the unique generic constituent of the L-packet.

By the additivity of trace characters and the parabolic induction formula, we have

$$S\Theta_\varphi(g) := (\Theta_{\pi^+} + \Theta_{\pi^-})(g) = \Theta_{I_B^G \chi}(g) = \begin{cases} 2\frac{\chi(g)}{|D_G(g)|^{1/2}}, & g \text{ split}, \\ 0 & \text{else.} \end{cases} \tag{4}$$

On the other hand, corresponding to the element

$$s := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in S_\varphi := \mathrm{Cent}(\varphi, \mathrm{PGL}_2(\mathbb{C}))$$

we have the factorization

$$\begin{tikzcd} \varphi\colon W_F \arrow[rr] \arrow[dr, "\varphi_S"'] & & {}^L G. \\ & {}^L S \arrow[ur, hook, "{}^L j_S"'] & \end{tikzcd} \tag{5}$$

Here ${}^L S = \mathbb{C}^\times \rtimes \Gamma$ where an element $\sigma \in \Gamma$ with non-trivial image in $\mathrm{Gal}(E/F)$ acts by inversion on $\mathbb{C}^\times$. This is the L-group of the unique non-split E-split F-torus $\mathbb{S}$ whose F points are

$$S = \mathbb{S}(F) = E^1 := \ker(\mathrm{Nm}_{E/F} : E^\times \to F^\times).$$

The maps are

$$\begin{aligned} \varphi_S &\colon \tau \mapsto (\chi(\tau), \tau) \\ {}^L j_S &\colon (z,1) \mapsto \left(\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, 1 \right), \\ &\quad (1,\sigma) \mapsto \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma \right), \qquad \sigma \in \mathrm{Gal}(E/F) \text{ non-trivial.} \end{aligned}$$

Proposition 1. The diagram

$$\begin{tikzcd} & {}^L T \arrow[dr, hook, "{}^L j_T"] & \\ \phi\colon W_F \arrow[rr] \arrow[ur, "\varphi_T"] \arrow[dr, "\varphi_S"'] & & {}^L G. \\ & {}^L S \arrow[ur, hook, "{}^L j_S"'] & \end{tikzcd}$$

commutes up to $\mathrm{PGL}_2(\mathbb{C})$-conjugacy.

Proof. ** □

Proposition 2. The parameter $\varphi_S : W_F \to {}^L S$ corresponds under the local Langlands correspondence for tori to the trivial character of $S(F)$.

Proof. We have that $(\varphi_S)|_{W_E} = 1$. Let $\sigma \in W_F \setminus W_E$, so φ_S is determined by $\varphi_S(\sigma) = (z, \sigma) \in {}^L S$ for some $z \in \mathbb{C}^\times$. For any $x \in \mathbb{C}^\times$, the inverse of (x, σ) in ${}^L S$ is (x, σ). It follows that the conjugate ${}^{(x,\sigma)}\phi$ satisfies $\phi(\sigma) = (x^2 z^{-1}, \sigma)$. By taking $x^2 = z$, we get back the trivial L-parameter as claimed above. □

To isolate the individual characters $\Theta_{\pi^\pm}$, we use the endoscopic character identity:

$$\Theta^{\mathfrak{w}}_{\varphi,s}(g) \coloneqq (\Theta_{\pi^+} - \Theta_{\pi^-})\,(g) = \sum_{\gamma \in S^{\mathrm{sr}}/\ \mathrm{st}} \Delta[\mathfrak{w}](\dot{\gamma}, g)\ S\Theta_{\varphi_S}(\gamma). \tag{6}$$

Here:

- The sum is over strongly regular semisimple elements of S up to stable conjugacy. As S is abelian, stable conjugacy classes are singletons.
- The Langlands-Shelstad transfer factor

$$\Delta[\mathfrak{w}, \eta] : S^{\mathrm{sr}} \times G^{\mathrm{sr}} \to \mathbb{C}$$

$$\Delta[\mathfrak{w}](\dot{\gamma}, g) = \epsilon\left(\frac{1}{2}, X^*(T)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda\right) **** \tag{7}$$

- the stable character $S\Theta_{\varphi_S}$ on S attached to $\varphi_S : W_F \to {}^L S$, which in our case is given by $S\Theta_{\varphi_S}(\gamma) = 1$ for all $\gamma \in S$.

The endoscopic character identity in this case gives

$$\Theta^{\mathfrak{w}}_{\varphi,s}(g) = (\Theta_{\pi^+} - \Theta_{\pi^-})(g) = \begin{cases} \Delta(g, g) + \Delta(g^{-1}, g), & g \in S \\ 0 & \text{else.} \end{cases}$$

Remark. Note that $g \in S$ is an abuse of notation, since we want to consider g stably conjugate to an element in S. The same holds for $g \in T$ later on.

Here $\Delta : S \times G \to \mathbb{C}$ is the Langlands-Shelstad transfer factor. We have that

$$\Delta(g^{-1}, g) = \kappa_{E/F}(-1)\Delta(g^{-1}, g^{-1}),$$

so we can apply the explicit Kaletha formula [Kal, §3.6]

$$\begin{aligned} \Delta(g, g) &= \lambda(E/F, \psi)\ \kappa_{E/F}\left(\frac{g - g^{-1}}{2\eta}\right) |g - \overline{g}|_F^{-1} \\ &= \lambda(E/F, \psi)\ \kappa_{E/F}\left(\mathrm{Im}(g)\right) |g - \overline{g}|_F^{-1} \end{aligned}$$

where $\lambda(E/F,\psi)$ is the Langlands constant and $\mathrm{Im}(g)$ is the imaginary part of g. Plugging in g^{-1}, we see that

$$\Delta(g^{-1},g^{-1}) = \kappa_{E/F}(-1)\Delta(g,g).$$

Thus

$$\Theta_{\pi^\pm}(g) = \begin{cases} \frac{\chi(g)}{|D_G(g)|^{1/2}} & g \in T, \\ \pm\lambda(E/F,\psi)\ \kappa_{E/F}\left(\mathrm{Im}(g)\right)|g-\overline{g}|_F^{-1}, & g \in S \end{cases}$$

Proposition 3. Viewing the regular supercuspidal character formula $\Theta_{\pi(S,\theta)}$ as a function of θ and formally inputting $\theta = 1$, we have

$$\Theta_{\pi^\pm} = \pm\frac{1}{2}\Theta_{\pi(S,1)}$$

on elements whose centralizer is S.

References

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