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Heegner Points and Generalised Heegner Cycles

Master's thesis

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Abstract. *We give an exposition to the classical theory of Heegner points, followed by proving norm relations between them and the construction of Kolyvagin's cohomology classes. Further, we introduce generalised Heegner cycles in motivic cohomology for modular forms of weight greater than* 2 *and prove norm relations for them.*

Dedicated to Nicki

Contents

1 Introduction

We will start this thesis by introducing the classical theory of the j-invariant and modular elliptic curves, followed by introducing the theory of complex multiplication. We want to show that the values of the j-invariant at CM-points are algebraic integers, and for that we rely on tools from class field theory. We will also heavily rely on this field of algebraic number theory in the following chapters, which is why we will lay out all the required preliminaries at the very beginning. That will occupy Chapter 2, where we mainly follow [DS], [Cox] and [Janusz].

Next, in Chapter 3, we will move on to introducing *Heegner points* in imaginary quadratic fields, as it was done by [Gross]. For a positive integer N, those are defined as pairs of N-isogenous curves with the same ring of complex multiplication. Assigning a lattice to each elliptic curve allows us to consider the j-invariant of the curve, which in turn means that our Heegner points will be characterised by values of the j-invariant on the curve $X_0(N)$. By assigning to each $n \in \mathbb{N}$ an order of index n (in the maximal order assigned to the field), and then to each order assigning a Heegner point, we obtain a whole set of Heegner points $\{x_n\}_n$. Those turn out to be rational over certain ring class fields K_n. Since the values of the j-invariant form a full set of Galois conjugates, we can take their sum to be the trace, which means that we can pull down our Heegner points from the ring class field they were initially defined in to a smaller one, by applying a trace operator. Via a modular parametrization we can map the points $\{x_n\}_n$, which initially lived in $X_0(N)(K_n)$, to points on an elliptic curve $E(K_n)$. For this we will mostly rely on [Gross], but also follow [DS], [Cox], [Janusz] and [Gross4].

Using the Hecke correspondence one can consider the Hecke operator T_l as a double coset operator on $\mathcal{H}$, which means that it acts on pairs of isogenous curves, so also on Heegner points. We will start Chapter 4 by describing this Hecke correspondence. It turns out that letting the Hecke operator T_l act on a Heegner point (almost) corresponds to taking its trace; and using the Eichler-Shimura theorem (§3.1.6), we obtain *norm relations* between Heegner points on elliptic curves (§4.2.1):

Proposition 1.0.1. Let $n = ml$. Then

$$\mathrm{Tr}_l y_n = \begin{cases} a_l \cdot y_m, & (l) \text{ inert}, \\ (a_l - \sigma_l - \sigma_l^{-1}) \cdot y_m, & (l) \text{ split}, \end{cases}$$

in $E(K_m)$.

Via the Kummer sequence in cohomology one can map Heegner points to cohomology classes, which together with our norm relations were used by Kolyvagin to construct an *Euler system*. This Euler system was used to prove much of the Birch–Swinnerton-Dyer conjecture for rank 1 elliptic curves. We will finish Chapter 4 by following [GrossK] for an introduction to the Kummer map and Kolyvagin's cohomology classes, together with a summary of group cohomology.

The construction of Heegner points above can be generalised to also considering modular forms weight greater than 2, as it was done in [JLZ] §2 and §3. In Chapter 5 we will construct generalised Heegner cycles in motivic cohomology. Let $a, b \geq 0$. We start our construction by projecting the Heegner class defined in relative Chow motives:

$$\Delta_{\phi_m} \in H^2_{\mathrm{mot}}\left(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathrm{TSym}^k\left(\mathfrak{h}^1(A)\right)(1)\right),$$

obtained from an elliptic curve and cyclic p^m-isogeny ϕ_m (as in [BDP] §2.3) to

$$z^{[a,b]}_{\mathrm{mot},m} \in H^2_{\mathrm{mot}}\left(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes h^{(a,b)}(A)(1)\right).$$

Here $\mathfrak{h}^1(\varepsilon)$, $\mathfrak{h}^1(A)$ denote the degree 1 part of the corresponding relative motives obtained by *Deninger-Murre* (§5.4.1), and $h^{(a,b)}(A)$ is a rank-1 direct summand on which the complex multiplication action of $\mathcal{O}_K$ is given by $[x]^* = x^a\overline{x}^b$. For a field E and an embedding $\sigma : K \hookrightarrow E$, we define a vector $e^{[a,b]}_m$ in the symmetrised tensor algebra $\mathrm{TSym}^{a+b}((E^2)^\vee)$. We will show that this vector is a basis vector of $\delta_m^*(V_{a,b}^\vee)$, which is a restriction of representations from $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ (denoted δ_m^*) of a dual representation $V_{a,b}^\vee$, living in the dual of $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$.

There is a functor constructed by [Ancona] from algebraic representations of $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ to relative Chow motives. This induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_K(\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)) & \longrightarrow & \mathrm{Rep}_K(\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)) \\ \downarrow{\scriptstyle \mathrm{Anc}} & & \downarrow{\scriptstyle \mathrm{Anc}} \\ \mathrm{CH}\mathcal{M}_K(Y_1(N) \times S_m) & \longrightarrow & \mathrm{CH}\mathcal{M}_K(S_m), \end{array}$$

between algebraic representations $\mathrm{Rep}_K(\cdot)$ to relative Chow motives (here S_m denotes the canonical model of the Shimura variety of $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ with level structure defined accordingly). Using this functor together with a pushforward (Gysin) map δ_{m*}, we will show how the pushforward of our vector $e_m^{[a,b]}$ is precisely the Heegner class defined above. For this construction we mainly follow [JLZ], but also rely on [MilSh] and [Rotger] to introduce Shimura varieties; [Fulton] and [CorHan] to introduce Chow motives; and [Ancona], [JLZ] and [BDP] and for an introduction to Grössencharacters and some representation theory.

Next, we will consider the realisation of $z_{\mathrm{mot},m}^{[a,b]}$ in étale cohomology, denoted $z_{\text{ét},m}^{[a,b]}$. Using the Hochschild-Serre spectral sequence we will project the Heegner classes $z_{\text{ét},m}^{[a,b]}$ into the group

$$H^1(F_m, V_p(f)^*(\sigma^a\overline{\sigma}^b)),$$

where $V_p(f)$ are Deligne's Galois representations associated to a cuspidal Hecke eigenform f of level $\Gamma_1(N)$ and weight greater than 2. For (f, χ) a Heegner pair (see 5.7.2) we will show in Proposition 5.7.3 how (similarly to Kolyvagin's cohomology classes) the class $z_{\text{ét},m}^{[f,j]}$ lies in

$$H^1(F_m, V_p(f)^*(\chi))^{\mathrm{Gal}(F_m/K_m)} \simeq H^1(K_m, V_p(f)^*(\chi)).$$

We will finish this Chapter by showing how the construction of generalised Heegner classes reduces to the case of Heegner *points* from Chapter §3 and §4 when we set $a = b = 0$ above.

Finally, in Chapter 6, we will prove norm relations for the set of generalised Heegner cycles $Z_{\text{ét},m,n}^{[a,b]}$ defined on the modular curve $Y_1(N(p^n))$, with $Z_{\text{ét},m,0}^{[a,b]} = z_{\text{ét},m}^{[a,b]}$ being our classes from Chapter 5. We will show in 6.2.3:

Proposition 1.0.2. Let U_p' be the dual of the (universal) Hecke operator U_p. For $n \geq 1$ the points $Z_{\text{ét},m,n}^{[a,b]}$ defined in (39) satisfy

$$\mathrm{norm}_{F_m}^{F_{m+1}}\left(Z_{\text{ét},m+1,n}^{[a,b]}\right) = U_p' \cdot Z_{\text{ét},m,n}^{[a,b]}. \tag{1}$$

2 Preliminaries

The goal of this Chapter is to lay out the basic tools of class field theory, together with a short introduction to the j-invariant and the theory of complex multiplication.

2.1 Complex tori

In this section we mainly rely on [DS] and [Cox].

For a lattice Λ a *complex torus* is given by $\mathbb{C}/\Lambda = \{z + \Lambda : z \in \mathbb{C}\}$.

Definition 2.1.1. An **isogeny** is a non-zero holomorphic homomorphism between complex tori.

Let $\mathbb{C}/\Lambda$ be a complex torus and $N \in \mathbb{N}$. We define the **multiply-by-integer map** as follows:

$$[N] : \begin{cases} \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda \\ z + \Lambda \mapsto Nz + \Lambda. \end{cases}$$

Note that $[N]$ is an isogeny. We say that the **N-torsion points of** $\mathbb{C}/\Lambda$ are given by

$$E[N] := \{z + \Lambda \in \mathbb{C}/\Lambda \mid [N](z + \Lambda) = 0\}.$$

Then $E[N]$ is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Next, we will show that we have a correspondence between complex tori and elliptic curves.

Definition 2.1.2 (***Weierstrass $\wp$-function***)**.** Take $z \in \mathbb{C} \setminus \Lambda$. Then we define the *Weierstrass $\wp$-function* as follows:

$$\wp(z) := \frac{1}{z^2} + \sum_{w \in \Lambda - \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

The *Weierstrass invariants* on a lattice Λ are given by:

$$\begin{cases} g_2(\Lambda) := 60 \displaystyle\sum_{w \in \Lambda - \{0\}} \frac{1}{w^4}, \\ g_3(\Lambda) := 140 \displaystyle\sum_{w \in \Lambda - \{0\}} \frac{1}{w^6}. \end{cases} \tag{2}$$

Proposition 2.1.3. Let Λ be a fixed lattice and $\wp(z)$ denote the *Weierstrass $\wp$-function* for Λ. Then for any $z, w, z + w \in \mathbb{C} \setminus \Lambda$ one has

(i) $\wp$ is an even elliptic function for Λ whose singularities are double poles at the points of Λ.

(ii) $\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$.

(iii) $\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2$.

(iv) Let $\Lambda = w_1\mathbb{Z} \oplus w_2\mathbb{Z}$ and let $w_3 = w_1 + w_3$. Then the cubic equation satisfied by $\wp$ and $\wp'$, $y^2 = 4x^3 - g_2(\Lambda)x^2 - g_3(\Lambda)$, is

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3), \quad e_i = \wp(w_i/2), \text{ for } i = 1, 2, 3.$$

Proof. See [Cox] pg. 200-205 for part (i) - (iii) and [DS] pg. 32 for part (iv). □

Part (ii) of Proposition 2.1.3 shows that there exists a map $z \mapsto (\wp(z), \wp'(z))$ mapping non-lattice points of $\mathbb{C}$ to a pair $(x, y) \in \mathbb{C}^2$ satisfying the elliptic equation of part (iv). This map

$$(\wp, \wp') : (\text{ complex torus }) \longrightarrow (\text{ elliptic curve })$$

actually gives a bijection between complex toruses and elliptic curves. Moreover, we explicitly have

Proposition 2.1.4. Let $y^2 = 4x^3 - a_2x - a_3$ be an elliptic curve with $a_2^3 = 27a_3^2 \neq 0$. Then there exists a lattice Λ with $a_2 = g_2(\Lambda)$ and $a_3 = g_3(\Lambda)$.

Proof. See [DS] pg. 35. □

Any map between elliptic curves $(x, y) \mapsto (m^{-2}x, m^{-3}y)$ comes from a holomorphic isomorphism between complex tori $z + \Lambda \mapsto mz + m\Lambda$. Thus, we can interchange elliptic curves with complex tori.

2.1.1 The j-invariant

Definition 2.1.5 (*j-invariant*). The *j-invariant* of a lattice Λ is given by

$$j(\Lambda) := 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2} = 1728 \frac{g_2(\Lambda)^3}{\Delta(\Lambda)},$$

where $\Delta(\Lambda) := g_2(\Lambda)^3 - 27g_3(\Lambda)^2$.

Remark 2.1.6. If Λ is a lattice, then $\Delta(\Lambda) \neq 0$.

Definition 2.1.7. Two lattices Λ, Λ' are called ***homothetic*** if $\Lambda' = \lambda\Lambda$, for some $\lambda \in \mathbb{C} - \{0\}$.

Remark 2.1.8. Homothety is an equivalence relation.

Note that for the *Weierstrass $\wp$-function* one has

$$\wp(\lambda z; \lambda\Lambda) = \lambda^{-2}\wp(z; \Lambda). \tag{3}$$

Moreover, if $f(z)$ is an arbitrary elliptic function for Λ, then $f(\lambda z)$ is an elliptic function for $1/\lambda\Lambda$.

Let Λ, Λ' be two lattices. Using equation (2) for $\lambda\Lambda$ instead of Λ one can see that $g_2(\lambda\Lambda) = \lambda^{-4}g_2(\Lambda)$ and similarly $g_3(\lambda\Lambda) = \lambda^{-6}g_3(\Lambda)$. Using these expressions we deduce that:

$$j(\Lambda) = j(\Lambda') \quad \Longleftrightarrow \quad \Lambda \text{ and } \Lambda' \text{ are homothetic.} \tag{4}$$

So far we have discussed the j-invariant of a lattice, but we can also consider the j-invariant of a complex number $z \in \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. By that we mean to consider the lattice generated by 1 and z, namely $\langle 1, z\rangle$ and then we define the ***j-function*** $j(z)$ as

$$j(z) := j(\langle 1, z\rangle).$$

Theorem 2.1.9. Let $\tau \in \mathcal{H}$. Then the following statements hold:

(i) $j(\tau)$ is a holomorphic function on $\mathcal{H}$,

(ii) if $\tau, \tau' \in \mathcal{H}$, then $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma\tau$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$; in particular $j(\tau)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant,

(iii) $j : \mathcal{H} \longrightarrow \mathbb{C}$ is surjective,

(iv) for $\tau \in \mathcal{H}$ one has $j'(\tau) \neq 0$, except in the following cases:

(a) $\tau = \gamma i$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $j'(\tau) = 0$ but $j''(\tau) \neq 0$,

(b) $\tau = \gamma\omega$, $\omega = e^{2\pi i/3}$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $j'(\tau) = j''(\tau) = 0$ but $j'''(\tau) \neq 0$.

Moreover, one can show that the *q-expansion* of $j(\tau)$ is given by

$$\begin{aligned} j(\tau) &= \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 \dots \\ &= \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n, \end{aligned}$$

where $q := q(\tau) = e^{2\pi i\tau}$, $c_n \in \mathbb{Z}$ for all $n \geq 0$.

The q-expansion of $j(\tau)$ shows that $j(\tau)$ is meromorphic at infinity. This shows that $j(\tau)$ is a *modular function* for $\Gamma_0(N)$, $N \in \mathbb{N}$.

Definition 2.1.10 (*Modular function*)**.** Let $N \in \mathbb{N}$. A complex-valued function $f(\tau)$ defined on the upper-half plane $\mathcal{H}$, except for isolated singularities, is called a ***modular function for*** $\Gamma_0(N)$ if the following hold:

(i) $f(\tau)$ is meromorphic on $\mathcal{H}$,

(ii) $f(\tau) = f(\gamma\tau)$, for all $\tau \in \mathcal{H}$, $\gamma \in \Gamma_0(N)$,

(iii) $f(\tau)$ is meromorphic at ∞, $\forall \gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Remark 2.1.11. We say that $f(\tau)$ is ***holomorphic at*** ∞ if its q-expansion has only non-negative powers of q.

Note that $j(\tau)$ satisfies conditions (i) and (ii) by Theorem 2.1.9.
The j-invariant has an even stronger connection to general modular functions for both $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(N)$. Namely, its the main building block for constructing all modular functions:

Theorem 2.1.12. Let $N \in \mathbb{N}$. Then

(i) $j(\tau)$ is a modular function for $\mathrm{SL}_2(\mathbb{Z})$, and every modular function for $\mathrm{SL}_2(\mathbb{Z})$ is a rational function in $j(\tau)$,

(ii) $j(\tau)$ and $j(N\tau)$ are modular functions for $\Gamma_0(N)$, and every modular function for $\Gamma_0(N)$ is a rational function of $j(\tau)$ and $j(N\tau)$.

Proof: We assume the following lemma:

Lemma 2.1.13. The following two statements hold:

(i) A holomorphic modular function for $\mathrm{SL}_2(\mathbb{Z})$ which is holomorphic at ∞ is constant.

(ii) A holomorphic modular function for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in $j(\tau)$.

Let $f(\tau)$ be an arbitrary modular function for $\mathrm{SL}_2(\mathbb{Z})$, possibly with poles on $\mathcal{H}$. If we can find a polynomial $B(x)$ such that $B(j(\tau))f(\tau)$ is holomorphic on $\mathcal{H}$, then the lemma above will imply that $f(\tau)$ is a rational function in $j(\tau)$. Since $f(\tau)$ has a meromorphic q-expansion, it follows that $f(\tau)$ has only finitely many poles in the region

$$R = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq 1/2, |\mathrm{Im}(\tau)| \geq 1/2\},$$

and since $f(\tau)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, every pole of $f(\tau)$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to one in R. Thus, if $B(j(\tau))f(\tau)$ has no poles in R, then it is holomorphic on the upper half-plane.
So, suppose that $f(\tau)$ has a pole of order m at $\tau_0 \in R$. If $j'(\tau_0) \neq 0$, then $(j(\tau) - j(\tau_0))^m f(\tau)$ is holomorphic at τ_0. In this way we can find a polynomial $B(x)$ such that $B(j(\tau))f(\tau)$ has no poles in R, except possibly for those where $j'(\tau_0) = 0$. When this is the case, by Theorem 2.1.9 we can assume that $\tau_0 = i$ or $\omega = e^{2\pi i/3}$.
When $\tau_0 = i$, we claim that m is even. To see this, note that in a neighborhood of i, $f(\tau)$ can be written in the form

$$f(\tau) = \frac{g(\tau)}{(\tau - i)^m}$$

where $g(\tau)$ is holomorphic and $g(i) \neq 0$. Now $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ fixes i, so that

$$f(\tau) = f(-1/\tau) = \frac{g(-1/\tau)}{(-1/\tau - i)^m}.$$

Comparing these two expressions for $f(\tau)$, we get

$$g(-1/\tau) = \frac{1}{(i\tau)^m} g(\tau)$$

Setting $\tau = i$ implies that $g(i) = (-1)^m g(i)$, and since $g(i) \neq 0$, it follows that m is even. By Theorem 2.1.9, $j(\tau) - 1728$ has a zero of order 2 at i, and hence $(j(\tau) - 1728)^{m/2} f(\tau)$ is holomorphic at i. The argument for $\tau_0 = \omega$ is similar and will be avoided here. This completes the proof of part (i) of Theorem 2.1.12.

Next, we turn to part (ii). We will study the following set of matrices

$$C(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = N, a > 0, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}.$$

The matrix $\sigma_0 = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in C(N)$ has two properties of interest: first, $\sigma_0 \tau = N\tau$, and second,

$$\Gamma_0(N) = \left(\sigma_0^{-1} \mathrm{SL}_2(\mathbb{Z}) \sigma_0\right) \cap \mathrm{SL}_2(\mathbb{Z}).$$

For the number of elements in $C(N)$ one can compute the formula:

$$|C(N)| = N \prod_{p|N} \left(1 + \frac{1}{p}\right). \tag{5}$$

Note that these two properties account for the $\Gamma_0(N)$ invariance of $j(N\tau)$ proved above. More generally, we have the following lemma:

Lemma 2.1.14. For $\sigma \in C(N)$, the set

$$\left(\sigma_0^{-1} \mathrm{SL}_2(\mathbb{Z}) \sigma\right) \cap \mathrm{SL}_2(\mathbb{Z})$$

is a right coset of $\Gamma_0(N)$ in $\mathrm{SL}(2, \mathbb{Z})$. This induces a one-to-one correspondence between right cosets of $\Gamma_0(N)$ and elements of $C(N)$.

We can now compute some q-expansions. Fix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and choose $\sigma \in C(N)$ so that γ lies in the right coset corresponding to σ in Lemma 2.1.14. This means that $\sigma_0 \gamma = \tilde{\gamma} \sigma$ for some $\bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z})$, and hence $j(N\gamma\tau) = j(\sigma_0 \gamma \tau) = j(\tilde{\gamma}\sigma\tau) = j(\sigma\tau)$ since $j(\tau)$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence

$$j(N\gamma\tau) = j(\sigma\tau) \tag{6}$$

Suppose that $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. The q-expansion of $j(\tau)$ is

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} c_n q^n, \quad c_n \in \mathbb{Z},$$

and since $\sigma\tau = (a\tau + b)/d$, it follows that

$$q(\sigma\tau) = e^{2\pi i (a\tau + b)/d} = e^{2\pi i b/d} q^{a/d}.$$

If we set $\zeta_N = e^{2\pi i/N}$, we can write this as

$$q(\sigma\tau) = \zeta_N^{ab} \left(q^{1/N}\right)^{a^2}$$

since $ad = N$. This gives us the q-expansion

$$j(N\gamma\tau) = j(\sigma\tau) = \frac{\zeta_N^{-ab}}{\left(q^{1/N}\right)^{a^2}} + \sum_{n=0}^{\infty} c_n \zeta_N^{abn} \left(q^{1/N}\right)^{a^2 n}, \quad c_n \in \mathbb{Z}. \tag{7}$$

Next, we want to introduce the *modular equation* $\Phi_N(X,Y)$. Let the right cosets of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ be $\Gamma_0(N)\gamma_i, i = 1, \ldots, |C(N)|$. Then consider the polynomial in the variable X

$$\Phi_N(X,\tau) = \prod_{i=1}^{|C(N)|} \left(X - j\left(N\gamma_i\tau\right)\right)$$

We will prove that this expression is a polynomial in X and $j(\tau)$. To see this, consider the coefficients of $\Phi_N(X,\tau)$. Being symmetric polynomials in the $j\left(N\gamma_i\tau\right)$'s, they are also holomorphic. To check invariance under $\mathrm{SL}_2(\mathbb{Z})$, pick $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then the cosets $\Gamma_0(N)\gamma_i\gamma$ are a permutation of the $\Gamma_0(N)\gamma_i$'s, and since $j(N\tau)$ is invariant under $\Gamma_0(N)$, the $j\left(N\gamma_i\gamma\tau\right)$'s are a permutation of the $j\left(N\gamma_i\tau\right)$'s. This shows that the coefficients of $\Phi_N(X,\tau)$ are invariant under $\mathrm{SL}_2(\mathbb{Z})$.
Next, we will show that the coefficients are meromorphic at infinity. Rather than expanding in powers of q, it suffices to expand in terms of $q^{1/N} = e^{2\pi i\tau/N}$ and show that only finitely many negative exponents appear.
By (6), we know that $j\left(N\gamma_i\tau\right) = j(\sigma\tau)$ for some $\sigma \in C(N)$, and then (7) shows that the q-expansion for $j\left(N\gamma_i\tau\right)$ has only finitely many negative exponents. Since the coefficients are polynomials in the $j\left(N\gamma_i\tau\right)$'s, they clearly are meromorphic at the cusps.
This shows that the coefficients of $\Phi_N(X,\tau)$ are holomorphic modular functions, and thus, by Lemma 2.1.13, they are polynomials in $j(\tau)$. This means that there is a polynomial

$$\Phi_N(X,Y) \in \mathbb{C}[X,Y]$$

such that

$$\Phi_N(X, j(\tau)) = \prod_{i=1}^{|C(N)|} \left(X - j\left(N\gamma_i\tau\right)\right).$$

The equation $\Phi_N(X,Y) = 0$ is called the ***modular equation*** or ***modular polynomial***. Using some simple field theory, it can be proved that $\Phi_N(X,Y)$ is irreducible as a polynomial in X.

By (6), each $j\left(N\gamma_i\tau\right)$ can be written as $j(\sigma\tau)$ for a unique $\sigma \in C(N)$. Thus we can also express the modular equation in the form

$$\Phi_N(X, j(\tau)) = \prod_{\sigma\in C(N)} (X - j(\sigma\tau)). \tag{8}$$

Note that $j(N\tau)$ is always one of the $j(\sigma\tau)$'s since $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in C(N)$. Hence $\Phi_N(j(N\tau), j(\tau)) = 0$, and the degree of $\Phi_N(X,Y)$ in X is $|C(N)|$.

Next, let $f(\tau)$ be an arbitrary modular function for $\Gamma_0(N)$. To prove that $f(\tau)$ is a rational function in $j(\tau)$ and $j(N\tau)$, consider the function

$$\begin{aligned} G(X,\tau) &= \Phi_N(X, j(\tau)) \sum_{i=1}^{|C(N)|} \frac{f\left(\gamma_i\tau\right)}{X - j\left(N\gamma_i\tau\right)} \\ &= \sum_{i=1}^{|C(N)|} f\left(\gamma_i\tau\right) \prod_{j\neq i} \left(X - j\left(N\gamma_j\tau\right)\right). \end{aligned} \tag{9}$$

This is a polynomial in X, and one can show that its coefficients are modular functions for $\mathrm{SL}_2(\mathbb{Z})$

(see [Cox] Chap. 11). Once the coefficients are modular functions for $\mathrm{SL}_2(\mathbb{Z})$, they are rational functions of $j(\tau)$ by what we proved above. Hence $G(X,\tau)$ is a polynomial $G(X, j(\tau)) \in \mathbb{C}(j(\tau))[X]$.

We can assume that γ_1 is the identity matrix. By the product rule, we obtain

$$\frac{\partial \Phi_N}{\partial X}(j(N\tau), j(\tau)) = \prod_{j\neq 1} \left(j(N\tau) - j\left(N\gamma_j\tau\right)\right).$$

Thus, substituting $X = j(N\tau)$ in (9) gives

$$G(j(N\tau), j(\tau)) = f(\tau)\frac{\partial \Phi_N}{\partial X}(j(N\tau), j(\tau))$$

Now $\Phi_N(X, j(\tau))$ is irreducible and hence separable, so that

$$\frac{\partial}{\partial X}\Phi_N(j(N\tau), j(\tau)) \neq 0.$$

Thus we can write

$$f(\tau) = \frac{G(j(N\tau), j(\tau))}{\frac{\partial \Phi_N}{\partial X}(j(N\tau), j(\tau))},$$

which proves that $f(\tau)$ is a rational function in $j(\tau)$ and $j(N\tau)$. □

Theorem 2.1.15. Let $N \in \mathbb{N}$. Then:

(i) $\Phi_N(X,Y) \in \mathbb{Z}[X,Y]$,

(ii) $\Phi_N(X,Y)$ is irreducible when regarded as a polynomial in X,

(iii) $\Phi_N(X,Y) = \Phi_N(Y,X)$ if $N > 1$,

(iv) if N is not a perfect square, then $\Phi_N(X,Y)$ is a polynomial of degree > 1 whose leading coefficient is ± 1,

(v) if N is a prime p, then $\Phi_p(X,Y) \equiv (X^p - Y)(X - Y^p) \bmod p\mathbb{Z}[X,Y]$.

Proof. See [Cox] Theorem 11.18. □

Lemma 2.1.16. Let $N \in \mathbb{N}$. If $x, y \in \mathbb{C}$, then $\Phi_N(x,y) = 0$ if and only if there exists a lattice Λ and a cyclic sublattice $\Lambda' \subset \Lambda$ of index N with $x = j(\Lambda')$ and $y = j(\Lambda)$

Proof. See [Cox] Theorem 11.23. □

2.1.2 Modular curves

Consider the congruence subrgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\},$$

which acts on the upper-half plane $\mathcal{H}$. Write $Y_0(N) = \Gamma_0(N) \setminus \mathcal{H}$ for the set of right cosets. Then $Y_0(N)$ is called the *modular curve* for $\Gamma_0(N)$. Similarly one can define modular curves for other congruence subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1.17. An **enchanced elliptic cruve for** $\Gamma_0(N)$ is a pair (E, C), where E is a complex elliptic curve (so a complex torus), and C is a cyclic subrgoup of E of order N.

We say that the enchanced elliptic curves (E, C) and (E', C') are equivalent, and write $(E, C) \sim (E', C')$, if there exists an isomorphism from E to E' taking C to C'. The set of equivalence classes will be denoted by

$$S_0(N) := \{\text{enchanced elliptic curves for } \Gamma_0(N)\}/\!\sim.$$

We will write $[E, C]$ for an element of $S_0(N)$ and call $S_0(N)$ the *space of moduli* or *moduli space*.

Theorem 2.1.18. For $\tau \in \mathcal{H}$ write $\Lambda_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$. Let $N \in \mathbb{N}$. The moduli space for $\Gamma_0(N)$ is

$$S_0(N) = \{[E_\tau, \langle 1/N + \Gamma_\tau \rangle] : \tau \in \mathcal{H}\}.$$

Two points $[E_\tau, \langle 1/N + \Lambda_\tau \rangle]$ and $[E_{\tau'}, \langle 1/N + \Lambda_{\tau'} \rangle]$ are equal if and only if $\Gamma_0(N)\tau = \Gamma_0(\tau')$. Thus we have a bijection

$$\begin{cases} S_0(N) \xrightarrow{\sim} Y_0(N) \\ \left[\mathbb{C}/\Lambda_\tau, \left\langle \frac{1}{N} + \Lambda_\tau \right\rangle\right] \mapsto \Gamma_0(N)\tau. \end{cases}$$

Each isomorphism class of complex elliptic curves has an associated orbit $\mathrm{SL}_2(\mathbb{Z})\tau$ in $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$ and thus we also get a well-defined j-invariant $j(\mathrm{SL}_2(\mathbb{Z})\tau)$. Going from complex tori to elliptic curves, we get that the value $j(\mathrm{SL}_2(\mathbb{Z})\tau)$ is also associated to any complex elliptic curve E in the isomorphism class, and we will denote it by $j(E)$.

The modular curve $Y_0(N)$ classifies pairs (E, E') of elliptic curves together with a cyclic isogeny $\phi : E \longrightarrow E'$ of degree N. According to Theorem 2.1.18, there exist complex tori $\mathbb{C}/M$ and $\mathbb{C}/M'$ corresponding to the elliptic curves E, E', and a point $\tau \in \mathbb{H}$ whose $\Gamma_0(N)$-orbit is well-determined by the point (E, E') in $Y(\mathbb{C})$.

Definition 2.1.19. For $N \in \mathbb{N}$ and $\mathcal{H}^* = \mathcal{H} \cup (\mathbb{Q} \cup \{\infty\})$ define the *modular curve*

$$X_0(N) = \Gamma_0(N) \setminus \mathcal{H}^*.$$

Lemma 2.1.20. The modular curve $X_0(N)$ is Hausdorff, compact and connected.

Proof. See [DS] Prop. 2.4.2. □

Moreover, one can show that $X_0(N)$ is a compact Riemann surface (see [DS] chapter 2.4). Part (ii) of Theorem 2.1.12 tells us that the field of meromorphic functions on $X_0(N)$ is $\mathbb{C}(j(\tau), j(N\tau))$. Then $X_0(N)$ is biholomorphic to the complex points of a non-singular projective curve, say X, which has $\mathbb{C}(j(\tau), j(N\tau))$ as its field of rational functions.

2.2 Complex multiplication

In this subsection we mainly follow [Cox] to introduce order in a quadratic field and the theory complex multiplication.

Definition 2.2.1. An ***order*** $\mathcal{O}$ in a quadratic field K is a subset $\mathcal{O} \subset K$ such that the following hold:

(i) $\mathcal{O}$ is a subring of K containing 1,

(ii) $\mathcal{O}$ is a finitely-generated $\mathbb{Z}$-module,

(iii) $\mathcal{O}$ contains a $\mathbb{Q}$-basis of K.

Remark 2.2.2. The ***maximal order*** will be denoted by $\mathcal{O}_K$.

Lemma 2.2.3. Let $\mathcal{O}$ be an order in a quadratic field K of discriminant d_K. Then $\mathcal{O}$ has finite index in $\mathcal{O}_K$, and if we set $\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}]$, then

$$\mathcal{O} = \mathbb{Z} + \mathfrak{f}\mathcal{O}_K = [1, \mathfrak{f}w_K],$$

with

$$w_K = \frac{d_K + \sqrt{d_K}}{2}$$

being the discriminant of K.

Proof. First, note that $\mathcal{O}$ and $\mathcal{O}_K$ are free $\mathbb{Z}$-modules of rank 2, so $[\mathcal{O}_K : \mathcal{O}] < \infty$. Setting $\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}]$, we have $\mathfrak{f}\mathcal{O}_K \subset \mathcal{O}$, and then $\mathbb{Z} + \mathfrak{f}\mathcal{O}_K \subset \mathcal{O}$ follows. However, since $\mathcal{O}_K = [1, w_K]$, we get that $\mathbb{Z} + \mathfrak{f}\mathcal{O}_K = [1, \mathfrak{f}w_K]$. Hence, to prove the lemma, we have to show that $[1, \mathfrak{f}w_K]$ has index $\mathfrak{f}$ in $\mathcal{O}_K = [1, w_K]$. This last fact is obvious, so we are done. □

Example 2.2.4. For n an integer, $\mathbb{Z}[\sqrt{-n}]$ is an order in $\mathbb{Q}(\sqrt{-n})$.

Remark 2.2.5. Given an order $\mathcal{O}$ in a quadratic field K, the index $\mathfrak{f} = [\mathcal{O}_K : \mathcal{O}]$ is called the *conductor* of the order.

Theorem 2.2.6 (***Complex multiplication***)**.** Let L be a lattice and $\wp(z)$ the $\wp$-function for L. For every $\alpha \in \mathbb{C} - \mathbb{Z}$ the following are equivalent:

(i) $\wp(\alpha z)$ is a rational function in $\wp(z)$,

(ii) $\alpha L \subset L$,

(iii) there exists an order $\mathcal{O}$ in an imaginary quadratic field K such that $\alpha \in \mathcal{O}$ and L is homothetic to a proper fractional $\mathcal{O}$-ideal.

If the above holds, then

$$\wp(\alpha z) = \frac{A(\wp(z))}{B(\wp(z))},$$

where $A(x), B(x)$ are relatively prime polynomials such that

$$\deg(A(x)) = \deg(B(x)) + 1 = [L : \alpha L] = N(\alpha).$$

Proof. See [Cox] Theorem 10.14. □

Remark 2.2.7. Theorem 2.2.6 tells us that, if an elliptic function has multiplication by some $\alpha \in \mathbb{C} - \mathbb{R}$, then it has multiplication by an entire order $\mathcal{O}$ in an imaginary quadratic field.

Corollary 2.2.8. Let $\mathcal{O}$ be an order in an imaginary quadratic field. Then there exists a one-to-one correspondence between the ideal class group $C(\mathcal{O})$ and the homothety classes of lattices with $\mathcal{O}$ as their full ring of complex multiplications.

If $h(\mathcal{O})$ is the *class number*, then by Corollary 2.2.8 it is equal to the number of homothety classes of lattices having $\mathcal{O}$ as their full ring of complex multiplication.

Note that we can now assign an order $\mathcal{O}$ to each elliptic curve.

2.3 Class field theory

The goal of this section is to introduce the necessary tools from class field theory in order to prove that the j-invariant is an algebraic integer. In this section we mainly follow [Cox] Chapters 5,7,8 and 11, and [Janusz] Chapter IV and V.

Let $\mathcal{O}_K$ be a maximal order in a quadratic field K.

Definition 2.3.1. Let K be a number field. A ***modulus*** of K is given by

$$\mathfrak{m} := \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}, \tag{10}$$

where the product is taken over all primes in K and the following three conditions hold:

(i) $n_{\mathfrak{p}} \geq 0$ and at most finitely many are non-zero,

(ii) $n_{\mathfrak{p}} = 0$ whenever $\mathfrak{p}$ is a complex infinite prime,

(iii) $n_{\mathfrak{p}} \leq 1$ whenever $\mathfrak{p}$ is a real infinite prime.

A modulus $\mathfrak{m}$ may be considered as a product $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ with

$$\mathfrak{m}_0 := \prod_{\mathfrak{p}\text{ finite}} \mathfrak{p}^{n_{\mathfrak{p}}}, \quad \mathfrak{m}_\infty := \prod_{\mathfrak{p}\text{ real}} \mathfrak{p}^{n_{\mathfrak{p}}} \tag{11}$$

We will refer to $\mathfrak{m}_0$ as the ***finite part*** of $\mathfrak{m}$, and $\mathfrak{m}_\infty$ as the ***infinite part*** of $\mathfrak{m}$.

Example 2.3.2. Let ∞ denote the unique infinite prime of $\mathbb{Q}$. The expression $\mathfrak{m} = (2)^3 \cdot (17)^2 \cdot (23) \cdot \infty$ is a modulus for $\mathbb{Q}$, where $\mathfrak{m}_0 = (2)^3 \cdot (17)^2 \cdot (23)$ and $\mathfrak{m}_\infty = \infty$.

Next, for a given modulus $\mathfrak{m}$, define $\mathcal{I}_K(\mathfrak{m})$ as the set of fractional $\mathcal{O}_K$-ideals relatively prime to $\mathfrak{m}_0$. Let $\mathcal{P}_{K,1}(\mathfrak{m})$ be a subgroup of $\mathcal{I}_K(\mathfrak{m})$ generated by the principal ideals $\alpha\mathcal{O}_K$, where we take $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv 1 \pmod{\mathfrak{m}}$ and such that for every infinite prime σ dividing $\mathfrak{m}_\infty$ we have $\sigma(\alpha) > 0$.
These two groups allow us to introduce ***congruence subgroups***, which are subgroups $H \subset \mathcal{I}_K(\mathfrak{m})$ such that

$$\mathcal{P}_{K,1}(\mathfrak{m}) \subset H \subset \mathcal{I}_K(\mathfrak{m}). \tag{12}$$

Finally, the ***generalised class group*** is defined as:

$$\mathcal{I}_K(\mathfrak{m})/H.$$

Definition 2.3.3 (***Artin symbol***)**.** Let L/K be an Abelian extension $L \supset K$ and $\mathfrak{m}$ a modulus divisible by all ramified primes of L/K. Given a prime $\mathfrak{p}$ not dividing $\mathfrak{m}$, the *Artin symbol*

$$\left[\frac{L/K}{\mathfrak{p}}\right] \in \mathrm{Gal}(L/K)$$

is given by the unique element $\sigma \in \mathrm{Gal}(L/K)$ such that for every $\alpha \in \mathcal{O}_K$ we have

$$\sigma(\alpha) \equiv \alpha^{N(\mathfrak{p})} \pmod{\mathfrak{p}},$$

and $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$.

Example 2.3.4. Let $p \neq q$ be primes and consider the abelian extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$. The Artin symbol $\left[\frac{\mathbb{Q}(\zeta_p)/\mathbb{Q}}{q}\right]$ coincides with the automorphism $\zeta_p \mapsto \zeta_p^q$ (cf. [FrLe] §3.2)

The *Artin symbol* allows us to define the ***Artin map***:

$$\varphi_\mathfrak{m} : \mathcal{I}_K(\mathfrak{m}) \longrightarrow \mathrm{Gal}(L/K). \tag{13}$$

This map will be useful in the *Existence theorem* (2.3.8), but one has to show that it is a homomorphism. For that we refer to [Cox] Chapter 5. Then *Artin's reciprocity* theorem ([Janusz] pg. 187, Theorem 5.8) shows that $\mathrm{Gal}(L/K)$ is a generalised ideal class group for certain modulus $\mathfrak{m}$ and that there exists an *isomorphism*:

$$\mathcal{I}_K(\mathfrak{m}) \big/ \ker(\varphi_\mathfrak{m}) \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

We will only give references for this here, namely [Janusz] Chap. V, Section 5.

Next, we want to define the *conductor*. An equivalence class of congruence subgroups is called an ***ideal group***.
Let $\mathbf{H}$ be an ideal group and let $\mathfrak{m}$ be a modulus such that there is some congruence subgroup defined (mod $\mathfrak{m}$) which belongs to $\mathbf{H}$. Then there is only one subgroup in $\mathbf{H}$ defined (mod $\mathfrak{m}$) and we denote it by $\mathbf{H}^\mathfrak{m}$.
One can show that whenever $\mathbf{H}^\mathfrak{m}$ and $\mathbf{H}^\mathfrak{n}$ belong to $\mathbf{H}$, then also $\mathbf{H}^{\gcd(\mathfrak{m},\mathfrak{n})} \in \mathbf{H}$. This tells us that there is a *unique* modulus $\mathfrak{f}$ such that

$$\begin{cases} \mathbf{H}^\mathfrak{f} \in \mathbf{H}, \\ \mathbf{H}^\mathfrak{m} \in \mathbf{H} \quad \Longrightarrow \quad \mathfrak{f} \mid \mathfrak{m}, \end{cases} \tag{14}$$

where we take $\mathfrak{f}$ to be the greatest common divisor of all $\mathfrak{m}$ for which $\mathbf{H}$ contains a congruence subgroup defined (mod $\mathfrak{m}$). We call the modulus $\mathfrak{f}$ the ***conductor*** of $\mathbf{H}$.

Definition 2.3.5 (*Reciprocity law*)**.** Let $\mathfrak{m} = \mathfrak{m}_o\mathfrak{m}_\infty$ be a modulus of a number field K. Let $K_{\mathfrak{m},1}$ be the *ray mod* $\mathfrak{m}$ (see [Janusz] Chap. IV, pg. 137). Also denote by $\varphi_\mathfrak{m}$ the Artin map.
We say that the ***reciprocity law*** holds for the triple $(L, K, \mathfrak{m})$ if L is an Abelian extension of K and $\mathfrak{m}$

is a modulus for K such that $\iota(K_{\mathfrak{m},1}) \subset \ker(\varphi_{L/K})$. Here ι sends an element of the group of units of K to the principal ideal it generates in the ring of integers of K.

Definition 2.3.6 (*Class field*)**.** Let L be a finite dimensional, Abelian extension of the number field K. Let $\mathbf{H}(L/K)$ be the ideal group consisting of all congruence subgroups $\mathbf{H}^{\mathfrak{m}}(L/K) = \ker(\varphi_{L/K}) \mid \mathcal{I}_K(\mathfrak{m})$, where we let $\varphi_{L/K}$ act on $\mathcal{I}_K(\mathfrak{m})$, with $\mathfrak{m}$ selected so that the reciprocity law holds for $(L, K, \mathfrak{m})$.
We call $\mathbf{H}(L/K)$ the ***class group*** of the extension L of K and L is called the ***class field*** to the ideal group $\mathbf{H}(L/K)$. The conductor of $\mathbf{H}(L/K)$ is denoted by $\mathfrak{f}(L/K)$ and is called the ***conductor of the extension*** L of K.

Theorem 2.3.7. Let $\mathfrak{f}(L/K)$ be the conductor of the Abelian extension L of K. Then $\mathfrak{f}(L/K)$ is divisible by every prime of K that ramifies in L and moreover, the reciprocity law holds for the triple $(L, K, \mathfrak{f}(L/K))$.

Proof. See [Janusz] Chapter V, 11.11 (a). □

Proposition 2.3.8. Let $K \subset L$ be an Abelian extension. Let $\mathfrak{m}$ be a modulus of K, and let $\mathbf{H}$ be a congruence subgroup for $\mathfrak{m}$. Then there exists a unique Abelian extension L of K, all of whose ramified primes (finite or infinite), divide $\mathfrak{m}$, such that if

$$\varphi_{\mathfrak{m}} : \mathcal{I}_K(\mathfrak{m}) \longrightarrow \mathrm{Gal}(L/K) \tag{15}$$

is the *Artin map* of $K \subset L$, then

$$\mathbf{H} = \ker(\varphi_{\mathfrak{m}}).$$

Proposition 2.3.8 is a consequence of one of the main theorems in class field theory, namely the *Existence theorem*:

Theorem 2.3.9 (***Existence theorem***)**.** Let K be any algebraic number field. The correspondence

$$L \longrightarrow \mathbf{H}(L/K)$$

is a one-to-one, inclusion preserving, correspondence between finite dimensional, Abelian extensions L of K and ideal groups of K.

Proof. See [Janusz] Chapter V, Theorem 9.9, pg. 215. □

Corollary 2.3.10. Let L and M be Abelian extensions of K. Then $L \subset M$ if and only if there is a moduls $\mathfrak{m}$, divisible by all primes of K ramified in either L or M, such that

$$\mathcal{P}_{K,1}(\mathfrak{m}) \subset \ker(\varphi_{M/K,\mathfrak{m}}) \subset \ker(\varphi_{L/K,\mathfrak{m}}). \tag{16}$$

Proof. See [Cox] pg. 163, Corollary 8.7. □

Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}$ in an imaginary quadratic field K. Let $\mathcal{P}_{K,\mathbb{Z}}(\mathfrak{f})$ be the subgroup of $\mathcal{I}_K(\mathfrak{f})$ generated by ideals of the form $\alpha\mathcal{O}_K$, where $\alpha \in \mathcal{O}_K$ satisfies $\alpha \equiv a \pmod{\mathfrak{f}}$ for some integer a relatively prime to $\mathfrak{f}$. Then the ideal class group can be written as:

$$C(\mathcal{O}) \simeq \mathcal{I}_K(\mathfrak{f}) \Big/ \mathcal{P}_{K,\mathbb{Z}}(\mathfrak{f}),$$

and moreover one has

$$\mathcal{P}_{K,1}(\mathfrak{f}) \subset \mathcal{P}_{K,\mathbb{Z}}(\mathfrak{f}) \subset \mathcal{I}_K(\mathfrak{f}).$$

By the *Existence theorem* (2.3.8) we know that the above data yields a unique Abelian extension L of K, and we will call that extension the ***ring class field*** of the order $\mathcal{O}$.

Definition 2.3.11. Let K be an imaginary quadratic field and let $\mathcal{O}_K$ be its ring of integers. For $n \in \mathbb{N}$ prime to N, define K_n to be the ring class field of conductor n over K.

Definition 2.3.12. Let X be a ringed space. Then the **Picard group** is given by the sheaf cohomology group

$$\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^*).$$

For an order $\mathcal{O}$ in an imaginary quadratic field K, the Picard group $\mathrm{Pic}(\mathcal{O})$ is just the *ideal class group*.

If $\mathcal{O}_n := \mathbb{Z} + n\mathcal{O}_K$ is the order of index n in $\mathcal{O}_K$, we similarly have

$$\mathrm{Pic}(\mathcal{O}_n) \simeq \mathrm{Gal}(K_n/K). \tag{17}$$

We can now determine the Galois groups for the ring class fields K_n over K. For that the next two results come in handy:

Lemma 2.3.13. Let L be the ring class field of an order $\mathcal{O}$ in an imaginary quadratic field K. Then L is a Galois extension of $\mathbb{Q}$, and its Galois group can be written as a semidirect product

$$\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(L/K) \rtimes (\mathbb{Z}/2\mathbb{Z})$$

where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Gal}(L/K)$ by sending σ to its inverse σ^{-1}.

Proof. See [Cox] Lemma 9.3. □

Proposition 2.3.14. For $n \in \mathbb{N}$ prime to N one has the exact sequence

$$1 \longrightarrow \mathcal{O}_n^\times \longrightarrow \mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/n\mathcal{O}_K)^\times \Big/ (\mathcal{O}_n/n\mathcal{O}_n)^\times \longrightarrow \mathrm{Pic}(\mathcal{O}_n) \longrightarrow \mathrm{Pic}(\mathcal{O}_K) \longrightarrow 1.$$

Proof. See [Neukirch] Prop. 12.9. and Prop. 12.11. □

We have already seen that $\mathrm{Gal}(K_1/K) = \mathrm{Pic}(\mathcal{O}_K)$. If τ denotes complex conjugation, we have $\mathrm{Gal}(K/\mathbb{Q}) = \langle 1, \tau \rangle$. From the exact sequence in 2.3.14 together with $\mathrm{Gal}(K_n/K) = \mathrm{Pic}(\mathcal{O}_n)$ one obtains

$$\mathrm{Gal}(K_n/K_1) \simeq (\mathcal{O}_K/n\mathcal{O}_K)^\times/(\mathbb{Z}/n\mathbb{Z})^\times. \tag{18}$$

Let l be a prime of n. We want to compute the order of the Galois group $\mathrm{Gal}(K_l/K_1)$ exactly. If (l) is inert, we have $\mathcal{O}_K/(l) \simeq \mathbb{F}_{l^2}$, and

$$\begin{aligned}|G_l| = |\mathrm{Gal}(K_l/K_1)| &= |(\mathcal{O}_K/l\mathcal{O}_K)^\times|/|(\mathbb{Z}/l\mathbb{Z})^\times| \\ &= |\mathbb{F}_{l^2}^\times|/(l-1) \\ &= (l^2-1)/(l-1) \\ &= l+1.\end{aligned}$$

But, for (l) split, $\mathcal{O}_K/(l) \simeq \mathbb{F}_l[x]^2$, and we compute:

$$\begin{aligned}|G_l| &= |(\mathcal{O}_K/l\mathcal{O}_K)^\times|/|(\mathbb{Z}/l\mathbb{Z})^\times| \\ &= |(\mathbb{F}_l[x]^2)^\times|/(l-1) \\ &= (l-1)^2/(l-1) \\ &= l-1.\end{aligned}$$

Theorem 2.3.15 (***Čebotarev density theorem***)**.** Let L be a Galois extension of K, and let $\langle\sigma\rangle$ be the conjugacy class of an element $\sigma \in \mathrm{Gal}(L/K)$. Then the set

$$\mathcal{S} = \left\{ \mathfrak{p} \in \mathcal{P}_K : \mathfrak{p} \text{ is unramified in L and } \left[\frac{L/K}{\mathfrak{p}}\right] = \langle\sigma\rangle \right\} \tag{19}$$

has Dirichlet density

$$\delta(\mathcal{S}) = \frac{|\langle\sigma\rangle|}{|\mathrm{Gal}(L/K)|} = \frac{|\langle\sigma\rangle|}{[L:K]}.$$

Proof. See [Janusz] Chapter V, Theorem 10.4. □

The quotient $\mathcal{I}_K(\mathfrak{m})/\mathcal{P}_{K,1}(\mathfrak{m})$ is called the **ray class group** $\pmod{\mathfrak{m}}$. A **ray class field** of K is the abelian extension of K associated to a ray class group by class field theory, such that its Galois group is isomorphic to the corresponding ray class group.

Proposition 2.3.16 (Example of a ray class field)**.** Let $m \in \mathbb{N}$ be odd or divisible by 4. The ray class field for (m) is $\mathbb{Q}[\zeta_m + \overline{\zeta}_m]$, and the ray class field for $\infty(m)$ is $\mathbb{Q}[\zeta_m]$.

Proof. Indeed, let $C_{\mathfrak{m}}$ be the ray class group, let $U = \mathcal{O}_K^{\times}$ be the group of units in K and denote by C the full ideal class group. There is a short exact sequence

$$0 \longrightarrow U/U_+ \longrightarrow K^{\times}/K_+ \longrightarrow C_{\mathfrak{m}} \longrightarrow C \longrightarrow 0, \tag{20}$$

where K_+ is the group of all elements and U_+ is the group of all units which are positive under all real embeddings. For teh field $\mathbb{Q}$ and the modulus (m), the sequence (20) becomes:

$$0 \longrightarrow \{\pm 1\} \longrightarrow (\mathbb{Z}/m\mathbb{Z})^{\times} \longrightarrow C_{\mathfrak{m}} \longrightarrow 0,$$

and for $\infty(m)$ it becomes

$$0 \longrightarrow \{\pm 1\} \longrightarrow \{\pm 1\} \times (\mathbb{Z}/m\mathbb{Z})^{\times} \longrightarrow C_{\mathfrak{m}} \longrightarrow 0.$$

If p is a prime such that $p \nmid m$, then $\sigma := \left[\dfrac{\mathbb{Q}[\zeta_m]/\mathbb{Q}}{p}\right]$ is the unique element of $\mathrm{Gal}(\mathbb{Q}[\zeta_m]/\mathbb{Q})$ such that

$$\sigma\alpha \equiv \alpha^p \pmod{\mathfrak{p}}, \text{ for all } \alpha \in \mathbb{Z}[\zeta_m],$$

for any prime ideal $\mathfrak{p}$ lying above p. Moreover, σ is the element of the Galois group such that $\sigma(\zeta_m) = \zeta_m^p$. From the short exact sequence above we obtain an explicit description for the $C_{\mathfrak{m}}$'s, from which we obtain ray class fields by considering the Frobenius element described above. □

2.3.1 Hilbert class field of an imaginary quadratic field

The ***Hilbert class field*** of K, denoted K_1, is the maximal abelian unramified extension of K.

Theorem 2.3.17. The Galois group $\mathrm{Gal}(K_1/K)$ is isomorphic to the class group C_K of K. In particular, the class number of K satisfies $h_K = [K_1 : K]$.

Proof. See [Janusz] Theorem 12.1. □

Given an order $\mathcal{O}$ in an imaginary quadratic field K, we can now determine the ray class field of the order $\mathcal{O}$ explicitly.

Theorem 2.3.18 (***j-invariant is an algebraic integer***)**.** Let $\mathcal{O}$ be an order in an imaginary quadratic field K and $\mathfrak{a}$ a proper fractional $\mathcal{O}$-ideal. Then $j(\mathfrak{a})$ is an algebraic integer and $K(j(\mathfrak{a}))$ is the ring class field of the order $\mathcal{O}$.

Proof. See [Cox] Theorem 11.1. □

Remark 2.3.19. Since the $j(\mathfrak{a})$'s are algebraic integers and hence are roots of the same irreducible polynomial over $\mathbb{Q}$, one can show that they form a full set of Galois conjugates. Taking the sum of those $j(\mathfrak{a})$'s leaves us with the trace.

Corollary 2.3.20. If K is an imaginary quadratic field, then $K(j(\mathcal{O}_K))$ is the Hilbert class field of K.

Proof. See [Cox] Theorem 11.34. □

Corollary 2.3.21. Let K be an imaginary quadratic field, and let $K \subset L$ be a finite extension. Then L is an abelian extension of K which is generalized dihedral over $\mathbb{Q}$ if and only if there is an order $\mathcal{O}$ in K such that $L \subset K(j(\mathcal{O}))$.

Proof. See [Cox] Corollary 11.35. ☐

Theorem 2.3.22. Let $\mathcal{O}$ be an order in an imaginary quadratic field K, and let L be the ring class field of $\mathcal{O}$. If $\mathfrak{a}$ is a proper fractional $\mathcal{O}$-ideal and $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, then

$$\left[\frac{L/K}{\mathfrak{p}}\right](j(\mathfrak{a})) = j(\overline{\mathfrak{p}\cap\mathcal{O}}\mathfrak{a}).$$

Proof. See [Cox] Theorem 11.36. ☐

Corollary 2.3.23. Let $\mathcal{O}$ be an order in an imaginary quadratic field K, and let L be the ring class field of $\mathcal{O}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be proper fractional $\mathcal{O}$-ideals, and define $\sigma_{\mathfrak{a}}(j(\mathfrak{b}))$ by

$$\sigma_{\mathfrak{a}}(j(\mathfrak{b})) = j(\overline{\mathfrak{a}}\mathfrak{b}).$$

Then $\sigma_{\mathfrak{a}}$ is a well-defined element of $\mathrm{Gal}(L/K)$, and $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$ induces an isomorphism

$$C(\mathcal{O}) \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

Proof. See [Cox] Corollary 11.37. ☐

3 Heegner points à la Gross

The aim of this Section is to introduce Heegner points as they were defined by Gross. For that we mainly follow [Gross], but also rely on [DS], [Cox], [Gross4] and [Janusz]. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with $D \neq 3, 4$. Let $N \in \mathbb{N}$ be such that all prime factors of N are split.

Definition 3.0.1. A *point* on the curve $X_0(N)$ is given by a pair (E, E') of elliptic curves and a cyclic N-isogeny $\phi : E \longrightarrow E'$. We will represent a *point* y by the diagram

$$(E \xrightarrow{\phi} E') = y.$$

We say that two diagrams represent the same point if they are isomorphic. The ring $\mathrm{End}(y)$ corresponding to the point y is given by pairs $(\alpha, \beta) \in \mathrm{End}(E) \times \mathrm{End}(E')$ which satisfy a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ E & \xrightarrow{\phi} & E' \end{array}$$

Let $\mathcal{O}$ be an order in an imaginary quadratic field K and let $\mathcal{O}_K$ be its full ring of integers. When $\mathrm{End}(y) \simeq \mathcal{O}$ we say that y has complex multiplication. Assume that this is the case. By Lemma 2.2.3 we know that $\mathcal{O} = \mathbb{Z} + \mathfrak{f}\mathcal{O}_K$, and the discriminants of $\mathcal{O}_K$ and $\mathcal{O}$ are d_K and $D = d_K c^2$ respectively.

Definition 3.0.2 (Heegner points). For $N \in \mathbb{N}$, consider the modular curve $X_0(N)$. **Heegner points** of $X_0(N)$ are pairs (E, E') of N-isogenous curves with the same ring $\mathcal{O}$ of complex multiplication.

Thus, y is a Heegner point if $\mathrm{End}(y) = \mathcal{O}$ and the conductor $\mathfrak{f}$ of $\mathcal{O}$ is relatively prime to N. This forces an equality: $\mathrm{End}(y) = \mathrm{End}(E) = \mathrm{End}(E')$. We have seen in Section 2.1.1 that the j-invariant classifies elliptic curves up to isomorphism. By Section 2.1 we can assign to each elliptic curve E a lattice Λ, and accordingly define $j(E) = j(\Lambda)$. Moreover, by Corollary 2.2.8, we have a correspondence between elliptic curves with complex multiplication and the ideal class group $\mathrm{Pic}(\mathcal{O})$ as follows:

$$\mathbb{C}/\mathfrak{a} \mapsto \mathfrak{a}.$$

From Theorem 2.3.17, we know that

$$\mathrm{Pic}(\mathcal{O}_K) \simeq \mathrm{Gal}(K(j(\mathcal{O}_K))/K). \tag{21}$$

For $n \in \mathbb{N}$ prime to N, we recall that K_n is the ring class field of conductor n over K. Let $\mathcal{O}_n$ be the order of index n in $\mathcal{O}_K$.

Proposition 3.0.3. Let $\mathcal{N}$ be an ideal of $\mathcal{O}_K$, with $\mathcal{O}/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$ and consider the Heegner point $x_1 = (\mathbb{C}/\mathcal{O}_K \longrightarrow \mathbb{C}/\mathcal{N}^{-1})$. Then x_1 is rational over the Hilbert class field $K_1 = K(j(\mathcal{O}_K))$.

Proof. Recall the results at the end of Section 2.1.1. Lemma 2.1.16 implies that $\Phi_N(j(\mathcal{O}_K), j(\mathcal{N}^{-1})) = 0$, so we can consider the point x_1 on $X_0(N)$ to have coordinates

$$(j(\mathcal{O}_K), j(\mathcal{N}^{-1})).$$

And since $K(j(\mathcal{O}_K)) = K(j(\mathcal{N}^{-1}))$ (by Theorem 2.3.18), $x_1 \in X_0(N)(K_1)$. □

We can extend this to ring class fields over K. Let $\mathcal{N}_n := \mathcal{N} \cap \mathcal{O}_n$, then $\mathcal{N}_n$ is an invertible $\mathcal{O}_n$-module with $\mathcal{O}_n/\mathcal{N}_n \simeq \mathbb{Z}/N\mathbb{Z}$. Similarly as before we obtain a Heegner point

$$x_n := (\mathbb{C}/\mathcal{O}_n \longrightarrow \mathbb{C}/\mathcal{N}_n^{-1})$$

on $X_0(N)$.

Proposition 3.0.4. The Heegner point x_n is rational over the ring class field K_n, so $x_n \in X_0(N)(K_n)$.

From the discussion in 2.3.19 we know that the h_K values of the j-invariant form a full set of Galois conjugates. Using (21) together with Corollary 2.3.23 one obtains for $\sigma \in \mathrm{Gal}(K_1/K)$:

$$\begin{aligned} \sigma x_1 &= \sigma(\mathbb{C}/\mathcal{O}_K \longrightarrow \mathbb{C}/\mathcal{N}^{-1}) \\ &= (\mathbb{C}/\mathfrak{a}^{-1} \longrightarrow \mathbb{C}/\mathcal{N}^{-1}\mathfrak{a}^{-1}), \end{aligned}$$

where σ corresponds to $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)$. Or in other words,

$$\sigma(j(\mathcal{O}_K), j(\mathcal{N}^{-1})) = (j(\mathfrak{a}^{-1}), j(\mathcal{N}^{-1}\mathfrak{a}^{-1})).$$

Taking the sum of the σx_1's over the whole Picard group (i.e. *taking the trace*), leaves us with a point in $X_0(N)(K)$.

3.1 The Jacobian

In this section we rely on the terminology introduced by [DS].

Definition 3.1.1. Let X be a complex elliptic curve $\mathbb{C}/\Gamma$. The **Jacobian** of X is defined as

$$\mathrm{Jac}(X) = \Omega^1_{\mathrm{hol}}(X)^\wedge / H_1(X, \mathbb{Z}).$$

We will often write $J_0(N) := \mathrm{Jac}(X_0(N))$.

Theorem 3.1.2 (*Mordell-Weil*)**.** For elliptic curves over the rationals $\mathbb{Q}$, the group of rational points is always finitely generated.

Proof. See [SiTa] §3.5. □

Theorem 3.1.3. The Jacobian associated to $\Gamma_1(n)$ is isogenous to a direct sum of Abelian varieties associated to equivalence classes of newforms.

Proof. See [DS] Chapter 6.6 Theorem 6.6.6. □

Thus by restricting to $\Gamma_0(N)$ we get that $J_0(N)$ is an abelian variety of dimension g, for $g = \text{genus}(X)$. By Theorem 3.1.2, the group of points $J(K(j(\mathcal{O})))$ is finitely-generated over $K(j(\mathcal{O}))$.

If we define

$$\text{Div}^0(X) := \left\{ \sum_{x \in X} n_x x : n_x \in \mathbb{Z}, n_x = 0 \text{ for almost all } x, \sum_x n_x = 0 \right\},$$

$$\text{Div}^l(X) := \{\delta \in \text{Div}^0(X) : \delta = \text{div}(f), \text{ for some } f \in \mathbb{C}(X)\}, \quad \text{div}(f) = \sum v_x(f)x.$$

Definition 3.1.4. Then the (degree-0) *Picard group* of X is defined as

$$\text{Pic}^0(X) := \text{Div}^0(X)/\text{Div}^l(X).$$

Moreover, we have the following theorem

Theorem 3.1.5 (*Abel*)**.** The map

$$\begin{cases} \text{Div}^0(X) \longrightarrow \text{Jac}(X) \\ \sum_x n_x x \mapsto \sum_x n_x \int_{x_0}^x \end{cases}$$

descends to divisor classes, including an isomorphism

$$\begin{cases} \text{Pic}^0(X) \xrightarrow{\sim} \text{Jac}(X) \\ \left[\sum_x n_x x\right] \mapsto \sum_x n_x \int_{x_0}^x \end{cases}$$

Proof. See [DS] Theorem 6.1.2. □

3.1.1 Modular parametrization

Theorem 3.1.6 (Eichler-Shimura). Let $f(\tau) = \sum_{n=1}^{\infty} c_n q^n$ be a newform in $S_2(\Gamma_0(N))$ normalized to have $c_1 = 1$, and suppose that all the c_n are in $\mathbb{Z}$. Then there exists a pair (E, ν) such that:

1. E is an elliptic curve defined over $\mathbb{Q}$.
2. E is a quotient of $\text{Jac}(X_0(N))$ by $(T_p - c_p \cdot \text{ id})\,\text{Jac}$, for all p, so that T_p acts on E as multiplication by the integers c_p, for all p.
3. The differential $\sum c_n q^n dq/q$ associated to f is a nonzero multiple of $\nu^*(\omega)$, where ω is the invariant differential of E.
4. If

$$\Lambda_f = \left\{ \int_{\tau_0}^{\gamma(\tau_0)} f(\xi)d\xi \mid \gamma \in \Gamma_0(N) \right\}$$

then Λ_f is a lattice in $\mathbb{C}$, and E is isomorphic to $\mathbb{C}/\Lambda_f$ over $\mathbb{C}$.

5. $L(s, E)$ coincides with $L(s, f)$, for almost all c_p.

Proof. See [DS] Chapter 8. □

Composing ν with the inclusion $X_0(N) \hookrightarrow \text{Jac}(X_0(N))$, one obtains a map

$$\varphi : X_0(N) \longrightarrow E,$$

over $\mathbb{Q}$. This map φ is called the ***modular parametrization*** of E.

Theorem 3.1.7 (*Modularity*)**.** Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. Then there exists a newform $f \in \text{Sl}_2(\Gamma_0(N))$ such that

$$L(s, f) = L(s, E).$$

Proof. See [DS] Theorem 8.8.3. □

Let E be a modular elliptic curve of conductor N over $\mathbb{Q}$. Fix a parametrization φ, which maps the cusp ∞ of $X_0(N)$ to the origin of E. Then there exists a unique differential w on E over $\mathbb{Q}$ such that $\varphi^*(w)$ is the differential $\sum a_n q^n \frac{dq}{q}$ associated to a normalized newform on $X_0(N)$.

Now, consider again the Heegner point x_1 we discussed in the previous Section. Since $x_1 \in X_0(N)(K_1)$, we have that $y_1 := \varphi(x_1) \in E(K_1)$. Similarly, $y_n := \varphi(x_n) \in E(K_n)$ and $y_K := \mathrm{Tr}_{K_1/K}(y_1) \in E(K)$ (where we add y_1 to its h_K conjugates, using the group law on E).

4 Euler system of Heegner points

Euler systems are tools for studying and controlling Selmer groups. Roughly speaking, an Euler system is a collection of points which live in cohomology groups satisfying certain relations between them; and these relations are called *norm relations*. In this Chapter we will first prove norm relations for the set of Heegner points we defined earlier and then introduce Kolyvagin's cohomology classes for our Heegner points. For that we mainly rely on [GrossK], [Serre], [Milne] and [Darmon]. From now on let l be a prime satisfying the Heegner condition (i.e. it does not divide $N \cdot D \cdot p$, for N, D, p defined in Section 3).

4.1 Hecke correspondence

Let E be a set and let X_E be the free abelian group generated by E. A *correspondence* on E is a homomorphism T of X_E into itself. We denote by T_l the correspondence on the set $\mathcal{R}$ of lattices of $\mathbb{C}$ which transforms a lattice Λ to the sum of its sub-lattices of index l. Then we have:

$$T_l \Lambda = \sum_{[\Lambda:\Lambda']=l} \Lambda'.$$

This action of T_l can be extended to an action on modular forms. Let F be a numerically valued function on E. We define the Hecke operator T_l acting on the space of homogeneous functions of weight k by:

$$(T_l F)(\Lambda) = l^{k-1} \sum_{\substack{\Lambda' \subset \Lambda \\ [\Lambda:\Lambda']=l}} F(\Lambda').$$

Then we define an action of T_l on the space of modular forms by

$$F_{T_l f} = T_l F_f,$$

where $F_f(\Lambda) = \omega_2^{-k} f(\omega_1/\omega_2)$, for $\Lambda = [\omega_1, \omega_2]$. The precise formula works out to be:

$$T_n f(z) = n^{k-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_n} f(\gamma z)(cz+d)^{-k},$$

where M_l is the set of matrices in $M_2(\mathbb{Z})$ of determinant l.

This means that we can consider T_l as a double coset operator on elements in $\mathcal{H}$. This gives us a modular curve interpretation of $T_l : \mathrm{Div} X_0(N) \longrightarrow \mathrm{Div} X_0(N)$. We know that modular curves correspond to moduli spaces. Let $[E, C] \in S_0(N)$, then

$$T_l : [E, C] \mapsto \sum_{\substack{[E[l]:\Lambda]=l \\ C \cap \Lambda = \{0\}}} [E/\Lambda, (C+\Lambda)/\Lambda].$$

We could have alternatively considered this sum as

$$\sum_{\phi : E \to E'} (E', \phi(C)),$$

where the sum runs over the l-th isogenies $\phi : E \longrightarrow E'$. The correspondence described above is called the **Hecke correspondence**. It allows us to define a Hecke operator on Heegner points. Let $x_m := (E \longrightarrow E')$ (together with an m-isogeny) be a Heegner point for $X_0(N)(K_m)$. We define a formal sum

$$T_l x_m = T_l(E \longrightarrow E') = \sum_{\substack{[E[l]:\Lambda]=l \\ C \cap \Lambda = \{0\}}} (E/\Lambda \longrightarrow E'/\Lambda).$$

If we write $E = \mathbb{C}/\Lambda_\tau$ and $C = \langle 1/N + \Lambda_\tau \rangle$, we get explicitly

$$T_l[\mathbb{C}/\Lambda_\tau, \langle 1/N + \Lambda_\tau \rangle] = \sum_{\mu=1}^{l-1} [\mathbb{C}/\Lambda_{(\tau+\mu)/l}, \langle 1/N + \Lambda_{(\tau+\mu)/l} \rangle] + [\mathbb{C}/\Lambda_{l\tau}, \langle 1/N + \Lambda_{l\tau} \rangle].$$

Definition 4.1.1. Let C be a curve defined over an algebraically closed field k of characteristic $p \neq 0$. Assume C is given by equations $\sum c_{i_0,i_1,\ldots} X_0^{i_0} X_1^{i_1} \cdots = 0$, and q is a power of p. Then $C^{(q)} = \sum c^q_{i_0,i_1,\ldots} X_0^{i_0} X_1^{i_1} \cdots = 0$. The **Frobenius map** $\mathrm{Fr}_q : C \longrightarrow C^{(q)}$ sends the point $(a_0 : a_1 : \ldots)$ to $(a_0^q : a_1^q : \ldots)$.

We want to consider the Frobenius map

$$\mathrm{Fr}_p : X_0(N) \longrightarrow X_0(N)$$

over $\mathbb{F}_p$. Denoting the map Fr_p also on divisors, we have

$$\mathrm{Fr}_p([E, C]) = [E^{(p)}, C^{(p)}].$$

We define the transpose of this map Fr_p^t as the correspondence in the opposite direction, i.e. the correspondence with respect to the transpose of the graph of Fr_p. On divisors we get

$$\mathrm{Fr}_p^t([E, C]) = \sum_{\substack{(E')^{(p)} = E \\ \mathrm{Fr}_p : E' \to E}} [E', \mathrm{Fr}_p^{-1}(C)]$$

Theorem 4.1.2 (**Eichler-Shimura congruence relation**)**.** For a prime p where $X_0(N)$ has good reduction one has

$$T_p = \mathrm{Fr}_p + \mathrm{Fr}_p^t,$$

over $\mathbb{F}_p$.

Proof. See [Milne] Theorem 10.3. □

4.2 Norm relations

Let l be a prime of n (as before, satisfying the Heegner condition). For now, assume further that the prime (l) remains inert in K: let $\mathrm{Frob}(l)$ be the conjugacy class in $\mathrm{Gal}(K(E[p])/\mathbb{Q})$ containing the Frobenius substitutions of the prime factors of l, such that

$$\mathrm{Frob}(l) = \mathrm{Frob}(\infty) \quad (\text{ as conjugacy classes in } \mathrm{Gal}(K(E[p])/\mathbb{Q})), \tag{22}$$

where $\mathrm{Frob}(\infty)$ is the conjugacy class of complex conjugation. Then equation (22) implies $\mathrm{Frob}(l) = \tau$ in $\mathrm{Gal}(K/\mathbb{Q})$ and (l) is inert.

Let K_l, K_1 be the ring glass fields from Section 3. Then the Galois group $G_l := \mathrm{Gal}(K_l/K_1)$ is cyclic, say generated by σ_l. Let $n \geq 1$ be square-free. We define

$$G_n = \prod_{l|n} G_l.$$

For $n = ml$ we define an operator:

$$\mathrm{Tr}_l := \sum_{\sigma \in G_l} \sigma = \sum_{i=0}^{l} \sigma_l^i,$$

which is an operator on $\mathrm{Div} X_0(N)(K_{ml}) \longrightarrow \mathrm{Div} X_0(N)(K_m)$.

Let x_n, x_m be Heegner points. Then $\mathcal{O}_n/\mathcal{O}_m$ is isomorphic to a cyclic subgroup of $\mathbb{C}/\mathcal{O}_n$ of order l.

We denote $C := (\mathbb{C}/\mathcal{O}_n)$. Then

$$\begin{aligned}\mathrm{Tr}_l(x_n) = \sum_{\sigma\in G_l} \sigma x_n &= \sum_{\sigma\in G_l} \sigma(\mathbb{C}/\mathcal{O}_n \longrightarrow \mathbb{C}/\mathcal{N}_n^{-1}) \\ &= \sum_{\sigma\in G_l} \sigma((\mathbb{C}/\mathcal{O}_m)/(\mathcal{O}_n/\mathcal{O}_m) \longrightarrow (\mathbb{C}/\mathcal{N}_m^{-1})/(\mathcal{O}_n/\mathcal{O}_m)) \\ &= \sum_{\sigma\in G_l} \sigma((\mathbb{C}/\mathcal{O}_m)/C \longrightarrow (\mathbb{C}/\mathcal{N}_m^{-1})/C) \\ &= \sum((\mathbb{C}/\mathcal{O}_m)/(C\mathfrak{a}^{-1}) \longrightarrow (\mathbb{C}/\mathcal{N}_m^{-1})/(C\mathfrak{a}^{-1})).\end{aligned}$$

We have that $C\mathfrak{a}^{-1}$ is again a cyclic subgroup of order l of $\mathbb{C}/\mathcal{O}_n$, so

$$\mathrm{Tr}_l(x_n) = T_l(x_m). \tag{23}$$

Here one can pose a natural question, which is *what happens if the prime* (l) *was not inert?* Assume (l) is split and write $l = \lambda\bar{\lambda}$. We still assume n to be square-free (so that $l \nmid \frac{n}{l}$). This means that $\mathcal{O}_K/(l) \simeq \mathbb{F}_l[x]^2$, whereas for the case of (l) inert, we had $\mathcal{O}_K/(l) \simeq \mathbb{F}_{l^2}$. Note that by (18),

$$G_l := \mathrm{Gal}(K_l/K_1) \simeq (\mathcal{O}_K/l\mathcal{O}_K)^\times/(\mathbb{Z}/l\mathbb{Z})^\times.$$

Recall from 2.3 that the size of the Galois group G_l depends on the prime l:

$$|G_l| = \begin{cases} l+1, & (l) \text{ inert} \\ l-1, & (l) \text{ split.}\end{cases}$$

This means that our Trace operator, $\mathrm{Tr}_l = \sum_{\sigma\in G_l}\sigma$, has two less terms in the split case. Also, the Galois group G_l is isomorphic to $(\mathbb{F}_l^2)^\times/\mathbb{F}_l^\times \simeq \mathbb{F}_l^\times$. Let $\{C_i\}_{i=0}^l$ denote the $l+1$ cyclic subgroups of order l of $E[l]$, where $C_0 := E[\lambda]$ and $C_l := E[\bar{\lambda}]$. Then G_l acts transitively on the cyclic groups $C_1, \dots, C_{l-1}$. Contrary to the inert case, the groups C_0 and C_l remain invariant. From our discussion of the Hecke correspondence in Section 4.1 we obtain:

$$T_l(x_m) = \mathrm{Tr}_l(x_n) + \sigma_l x_m + \sigma_l^{-1} x_m,$$

i.e.

$$\mathrm{Tr}_l(x_n) = (T_l - \sigma_l - \sigma_l^{-1})x_m.$$

Let $\varphi : X_0(N) \longrightarrow E$ be the modular parametrization of E from Section 3.1.1, which sends ∞ to 0. By Theorem 3.1.6 2. the operator T_l acts on an elliptic curve E as multiplication by a_l and thus for any divisor D on $X_0(N)$ we have $\varphi(T_l D) = a_l\varphi(D)$. After applying our modular parametrization on equation (23) we obtain the following result:

Proposition 4.2.1 (*Norm relations for* y_n)**.** Let $n = ml$. Then

$$\mathrm{Tr}_l y_n = \begin{cases} a_l \cdot y_m, & (l) \text{ inert,} \\ (a_l - \sigma_l - \sigma_l^{-1})\cdot y_m, & (l) \text{ split,}\end{cases}$$

in $E(K_m)$.

Remark 4.2.2. One can extend Proposition 4.2.1 to the case of l being ramified and the case of $l \mid m$ (which was here not possible, as n is square-free).

Proposition 4.2.3. Let $n = ml$. Each prime factor λ_n of l in K_n divides a unique prime λ_m of K_m, and one has the congruence

$$x_{ml} \equiv \mathrm{Frob}(\lambda_m)(x_m) \pmod{\lambda_n},$$

on $X_0(N)$.

Proof. Write $\lambda = l\mathcal{O}_K$. Then λ splits completely in K_m/K and is totally ramified in K_{ml}/K_m. Let λ_m be a prime of K_m above λ and λ_{ml} a prime of K_{ml} over λ_m. We have for the residue fields

$$\mathbb{F}_{\lambda_m} \simeq \mathbb{F}_{\lambda_{ml}} \simeq \mathbb{F}_\lambda \simeq \mathbb{F}_{l^2}.$$

By equation (23), we obtain $T_l(x_m) = \mathrm{Tr}_l x_n \equiv x_n \pmod{\lambda_n}$. On the other hand, by Theorem 4.1.2, $T_l(x_m) \equiv \mathrm{Frob}(\lambda_m)(x_m) \pmod{\lambda_n}$. □

Applying our modular parametrization to Proposition 4.2.3 yields the congruence

$$y_{ml} \equiv \mathrm{Frob}(\lambda_m)(y_m) \pmod{\lambda_n}. \tag{24}$$

4.3 Group cohomology

Here we mainly rely on [MilCFT] Chapter II to give an introduction to the cohomology of groups.

Let G and H be groups and M an H-module. Define

$$\mathrm{Ind}_H^G(M) := \{ \text{ maps } \varphi : G \longrightarrow M \mid \varphi(hg) = h\varphi(g), \forall h \in H\}.$$

Note that Ind_H^G can be made into a G-module. For M a G-module, we also define

$$M^G := \{m \in M \mid gm = m, \forall g \in G\}.$$

Then, the functor

$$\begin{cases} \mathrm{Mod}_G & \longrightarrow \mathrm{Ab} \\ M & \mapsto M^G \end{cases}$$

is left exact. This follows from the fact that $(\cdot)^G$ is isomorphic to the left exact functor $\mathrm{Hom}_G(\mathbb{Z}, \cdot)$.

Definition 4.3.1. A G-module $\mathcal{I}$ is said to be *injective* if $\mathrm{Hom}_G(\cdot, \mathcal{I})$ is an exact functor.

Definition 4.3.2 (*Cohomology group*)**.** Let M be a G-module and let

$$0 \longrightarrow M \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \cdots$$

be an injective resolution of M. Then we obtain a complex

$$0 \xrightarrow{d^{-1}} (\mathcal{I}^0)^G \xrightarrow{d^0} (\mathcal{I}^1) \xrightarrow{d^0} (\mathcal{I}^1)^G \longrightarrow \cdots \xrightarrow{d^{r-1}} (\mathcal{I}^{r+1})^G \longrightarrow \cdots$$

The r**-th cohomology group of** G with coefficients in M is defined as

$$H^r(G, M) := \frac{\ker(d^r)}{\mathrm{im}(d^{r-1})}.$$

Proposition 4.3.3 (*Shapiro's lemma*)**.** Let $H < G$. For every H-module N, there exists a canonical isomorphism:

$$H^r(G, \mathrm{Ind}_H^G(N)) \xrightarrow{\sim} H^r(H, N),$$

for every $r \geq 0$.

Proof. See [MilCFT] §1.11. □

Definition 4.3.4 (*Restriction homomorphism*)**.** Let H be a subgroup of G. Let $M \longrightarrow \mathrm{Ind}_H^G(M)$ be the homomorphism sending m to the map $g \mapsto gm$. This homomorphism defines a homomorphism on cohomology

$$H^r(G, M) \longrightarrow H^r(G, \mathrm{Ind}_H^G(M)).$$

Together with 4.3.3, we obtain a **restriction homomorphism**

$$\mathrm{Res}: H^r(G, M) \longrightarrow H^r(H, M).$$

Definition 4.3.5 (*Inflation homomorphism*)**.** Let H be a normal subgroup of G. The quotient map $G \to G/H$, together with the inclusion $M^H \hookrightarrow M$ induces a homomorphism in cohomology:

$$\mathrm{Inf}: H^r(G/H, M^H) \longrightarrow H^r(G, H),$$

called the ***inflation homomorphism.***

Remark 4.3.6. More generally, if M is a G-module and M' is a G'-module, together with homomorphisms $\alpha : G' \longrightarrow G$, $\beta : M \longrightarrow M'$ such that

$$\beta(\alpha(g)m) = g(\beta(m)),$$

(such homomorphisms are called *compatible*), we obtain a homomorphism of chain complexes

$$\begin{cases} C^\bullet(G, M) & \longrightarrow C^\bullet(G', M') \\ \quad \varphi & \mapsto \beta \circ \varphi \circ \alpha^r. \end{cases}$$

This defines a homomorphism between cohomology groups

$$H^r(G, M) \longrightarrow H^r(G', M').$$

When $H = \{1\}$, an H-module is an abelian group and we will write

$$\mathrm{Ind}^G(M_0) = \{\text{maps } \varphi : G \longrightarrow M_0\},$$

where M_0 is M regarded as an abelian group. For a G-module M we have the sequence:

$$0 \longrightarrow M \longrightarrow \mathrm{Ind}^G(M_0) \longrightarrow \mathrm{Ind}^G(M_0)/M \longrightarrow 0,$$

which yields a cohomology sequence:

$$0 \longrightarrow M^G \longrightarrow (\mathrm{Ind}^G(M_0))^G \longrightarrow (\mathrm{Ind}^G(M_0)/M)^G \longrightarrow H^1(G, M) \longrightarrow 0,$$

together with a collection of isomorphisms:

$$H^r(G, (\mathrm{Ind}^G(M_0)/M)) \xrightarrow{\sim} H^{r+1}(G, M), r \geq 1,$$

(cf. [MilCFT] pg. 62). Using induction, one can show that the inflation and restriction homomorphisms induce an exact sequence

$$0 \longrightarrow H^{r-1}(G/H, (\mathrm{Ind}^G(M_0)/M)^H) \xrightarrow{\mathrm{Inf}} H^{r-1}(G, \mathrm{Ind}^G(M_0)/M) \xrightarrow{\mathrm{Res}} H^{r-1}(H, \mathrm{Ind}^G(M_0)/M).$$

This sequence is isomorphic to the exact sequence:

$$0 \longrightarrow H^r(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^r(G, M) \xrightarrow{\mathrm{Res}} H^r(H, M). \tag{25}$$

Definition 4.3.7 (*Hochschild-Serre exact sequence*)**.** The sequence in (25) yields an exact sequence:

$$0 \longrightarrow H^1(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^1(G, M) \xrightarrow{\mathrm{Res}} H^1(H, M)^{G/H} \longrightarrow H^2(G/H, M^H) \xrightarrow{\mathrm{Inf}}$$
$$\to \ker(H^2(G, M) \xrightarrow{\mathrm{Res}} H^2(H, M)) \longrightarrow H^1(G/H, H^1(H, M)) \longrightarrow \cdots$$

4.3.1 Kummer map

For $p \in \mathbb{N}$ and K a field with algebraic closure $\overline{K}$, there exists a short exact sequence

$$0 \longrightarrow \overline{K}^{\times}[p] \longrightarrow \overline{K}^{\times} \xrightarrow{z \mapsto z^p} \overline{K}^{\times} \longrightarrow 0.$$

Let L/K be a field extension. Taking $\mathrm{Gal}(\overline{K}/L)$-cohomology, one obtains the sequence in cohomology:

$$0 \longrightarrow L^{\times}/(L^{\times})^p \longrightarrow H^1(L, \overline{K}^{\times}[p]) \longrightarrow H^1(L, \overline{K}^{\times})[p] \longrightarrow 0.$$

Proposition 4.3.8 (*Hilberts Satz 90*)**.** Let L/K be a finite Galois extension with Galois group G. Then $H^1(G, L^{\times}) = 0$.

Proof. See [MilCFT] §2, Proposition 1.22. □

Remark 4.3.9. Proposition 4.3.8 is Emmy Noether's generalization of Hilbert's theorem 90.

Definition 4.3.10 (*Kummer map*)**.** By Proposition 4.3.8, $H^1(L, \overline{K}^{\times}) = 0$, so we obtain an isomorphism

$$\delta : L^{\times}/(L^{\times})^p \xrightarrow{\sim} H^1(L, \overline{K}^{\times}[p]),$$

which is called the *Kummer map*.

For an elliptic curve E/K we have a short exact sequence:

$$0 \longrightarrow E[p] \longrightarrow E \xrightarrow{P \mapsto m \cdot P} E \longrightarrow 0.$$

Then, for an algebraic extension L/K we obtain a Kummer sequence in cohomology

$$0 \longrightarrow (E(L)/pE(L)) \longrightarrow H^1(L, E[p]) \longrightarrow H^1(L, E)[p] \longrightarrow 0.$$

For K_n the ring class field of conductor n over K, we thus obtain an exact sequence

$$0 \longrightarrow (E(K_n)/pE(K_n))^{\mathcal{G}_n} \xrightarrow{\delta_n} H^1(K_n, E[p])^{\mathcal{G}_n} \longrightarrow H^1(K_n, E)[p]^{\mathcal{G}_n},$$

where $\mathcal{G}_n := \mathrm{Gal}(K_n/K)$.

Proposition 4.3.11. The curve E has no p-torsion rational over Q, so $E[p](K_n) = 0$.

Proof. See [GrossK] §4, pg. 241. □

Applying 4.3.11 together with 4.3.7,

$$0 \longrightarrow H^1(K_n/K, E[p](K_n)) \xrightarrow{\mathrm{Inf}} H^1(K_n, E[p]) \xrightarrow{\mathrm{Res}} H^1(K_n, E[p])^{\mathcal{G}_n} \longrightarrow H^2(K_n/K, E[p](K_n)),$$

gives us an isomorphism

$$H^1(K, E[p]) \xrightarrow{\sim} H^1(K_n, E[p])^{\mathcal{G}_n}.$$

Alltogether, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & H^1(K_n/K, E)[p] & & \\
 & & & & & & \downarrow{\scriptstyle \mathrm{Inf}} & & \\
0 & \longrightarrow & E(K)/pE(K) & \xrightarrow{\delta} & H^1(K, E[p]) & \longrightarrow & H^1(K, E)[p] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathrm{Res}} & & \\
0 & \longrightarrow & (E(K_n)/pE(K_n))^{\mathcal{G}_n} & \xrightarrow{\delta_n} & H^1(K_n, E[p])^{\mathcal{G}_n} & \longrightarrow & H^1(K_n, E)[p]^{\mathcal{G}_n}. & &
\end{array}$$

4.4 Kolyvagin's cohomology classes

As before, let n be square-free such that the primes l of n are unramified in $K(E[p])$ and (l) remains inert in K. Consider again our Heegner point y_n defined in 3.1.1. For G_n the Galois group of the extension K_n/K_1, and $G_n = \prod_{l|n} G_l$, where for each $l \mid n$, G_l is the subgroup fixing $K_{n/l}$. Kolyvagin defines an operator $D_l := \sum i \cdot \sigma_l^i$, which is a solution of

$$(\sigma_l - 1) \cdot D_l = l + 1 - \sum_{\sigma \in G_l} \sigma.$$

Write $D_n := \prod D_l$. For S a set of coset representatives for G_n in $\mathcal{G}_n = \mathrm{Gal}(K_n/K)$, define

$$P_n := \sum_{\sigma \in S} \sigma(D_n y_n) \in E(K_n).$$

Then one can show (cf. [GrossK] §3.6) that the class of $[P_n]$ in $E(K_n/pE(K_n))$ is fixed by $\mathcal{G}_n$, i.e.

$$[P_n] \in (E(K_n)/pE(K_n))^{\mathcal{G}_n}.$$

Definition 4.4.1 (*Kolyvagin's cohomology classes*)**.** By the diagram 4.3.1, there exists a unique class $c(n)$ in $H^1(K, E[p])$ such that

$$\mathrm{Res}\ c(n) = \delta_n[P_n],$$

in $H^1(K_n, E[p])^{\mathcal{G}_n}$.

5 Generalised Heegner cycles

After looking at Heegner points related to modular forms of weight 2, the next natural question to ask is: *how could we generalise this to higher weights*?

In this Chapter we refer to the Appendix 7 for an introduction to étale cohomology, realisations of relative motives and spectral sequences, which will be used in the upcoming sections.

5.1 Grössencharacters and field extensions

Let K be an imaginary quadratic field satisfying the Heegner condition (i.e. all primes dividing N are split in K). Write $A := \mathbb{C}/\mathcal{O}_K$ and let t be a generator of $\mathcal{N}^{-1}/\mathcal{O}_K$. By [JLZ] we can choose a point $\tau \in \mathcal{H} \cap K$ whose $\Gamma_1(N)$-orbit represents the pair (A, t) and which

(i) generates $(\mathcal{O}_K \otimes \mathbb{Z}_p)/\mathbb{Z}_p$,

(ii) is a unit at all primes above p.

Note that (A, t) defines a point of $Y_1(N)(\mathbb{C})$. Similarly, we define $\tau_m := p^{-1}\tau$ and $A_m := \mathbb{C}/(\mathbb{Z}\tau_m + \mathbb{Z})$. There is a canonical cyclic p^m-isogeny $\phi_m : A \longrightarrow A_m$. Let t_m be the N-torsion point $\phi_m(\tau)$ of A_m.

Definition 5.1.1. For A_1, A_2 elliptic curves, and $\varphi_1 : A_1 \longrightarrow A_1'$, $\varphi_2 : A_2 \longrightarrow A_2'$ isogenies, we say that (A_1', φ_1) and (A_2', φ_2) are **equivalent**, $(A_1', \varphi_1) \sim (A_2', \varphi_2)$, if there exists an isomorphism $\iota : A_1' \longrightarrow A_2'$ with $\iota\varphi_1 = \varphi_2$. For an elliptic curve A, write $\operatorname{Isog}(A) := \{(A', \varphi)\}/\sim$. Also define

$$\begin{aligned}
\operatorname{Isog}_c(A) &:= \{(A', \varphi) \in \operatorname{Isog}(A) : \operatorname{End}(A') = \mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K\}, \\
\operatorname{Isog}^{\mathcal{N}}(A) &:= \{(A', \varphi) \in \operatorname{Isog}(A) : \ker(\varphi) \cap A[\mathcal{N}] = \{0\}\}, \\
\operatorname{Isog}^{\mathcal{N}}(A)_c &:= \operatorname{Isog}^{\mathcal{N}}(A) \cap \operatorname{Isog}_c(A),
\end{aligned}$$

where $A[\mathcal{N}]$ denotes $\mathcal{N}$-torsion in A.

In our case, since $\operatorname{End}(A_m) = \mathcal{O}_{p^m}$, the pair (A_m, ϕ_m) defines an element of $\operatorname{Isog}^{\mathcal{N}}_{p^m}(A)$.

Fix an embedding $K \subset \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. This gives us two canonical characters

$$\sigma, \bar{\sigma} : (K \otimes \mathbb{Q}_p)^\times \longrightarrow \overline{\mathbb{Q}}_p^\times.$$

As before, let $\mathcal{N} \subset \mathcal{O}_K$ be an ideal such that $\mathcal{O}_K/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$. Let c be an integer coprime to N. Also, let ε be a Dirichlet character modulo N. Since $\mathcal{O}_c/(\mathcal{N} \cap \mathcal{O}_c) \cong \mathbb{Z}/N\mathbb{Z}$, we can regard ε as a character of $(\mathcal{O}_c/(\mathcal{N} \cap \mathcal{O}_c))^\times$, and thus also of the completion $\hat{\mathcal{O}}_c^\times$.

Definition 5.1.2. Let (l_1, l_2) be a pair of integers. A **Hecke character** of K of type (l_1, l_2) is a continous homomorphism

$$\chi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

satisfying

$$\chi(\alpha \cdot x \cdot z_\infty) = \chi(x) \cdot z_\infty^{-l_1} \cdot \overline{z}_\infty^{-l_2},$$

for all $\alpha \in K^\times$, $z_\infty \in K_\infty^\times$.

Definition 5.1.3 (*Grössencharacters*)**.** An **algebraic Grössencharacter** of K of finite type $(c, \mathcal{N}, \varepsilon)$ and of ∞-type (a, b) is a continous homomorphism

$$\chi : \mathbb{A}_{K,f}^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

satisfying

$$\chi(x) = x^a \overline{x}^b \tag{26}$$

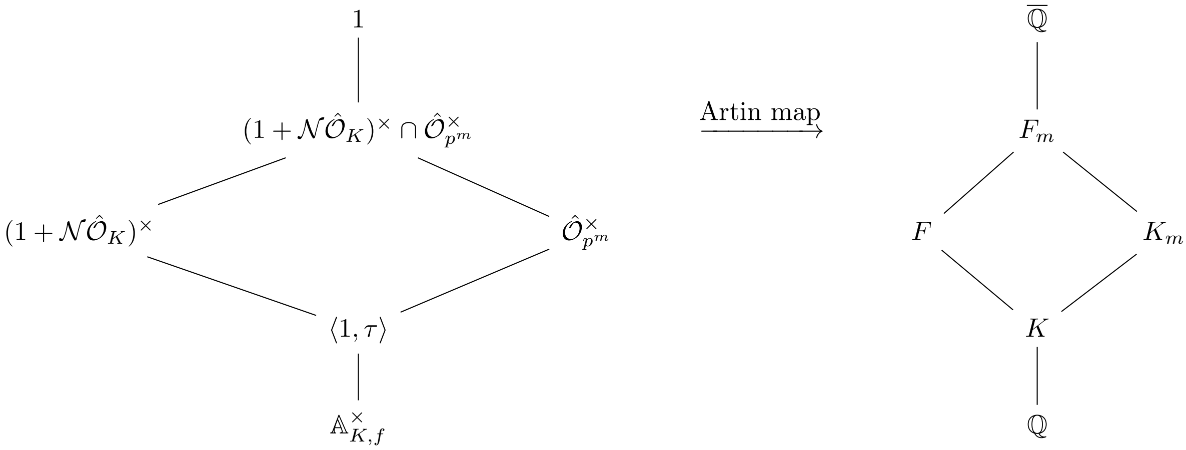
for $x \in K^\times$, whose conductor is divisible by c, and whose restriction to $\hat{\mathcal{O}}_c^\times$ is ε^{-1}.

Let $\chi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$ be an algebraic Grössencharacter of ∞ -type (a, b). Define a map

$$\begin{cases} \mathbb{A}_{K,f}^\times & \longrightarrow \overline{\mathbb{Q}}_p^\times \\ \quad x & \mapsto \chi(x)\sigma(x_p)^{-a}\overline{\sigma}(x_p)^{-b}. \end{cases}$$

Then, by (26), we have that the map above is trivial on $K^\times$, hence is a Galois character.

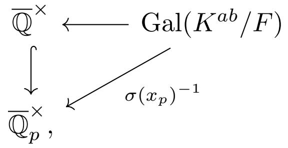
Next, consider the Artin map $\mathbb{A}_{K,f}^\times / K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$, where we normalise it so that the uniformisers map to the geometric Frobenius elements. Via the Artin map we consider the following abelian extensions of K:



for $m \geq 0$. Note that F is the ray class field modulo $\mathcal{N}$ and K_m is the ring class field modulo p^m. Since $\mathcal{O}_K^\times \cap (1 + \mathcal{N}) = \{1\}$, considering the Artin map, we obtain an isomorphism

$$(1 + \mathcal{N}\hat{\mathcal{O}}_K)^\times \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/F).$$

Going from right to left, we get a map



and similarly for $\overline{\sigma}$. Using a similar argument, we obtain that the restriction of χ to $\mathrm{Gal}(K^{ab}/F_m)$ is

$$\sigma(x_p)^{-a}\overline{\sigma}(x_p)^{-b},$$

and to $\mathrm{Gal}(K^{ab}/K_m)$ is

$$\varepsilon(x \mod \mathcal{N})^{-1}\sigma(x_p)^{-a}\overline{\sigma}(x_p)^{-b}.$$

5.2 Shimura varieties

Here we mainly rely on [MilSh] and [Rotger]. Let $\mathbb{G}_m = \mathrm{Spec}[X, Y]/(XY - 1)$. Let $\mathbb{S}$ be the algebraic torus over $\mathbb{R}$ obtained from $\mathbb{G}_m$ over $\mathbb{C}$ by restriction of scallars.

Let V be a real vector space. Complex conjugation on $V(\mathbb{C}) := \mathbb{C} \otimes_{\mathbb{R}} V$ is defined by

$$\overline{z \otimes v} = \overline{z} \otimes v.$$

Definition 5.2.1. A ***Hodge decomposition*** of a real vector space V is a decomposition

$$V(\mathbb{C}) = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q},$$

such that $V^{p,q}$ is the complex conjugate of $V^{p,q}$. A ***Hodge structure*** is a real vector space together with a Hodge decomposition. The set of pairs (p,q) for which $V^{p,q} \neq 0$ is called the *type* of Hodge structure.

Let D be a hermitian symmetric domain and Γ a torsion-free arithmetic subgroup of $\mathrm{Hol}(D)^+$, where $\mathrm{Hol}(\cdots)$ denotes the group of automorphisms. Write $D(\Gamma) = \Gamma \setminus D$ for the quotient.

Definition 5.2.2 (*Shimura datum*)**.** A **Shimura datum** is a pair (G,X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ satisfying the following conditions:

(i) for all $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1,1),(0,0),(1,-1)\},$$

(ii) for all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}^{\mathrm{ad}}}$,

(iii) $G^{\mathrm{ad}}_{\mathbb{R}}$ has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Consider the ring of finite adèles,

$$\mathbb{A}_f = \prod_l (\mathbb{Q}_l, \mathbb{Z}_l),$$

whre l runs over all finite primes of $\mathbb{Q}$. Note that $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For an affine variety V over Q write

$$V(\mathbb{A}_f) = \prod_l (V(\mathbb{Q}_l), V(\mathbb{Z}_l)).$$

Example 5.2.3.

$$\mathbb{G}_m(\mathbb{A}_f) = \prod_l (\mathbb{Q}_l^\times, \mathbb{Z}_l^\times) = \mathbb{A}_f^\times.$$

Let $K < G(\mathbb{A}_f)$ be a compact open subgroup, and write

$$\mathrm{Sh}_k(G,X) := G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K.$$

This is a double coset space in which $G(\mathbb{Q})$ acts on X and $G(\mathbb{A}_f)$ on the left, and K acts on $G(\mathbb{A}_f)$ on the right.

Definition 5.2.4 (*Shimura variety*)**.** Let (G,X) be a Shimura datum. A **Shimura variety** relative to (G,X) is a variety of the form $\mathrm{Sh}_K(G,X)$, for some (small) compact open subgroup $K \subset G(\mathbb{A}_f)$. The Shimura variety $\mathrm{Sh}(G,X)$ attached to a Shimura datum (G,X) is the inverse system of varieties $(\mathrm{Sh}_K(G,X))_K$ endowed with an action of $G(\mathbb{A}_f)$. Here K runs through the sufficiently small compact open subgroups of $G(\mathbb{A}_f)$.

Proposition 5.2.5. Let (G,D) be a Shimura datum. Let D_0 be a connected component of of D and $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then the set

$$G(\mathbb{Q})_0 \setminus G(\mathbb{A}_f)/K$$

is finite. Moreover, let $\mathcal{C}$ be a set of representatives of the double coset. There exists a homeomorphism

$$\bigcup_{c\in\mathcal{C}} \Gamma_c \setminus D_0 \xrightarrow{\sim} G(\mathbb{Q}) \setminus D \times G(\mathbb{A}_f)/K,$$

$$x \mapsto (x, c_x),$$

where $\Gamma c = c \cdot K \cdot c^{-1} \cap G(\mathbb{Q})_0$ and c_x denotes the connected component to which x belongs.

Proof. See [Rotger] Prop. 3.2.2. $\square$

We will write S_m for the canonical model over K of the Shimura variety for $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ of level

$$U = U_{\mathcal{N},p^m} := \{x \in \hat{\mathcal{O}}_{p^m}^\times : x \equiv 1 \pmod{\mathcal{N}}\}.$$

Remark 5.2.6. A model of a Shimura variety (over a reflex field E) is called *canonical* if it is uniquely characterized by the reciprocity laws at the special points (see [GeNg] for more details).

Definition 5.2.7 (*PEL-type*)**.** A **Shimura-PEL-datum** (PEL standing for polarization, endomorphism and level structure) is defined as a collection of data

$$\mathcal{D} := (B, *, V, \langle \cdot, \cdot \rangle, \mathcal{O}_B, \Lambda, h),$$

defined as follows:

(i) B is a finite-dimensional semi-simple $\mathbb{Q}$-algebra, such that $B_{\mathbb{Q}_p}$ is isomorphic to a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$;

(ii) * is a $\mathbb{Q}$-linear positive involution on B;

(iii) V is a finitely generated faithful left B-module;

(iv) $\langle , \rangle : V \times V \to \mathbb{Q}$ is a symplectic form on V such that $\langle bv, w \rangle = \langle v, b^* w \rangle$ for all $v, w \in V$ and $b \in B$;

(v) $\mathcal{O}_B$ is a *-invariant $\mathbb{Z}_{(p)}$-order of B such that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order of $B \otimes \mathbb{Q}_p$;

(vi) Λ is an $\mathcal{O}_B$-invariant $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$, such that $\langle \ , \ \rangle$ induces a perfect pairing $\Lambda \times \Lambda \to \mathbb{Z}_p$;

(vii) G is the $\mathbb{Q}$-group of B-linear symplectic similitudes of $(V, \langle \ , \ \rangle)$, i.e. for any $\mathbb{Q}$-algebra R we have

$$G(R) = \left\{ g \in \mathrm{GL}_B(V \otimes R) \mid \langle gv, gw \rangle = c(g) \cdot \langle v, w \rangle \text{ for some } c(g) \in R^\times \right\};$$

(viii) $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to G_{\mathbb{R}}$ is a homomorphism that defines a Hodge structure of type $(-1,0)+(0,-1)$ on $V \otimes \mathbb{R}$ such that there exists a square root $\sqrt{-1}$ of -1 such that $2\pi\sqrt{-1}\langle \ , \ \rangle$ is a polarization form; let $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the cocharacter such that $\mu_h(z)$ acts on $V^{(-1,0)}$ (resp. $V^{(0,-1)}$) via z (resp. via 1).

(ix) $[\mu]$ is the $G(\mathbb{C})$-conjugacy class of the cocharacter μ_h associated with h. Then $V_{\mathbb{C}}$ has only weights 0 and 1 with respect to any $\mu \in [\mu]$.

Example 5.2.8 (Shimura data for GL_2). Take $G = \mathrm{GL}_2$ for example and let

$$h : \begin{cases} \mathbb{S}_{\mathbb{R}} & \longrightarrow \mathrm{GL}_{2,\mathbb{R}}, \\ a+bi & \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \end{cases}$$

Then

$$D := \{h_g = ghg^{-1}\}_{g \in \mathrm{GL}_2(\mathbb{R})} \xrightarrow{\sim} \mathbb{C}/\mathbb{R}$$
$$g \mapsto p,$$

where p is the fixed point of $h_g(\mathbb{C}^\times)$ on $\mathbb{C}$ such that $h(z)$ acts on the tangent space of p as $z/\overline{z}$, gives us a Shimura datum (G, D). Indeed, let $z := re^{it} \in \mathbb{C}^\times$. Then there exists $g \in \mathrm{GL}_2(\mathbb{C})$ such that

$$gh(re^{it})g^{-1} = \begin{pmatrix} re^{it} & \\ & re^{-it} \end{pmatrix}.$$

This shows that the eigenvalues of $ad(h(z))$ acting on $\mathfrak{gl}_2(\mathbb{C})$ are $e^{2it} = \frac{z}{\overline{z}}$, $e^{-2it} = \frac{\overline{z}}{z}$ and 1. Moreover, the stabilizer of h is

$$\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{R})}(h) = \{g \mid g \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g^{-1} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\} \simeq \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^{\times}.$$

Thus $D \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R})/(\mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^{\times}) \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \times \{\pm 1\} \simeq \mathcal{H}^{\pm}$, by sending $\mathrm{Ad}_g(h)$ to $gi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} i = \frac{ai+b}{ci+d}$. Now, the tuple (V, ψ) from Definition 5.2.7 is in this case given by the bilinear form defined by the matrix $J := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ acting on the vector space E^2. We have $(B, *) = (E, *)$, where $* : z \longrightarrow \overline{z}$.

Then the Shimura data for GL_2 given above is of PEL-type.

Example 5.2.9. Let $G = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ be the torus for $K/\mathbb{Q}$ a finite Galois extension. Take

$$h : \mathbb{S} \longrightarrow \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$$

to be any morphism. Then $(\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m))^{ad} = \{1\}$. Let $D = \{x\}$ be a single point. Then for any compact open subgroup $K \subset G(\mathbb{A}_f)$ and $T := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$, we have that

$$\mathrm{Sh}_K(T, x) = T(\mathbb{Q}) \setminus \{x\} \times T(\mathbb{A}_f)/K \simeq T(\mathbb{Q}) \setminus T(\mathbb{A}_f)/K$$

is a finite set of points by Proposition 5.2.5.

The Shimura data for $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ is also of PEL-type.

5.2.1 Modular interpretation of Shimura data

Let (V, ψ) be a symplectic space over $\mathbb{Q}$ of dimension $2n$. Let $\mathrm{GS}_p(\psi)$ be the subgroup of $\mathrm{GL}(V)$ such that for any $\mathbb{Q}$-algebra R we have:

$$\mathrm{GS}_p(\psi)(R) := \{g \in \mathrm{GL}(V)(R) \mid \psi(gu, gv) = v(g) \cdot \psi(u, v), \quad v(g) \in R^{\times}\}.$$

Also, let

$$X^+ := \{J \mid \psi(Ju, Jv) = \psi(u, v), \quad \psi(\cdot, J\cdot) \text{ positive-definite}\},$$

and define X^- as those J for which $\psi(\cdot, J\cdot)$ is negative-definite. Write

$$X(\psi) := X^+ \amalg X^-.$$

Proposition 5.2.10. Let (V, ψ) be a symplectic space of dimension $2n$ over $\mathbb{Q}$. Then the pair $(\mathrm{GS}_p(\psi), X(\psi))$ is a Shimura datum.

Proof. See [MilSh] Section 5. ☐

Proposition 5.2.10 allows us to attach a Shimura variety $\mathrm{Sh}_K(G, D)$ to a Shimura datum (G, D) and an open compact subgroup $K \subset G(\mathbb{A}_f)$.

Let $A/\mathbb{C}$ be a complex abelian variety of dimension n. There exists a lattice $\Lambda \subset T_0(A)$ (recall that $T_p(A)$ is the Tate module) of rank $2n$ over $\mathbb{Z}$ such that $A(\mathbb{C}) \xrightarrow{\sim} T_0(A)/\Lambda$. Then $H_1(A, \mathbb{Z}) = \Lambda$.

Definition 5.2.11. Let $s : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ be a non-degenerate form such that $s(Ju, Jv) = \psi(u, v)$, for all $u, v \in V(\mathbb{R})$, and $s(u, Jv)$ is positive-definite. Let $V(\mathbb{A}_f) = \Lambda \otimes_{\mathbb{Z}} \mathbb{A}_f$ and $V_f(A) = H_1(A, \mathbb{A}_f) \simeq \Lambda \otimes \mathbb{A}_f$ be the Tate module of A. Let AV^0 be the category of abelian varieties up to isogeny and let $M_K(G, D)$ be the set of triples $\{A, s, \eta \cdot K\}$, where

(i) A is a complex abelian variety of dimension n, $A(\mathbb{C}) = V(\mathbb{R})/\Lambda$;

(ii) s is an alternating bilinear form on $H_1(A, \mathbb{Z})$ such that s or $-s$ is a polarization on A ;

(iii) $\eta : V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A)$ such that $\eta_*(\psi) = a \cdot s$, for some $a \in \mathbb{A}_f^*$.

Remark 5.2.12. Two triples $(A, s, \eta \cdot K)$, $(A', s', \eta' \cdot K)$ in $M_K(G, D)$ are isomorphic if there exists an isogeny $f : A \longrightarrow A'$, such that $f^*(s') = q \cdot s$, $q \in \mathbb{Q}^*$ and $f^*(\eta' K) = \eta K$.

Theorem 5.2.13. The Shimura variety $\mathrm{Sh}_K(G, D)$ is the coarse moduli space over $\mathbb{C}$ that classifies triples in $M_K(G, D)$ up to isomorphism. In particular, there is a canonical bijection of sets

$$M_K(G, D)/\!\sim \;\longleftrightarrow\; G(\mathbb{Q}) \setminus D \times G(\mathbb{A}_f)/K.$$

Proof. See [Rotger] Section 3.3. □

Now, recall that the modular curve $Y_1(N)$ can be defined as the moduli space of pairs (E, P) with E being an elliptic curve and P being a section of order N.

Definition 5.2.14. An *enchanced elliptic curve for* $\Gamma_1(N)$ is a pair (E, Q), where E is a complex elliptic curve and Q is a point of E of order N. Two such pairs (E, Q) and (E', Q') are said to be equivalent, written $(E, Q) \sim (E', Q')$, if there exists an isomorphism $E \xrightarrow{\sim} E'$ which takes Q to Q'.

Let $S_1(N)$ denote the set of enchanced elliptic curves for $\Gamma_1(N)$ modulo the equivalence relation $\sim$. Write $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$. Then there is a natural map

$$S_1(N) \xrightarrow{\sim} Y_1(N)$$
$$[\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau] \mapsto \Gamma_1(N)\tau,$$

which allows us to identify $\Gamma_1(N) \setminus \mathcal{H} \simeq Y_1(N)(\mathbb{C})$.

5.2.2 Representation theory

For any $\tau \in K - \mathbb{Q}$, there exists a unique embedding of $\mathbb{Q}$-algebras $\iota : K \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ such that $\iota(K)$ fixes the line spanned by $\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ in K^2 and acts on this line by scalar multiplication. Explicitly, ι is given by:

$$\iota : a + b\tau \mapsto \begin{pmatrix} a + b(\tau + \overline{\tau}) & -b\tau\overline{\tau} \\ b & a \end{pmatrix},$$

since $(a + b\tau) \cdot \tau = a\tau + b\tau^2 = (a + b(\tau + \overline{\tau}))\tau - b\tau\overline{\tau}$. Let $H := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ be the Weil restriction of the algebraic torus. Recall that

$$\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) : T \mapsto \mathrm{Hom}_K(T \times_{\mathbb{Q}} K, \mathbb{G}_m).$$

Consider representations of algebraic tori, $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \longrightarrow \mathrm{GL}_2$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}) = \langle 1, \tau \rangle$ has only one non-trivial element (the one corresponding to sending τ to $\overline{\tau}$), so $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathbb{Z}$ either as the identity or as -1. Hence, we can regard ι as an embedding of $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \hookrightarrow \mathrm{GL}_2$.

Recall the embedding

$$\iota_m := \begin{pmatrix} p^{-m} & 0 \\ 0 & 1 \end{pmatrix} \iota \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}.$$

Write $\delta := (\iota, \mathrm{id})$ for the diagonal embedding $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m) \hookrightarrow \mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ and analogously, $\delta_m := (\iota_m, \mathrm{id})$. Also, write δ_m^* for the restriction of representations from $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ to $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ via δ_m. Define $\tau^* := -1/\overline{\tau}$ and let $e_m := \begin{pmatrix} \sigma(\tau_m^*) \\ 1 \end{pmatrix}$, for $\tau_m^* = p^m \tau^*$. Then

$$({}^t\iota_m{}^{-1})e_m = \begin{pmatrix} a & -bp^m \\ p^{-m}b\tau\overline{\tau} & a + b(\tau + \overline{\tau}) \end{pmatrix} \begin{pmatrix} \tau_m^* \\ 1 \end{pmatrix} = (a + b\overline{\tau}) \begin{pmatrix} \tau_m^* \\ 1 \end{pmatrix},$$

and the group $\iota_m(\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m))$ acts as σ^{-1}.

Definition 5.2.15. A *morphism* of Shimura data

$$(G_1, D_1) \longrightarrow (G_2, D_2)$$

is a group homomorphism $G_1 \longrightarrow G_2$ mapping D_1 into D_2.

Similarly one defines a morphism of *Shimura varieties* by taking the PEL data from one variety to another. Next we will explain how one can define a morphism between the PEL data of H to PEL data of GL_2.

Remark 5.2.16. Write $G := \mathrm{GL}_2 \times H$. The map δ_m defines a morphism of Shimura data $(H, D_H) \xrightarrow{\delta} (G, D_G)$. Indeed, for $a + b\tau \in K^\times$, we have $(a + b\tau)(a + b\overline{\tau}) \neq 0$, so

$$\det \begin{pmatrix} a + b(\tau + \overline{\tau}) & -b\tau\overline{\tau} \\ b & a \end{pmatrix} \neq 0,$$

and δ defines a group homomorphism $H \longrightarrow \iota(H) \times H \subset \mathrm{GL}_2 \times H$. Similarly, consider $D_H = \{x\}$ from Example 5.2.9. Taking $x = i$ in the canonical model, we send

$$D_H \xrightarrow{\delta} \delta(\{i\}) \times D_H = \left\{ \begin{pmatrix} a + b(i + \overline{i}) & -bi\overline{i} \\ b & a \end{pmatrix} \right\} \times D_H = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\} \times D_H.$$

Similar computations can be done with δ_m instead of δ to show that it is also a morphism of Shimura data.

Definition 5.2.17 (*Symmetrised tensor product*)**.** Let $\mathrm{TSym}^k(\cdot)$ denote the k-th symmetric tensor algebra. For $a, b \geq 0$, M a vector space and $\otimes_{i=1}^l e_i^l \in \mathrm{TSym}^l(M)$, with $l = a, b$, we define the ***symmetrised tensor product***

$$\begin{cases} \mathrm{TSym}^a(M) \times \mathrm{TSym}^b(M) & \longrightarrow \mathrm{TSym}^{a+b}(M) \\ \quad (\otimes_{i=1}^a e_i^a, \otimes_{j=1}^b e_j^b) & \mapsto \frac{(a+b)!}{a!b!} (\otimes_{i=1}^a e_i^a) \otimes (\otimes_{j=1}^b e_j^b), \end{cases}$$

in the algebra $\mathrm{TSym}^\bullet(M) = \oplus_{k \geq 0} \mathrm{TSym}^k(M)$.

Let E be a field and $K \hookrightarrow E$ a fixed embedding. For any $a, b \geq 0$, consider the vector

$$e_m^{[a,b]} = (e_m)^{\otimes a} \cdot (\overline{e_m})^{\otimes b} \in \mathrm{TSym}^{a+b}((E^2)^\vee),$$

where $\cdot$ is the symmetrised tensor product. Then the vector $e_m^{[a,b]}$ transforms under $\delta_m(H)$ via $\sigma^{-a}\overline{\sigma}^{-b}$.

Definition 5.2.18. Let E be a commutative ring. We define $\mathrm{Sym}^k E^2$ as the set of left representations of $\mathrm{GL}_2(E)$ afforded by the space of homogeneous polynomials of degree k over E in two variables X, Y, with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f = f\left((X, Y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = f(aX + cY, bX + dY).$$

Remark 5.2.19. In general, the *k'th symmetric power* of an n-dimensional representation L can be thought of as having the basis

$$e_{i_1} \cdots e_{i_k} = \frac{1}{k!} \sum_\sigma e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_k)},$$

where $1 \leq i_1 \leq \cdots \leq i_k \leq n$ and σ ranges over all permutations of the k indices. In our case, we consider representations of $\mathrm{GL}_2(E)$ on E^2, so $n = 2$. If, for example, we also take $k = 2$, then we obtain the basis e_1^2, e_1e_2, e_2^2. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(E)$, then this matrix sends

$$e_1 \mapsto ae_1 + be_2,$$
$$e_2 \mapsto ce_1 + de_2.$$

Then it will act on the basis e_1^2, e_1e_2, e_2^2 as

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

This allows us to think of the space of k'th symmetric powers in terms of homogeneous polynomials, as in Definition 5.2.18.

For $e_1 \otimes \cdots \otimes e_k \in (E^2)^{\otimes k}$, the space $\mathrm{Sym}^k(E^2)$ is generated by the elements in $(E^2)^{\otimes k}$ modulo elements of the form $e_1 \otimes \cdots \otimes e_k - e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}$, where σ goes over all permutations of k letters. Write C for the set of elements of this form. Then $\mathrm{Sym}^k(E^2) \simeq (E^2)^{\otimes k}/C$ and the dual of $\mathrm{Sym}^k(E^2)$ is isomorphic to the vector space of symmetric multilinear maps from $(E^2)^k$ to E, denoted $S((E^2)^k, E)$. Moreover, we have

$$(\mathrm{Sym}^k(E^2))^\vee = \mathrm{Hom}_E(\mathrm{Sym}^k(E^2)^k, E) \simeq S((E^2)^k, E) \simeq S^k(E^2, E).$$

Then, the dual of $\mathrm{Sym}^k E^2$ as a $\mathrm{GL}_2(E)$-representation is given by $\mathrm{TSym}^k((E^2)^\vee)$, with $(E^2)^\vee$ being the dual of the standard representation.

Definition 5.2.20 (*External tensor product*)**.** Consider an indexed monoidal category given by a Cartesian fibration $Mod(\cdot) \to \mathbf{H}$ over a cartesian monoidal category $\mathbf{H}$ (cf. [nLab]). Given $X_1, X_2 \in \mathbf{H}$, the external tensor product over these is the functor

$$\boxtimes : Mod(X_1) \times Mod(X_2) \longrightarrow Mod(X_1 \times X_2)$$

given on $A_1 \in Mod(X_1)$ with $A_2 \in Mod(X_2)$ by

$$A_1 \boxtimes A_2 := (p_1^* A_1) \otimes_{X_1 \times X_2} (p_2^* A_2) \in Mod(X_1 \times X_2)\,,$$

where p_1, p_2 denote the projection maps out of the Cartesian product $X_1 \times X_2 \in \mathbf{H}$.

Let $a, b \geq 0$. Consider the following representation over $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$:

$$V_{a,b} = \mathrm{Sym}^{a+b}(E^2) \boxtimes \left(\sigma^{-a} \otimes \bar{\sigma}^{-b}\right). \tag{27}$$

Then we can explicitly write for the dual:

$$V_{a,b}^\vee = \left(\mathrm{p}_1^*(\mathrm{TSym}^{a+b}((E^2)^\vee))\right) \otimes_{\mathrm{TSym} \times (\sigma \otimes \bar{\sigma})} \left(\mathrm{p}_2^*((\sigma^{-a} \otimes \bar{\sigma}^{-b})^\vee)\right),$$

and considering the restriction δ_m^* of representations from $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ to $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$, for any m the representation $\delta_m^*(V_{a,b}^\vee)$ has a unique summand isomorphic to the trivial representation of $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$. Then $e_m^{[a,b]}$ is a basis of this trivial summand, since it transforms under $\delta_m(H)$ via $\sigma^{-a}\overline{\sigma}^{-b}$ and we twist by $\sigma^{-a} \otimes \overline{\sigma}^{-b}$ in $V_{a,b}$. Thus, the vector $e_m^{[a,b]}$ is a basis vector of $\delta_m^*(V_{a,b}^\vee)$.

5.3 Chow motives

Following [Fulton] we start by giving an introduction to Chow groups after which we move on to introducing Chow motives (following [CorHan]).

5.3.1 What is a Chow group?

Let X be an algebraic scheme. A k-cycle on X is a finite formal sum $\sum n_i[V_i]$, with $n_i \in \mathbb{Z}$, and V_i k-dimensional subvarieties on X. Let $Z_k(X)$ be the free abelian group on k-dimensional subvarieties of X, called the *group of k-cycles on* X. If V is a subvariety of X, then $[V]$ is in $Z_k(X)$. For W a $k+1$-dimensional subvariety of X, let $R(W)^*$ denote the non-zero elements of the field of rational functions of

W. For $r \in R(W)^*$, a k-cycle $[\mathrm{div}(r)]$ on X is given by

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V],$$

where the sum is taken over all codimension 1 subvarieties V of W, and $\mathrm{ord}_V(\cdot)$ is the order function on $R(W)^*$. A k-cycle α is *rationally equivalent to zero*, $\alpha \sim 0$, if there exists finitely many $(k+1)$-dimensional subvarieties W_i of X and $r_i \in R(W_i)^*$, such that

$$\alpha = \sum [\mathrm{div}(r_i)].$$

Since $[\mathrm{div}(r^{-1})] = -[\mathrm{div}(r)]$, the k-cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_k(X)$ of $Z_k(X)$. Finally, write

$$A_k(X) := Z_k(X)/\mathrm{Rat}_k(X)$$

for the *group of k-cycles modulo rational equivalence on X*. Then we define the rational Chow group $\mathrm{CH}_k(X)$ as:

$$\mathrm{CH}_k(X) := A_k(X) \otimes \mathbb{Q}.$$

5.3.2 Motives

Next, we will shortly summarise Grothendieck's classical construction of motives. For that we need to first construct a correspondence category.

Fix a field k, which is finitely generated over its prime field. Consider smooth and projective varieties X over k, and denote by C^iX the group of algebraic cycles of codimension i on X modulo a suitable equivalence relation. Note that for $C^iX = CH^iX$, we get the Chow group of cycles modulo rational equivalence. Next, we will construct the categories $C\mathcal{C}$ of *C-correspondences.*

Definition 5.3.1. An *object* of $C\mathcal{C}$ is a smooth and projective variety X. *Morphisms* in $C\mathcal{C}$ are correspondences:

$$\mathrm{Hom}_{C\mathcal{C}}(X, Y) = \oplus C^{\dim X_\alpha} X_\alpha \times Y,$$

where $X = \bigsqcup X_\alpha$ is the decomposition of X into its connected components X_α. For correspondences $u : X_1 \to X_2$ and $v : X_2 \to X_3$, one defines the *composition* as follows:

$$u \circ v = p_{13*}(p_{23}^* v \cdot p_{12}^* u),$$

where $p_{ij} : X_1 \times X_2 \times X_3 \longrightarrow X_i \times X_j$ is the projection.

Remark 5.3.2. Since the intersection product for Chow groups is compatible with the cup product for cohomology classes, we have a forgetful functor $\mathrm{CH}\mathcal{C} \to \mathrm{A}\mathcal{C}$ from the category of Chow correspondences to the category of homological correspondences.

Next, we would like to define the category $\mathrm{C}\mathcal{M}$ of *C-motives*, which will be obtained by taking the pseudo-abelianization of $\mathrm{C}\mathcal{C}$, followed by inserting Tate objects and then twisting by them.

Definition 5.3.3. An *object* of $\mathrm{C}\mathcal{M}$ is a triple

$$(X, P, r),$$

where X is smooth projective, $P \in \mathrm{End}_{\mathrm{C}\mathcal{C}}(X, X)$ is a projector (i.e. $P^2 = P$), and $r \in \mathbb{Z}$. *Morphisms* in $\mathrm{C}\mathcal{M}$ are defined as

$$\mathrm{Hom}_{\mathrm{C}\mathcal{C}}\left((X, P, r), (Y, Q, s)\right) = Q \circ \left(\oplus C^{\dim X_\alpha + s - r}(X_\alpha \times Y)\right) \circ P, \tag{28}$$

where $X = \bigsqcup X_\alpha$ is the decomposition of X into its connected components X_α. Composition is obtained the same way that we composed correspondences above.

Definition 5.3.4. For $\mathrm{C} = \mathrm{CH}$, the category $\mathrm{CH}\mathcal{M}$ is called the category of **Chow motives**.

Now we are equipped to introduce *relative* motives. For that we follow [Kings]. Let S be a smooth scheme over a field and such that (for simplicity) all connected components of S have the same dimension. Write $V(S)$ for the category of smooth projective S schemes. For $X, Y \in V(S)$, with $X = \bigsqcup X_i$, for X_i connected components (as before), let

$$\mathrm{Corr}^r(X, Y) := \oplus_i \mathrm{CH}^{\dim X_i - \dim S + r}(X_i \times_S Y, \overline{\mathbb{Q}}),$$

be the correspondences of degree r with coefficients in $\overline{\mathbb{Q}}$. Define composition in the usual way as before.

Let $M(S, \overline{\mathbb{Q}}) := \mathrm{CH}\mathcal{M}(S_{\overline{\mathbb{Q}}})$ be the category of ***relative motives*** with coefficients in $\overline{\mathbb{Q}}$, which looks as follows: it has objects (X, P, r), with $X \in V(S)$, $P \in \mathrm{Corr}^0(X, X)$ and $P^2 = P$, $r \in \mathbb{Z}$ and $\mathrm{Hom}_M((X, P, r), (Y, Q, s))$ defined as in (28).

For objects $(X, P, r), (Y, Q, s)$ we further define the tensor product and dual as:

$$(X, P, r) \otimes (Y, Q, s) := (X \times_S Y, P \times_S Q, r + s),$$
$$(X, P, n)^\vee := (X, {}^tP, \dim(X) - \dim(S) - n).$$

By applying the tensor product above multiple times, we define:

$$\mathrm{Sym}^i(X, P, n) := (X^i, \lambda_i \circ (P \times_S \cdots \times_S P), n \cdot i),$$

where $X^i := X \times_S \cdots \times_S X$, and for $\Gamma_f \in \mathrm{Corr}^0(X, Y)$ being the graph of $f : X \longrightarrow Y$, we write

$$\lambda_i := \frac{1}{i!} \cdot \sum_{\sigma \in \sum_i} \sigma \cdot {}^t\Gamma_\sigma.$$

Here $\sum_i$ denotes the symmetric group in i letters permuting the factors of X_i.

5.4 Cycles in motivic cohomology

Theorem 5.4.1 (Deninger-Murre). Let X be an abelian variety of dimension d and let $\mathfrak{h}(X)$ be its Chow motive. Then there is a unique, multiplicative Chow-Künneth decomposition:

$$\mathfrak{h}(X) = \oplus_{i=0}^{2d} \mathfrak{h}^i(X),$$

such that for every $N \in \mathbb{Z}$, $[N] : X \longrightarrow X$ acts on $\mathfrak{h}^i(X)$ by $[N]^* = N^i$.

Proof. See [DenMurr]. □

Now, considering the relative motive of $\epsilon/Y_1(N)$, one can show that the image of the projector $\epsilon_W \epsilon_A$ (from [BDP] §2) acting on $\mathrm{CH}^{k+1}((\epsilon^k \times A^k)_{F_m})_{\mathbb{Q}}$ is given by

$$H^2_{\mathrm{mot}}\left(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathrm{TSym}^k\left(\mathfrak{h}^1(A)\right)(1)\right), \tag{29}$$

with $\mathfrak{h}^1(\epsilon)$ being the degree 1 part in the Chow-Künneth decomposition of $\epsilon/Y_1(N)$, and similarly for $\mathfrak{h}^1(A)$.

Definition 5.4.2. The pair (ϕ_m, A_m) determines a cycle Δ_{ϕ_m} in the 2-motivic cohomology group 29 above.

Remark 5.4.3. By [BDP] §2, Δ_{ϕ_m} is a cycle in $\epsilon_W \epsilon_A \mathrm{CH}^{k+1}((W_k \times A^k)_{F_m})_{\mathbb{Q}}$.

Take $a, b \geq 0$ with $a + b = k$. Consider $\mathrm{TSym}^k\left(\mathfrak{h}^1(A)\right)$. By 5.4.1 we have a rank-1 direct summand $\mathfrak{h}^{(a,b)}(A)$, in $\mathrm{TSym}^k\left(\mathfrak{h}^1(A)\right) \otimes K$, on which the action of complex multiplication is given by $[x]^* = x^a \bar{x}^b$.

Now, instead of dealing with projectors on Chow motives, we would like to work with elements in the symmetrised tensor product defined above and reduce our problem of working with cycles to representation theory. To get elements in motivic cohomology from algebraic representations we will use a functor constructed by Ancona.

5.5 From representations to motives: Ancona's functor

Next, we would like to relate $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$-representations to relative Chow motives. For that we rely on [Ancona].

Let $S := S_K(G, X)$ be a Shimura variety of PEL-type. Write S_0 for the canonical model of the reflex field, and A_0 for the universal abelian scheme of S_0. Let $f : A_0^r \longrightarrow S_0$ be the r-th fiber product of A_0 above S_0. Specifically, in our case we have $\pi : \epsilon^r \longrightarrow Y_1(N)$.

Theorem 5.5.1. (i) For all $0 \leq i \leq 2g$, there exists a canonical isomorphism

$$\mathrm{Sym}^i \mathfrak{h}^1(A) \xrightarrow{\sim} \mathfrak{h}^i(A)$$

(ii) One has the Poincaré duality

$$\mathfrak{h}^{2g-i}(A)^\vee = \mathfrak{h}^i(A)(g),$$

and the non-canonical *Liefschetz* isomorphism

$$\mathfrak{h}^i(A) \xrightarrow{\sim} \mathfrak{h}^{2g-i}(A)(g-i).$$

Proof. See [Ancona] pg. 312. □

Let $\mathrm{Rep}(G_F)$ denote the algebraic representations of G_F and $\mathrm{CH}\mathcal{M}(S)_F$ the category of F-linear, tensor, symmetric and pseudo-abelian relative Chow motives (with coefficients in F), equipped with a contravariant functor.

Theorem 5.5.2 (*Ancona*)**.** There exists an F-linear functor $\mathrm{Anc}_{G_F} : \mathrm{Rep}(G_F) \longrightarrow \mathrm{CH}\mathcal{M}(S_0)_F$, which is compatible with tensors and commutes with Tate twists. It sends $V^\vee$ to $\mathfrak{h}^1(A)$.

Proof. See [Ancona] Theorem 8.6. □

Remark 5.5.3. One can show that Ancona's functor is natural with respect to morphisms of PEL data (cf. [Tor]).

Remark 5.5.4. Instead of working with a functor which sends $V^\vee$ to $\mathfrak{h}^1(A)$ (as it was done by Ancona), we will slightly modify the functor such that V gets sent to $\mathfrak{h}^1(A)$. From this point on we will denote the modified functor by Anc_{G_F}.

Let $\mathfrak{h}^{(a,b)}(A)$ be the rank-1 direct summand in $\mathrm{TSym}^k(\mathfrak{h}^1(A)) \otimes K$ on which the complex multiplication action of $\mathcal{O}_K$ is given by $x^a \overline{x}^b$. Since the Shimura data for $\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ and GL_2 are of PEL-type, we can apply Theorem 5.5.2 to obtain a functor from algebraic representations of $\mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ to relative Chow motives.

Consider an irreducible algebraic representation $\mathrm{Sym}^k(E^2) \otimes \det$ of GL_2. Then Ancona's (modified) functor maps $\mathrm{Sym}^k(E^2)$ to $\mathrm{Sym}^k(\mathfrak{h}^1(\epsilon))$. Note that the dual of $\mathrm{Sym}^k(E^2)$ is given by $\mathrm{TSym}^k((E^2)^\vee)$, which is mapped via Ancona's functor to $\mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)^\vee)$, and by Poincaré duality, $\mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)^\vee) = \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1))$.

For $a + b = k$ and $G := \mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$, consider $\mathrm{Rep}_K(G) \xrightarrow{\mathrm{Anc}_G} \mathrm{CH}\mathcal{M}_K(Y_1(N) \times S_m)$. Then

$$(\mathrm{Anc}_G(V_{a,b}))^\vee = (\mathrm{Anc}_G(\mathrm{Sym}^{a+b}(E^2) \boxtimes (\sigma^{-a} \otimes \overline{\sigma}^{-b})))^\vee \tag{30}$$
$$= (\mathrm{Sym}^k(\mathfrak{h}^1(\epsilon)))^\vee \otimes \mathfrak{h}^{(a,b)}(A) \tag{31}$$
$$= \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)^\vee) \otimes \mathfrak{h}^{(a,b)}(A). \tag{32}$$

And by Poincaré duality, $\mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)^\vee) = \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1))$.

Recall our restriction δ_m^* of representations from G to H via $\delta_m = (\iota_m, \mathrm{id})$. By Section 5.2, we have that δ_m is an admissible morphism of PEL-type Shimura varieties, so by a result of Torzewski (cf. [Tor] Theorem 1.2), we obtain a commutative diagram:

$$\begin{array}{ccc} \mathrm{Rep}_K(G) & \xrightarrow{\delta_m^*} & \mathrm{Rep}_K(H) \\ \downarrow{\scriptstyle \mathrm{Anc}_G} & & \downarrow{\scriptstyle \mathrm{Anc}_H} \\ \mathrm{CH}\mathcal{M}_K(Y_1(N) \times S_m) & \xrightarrow{\delta_m^*} & \mathrm{CH}\mathcal{M}_K(S_m) \end{array}$$

with the bottom row denoting pullback of relative Chow motives.

Now, note that $S_m \subset Y_1(N) \times S_m$ is a subvariety of codimension 1.

Proposition 5.5.5 (*Pushforward map*)**.** If $Z \subset X$ is a closed subvariety of codimension d (and X and Z are both smooth), then there are pushforward maps

$$H^i(Z, \mathbb{Q}_p(i)) \longrightarrow H^{i+2d}(X, \mathbb{Q}_p(n+d)).$$

Proof. See [Arizona] §2. □

By 5.5.5 there is a pushfroward (Gysin) map

$$H^0_{\mathrm{mot}}\left(S_m, \delta_m^*\left(\mathrm{Anc}_G(V_{a,b})\right)^\vee\right) \xrightarrow{\delta_{m*}} H^{0+2\cdot 1}_{\mathrm{mot}}\left(Y_1(N)_K \times S_m, \mathrm{Anc}_G(V_{a,b})^\vee(1)\right).$$

By equation (32), we get

$$H^2_{\mathrm{mot}}\left(Y_1(N)_K \times S_m, \mathrm{Anc}_G(V_{a,b})^\vee(1)\right) = H^2_{\mathrm{mot}}\left(Y_1(N)_K \times S_m, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathfrak{h}^{(a,b)}(A)(1)\right) \quad (33)$$

Next, recall that the field extension F_m corresponds to $(1+\mathcal{N}\hat{\mathcal{O}}_K)^\times \cap \mathcal{O}_{p^m}^\times$, and consider the Shimura datum (S_m, D) for S_m the canonical model over K of the Shimura variety for the torus $H = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$. Since H is of level $U_{\mathcal{N},p^m}$, S_m is isomorphic as a K-variety, to the $\mathrm{Gal}(F_m/K)$-orbit of τ_m.

Then (33) is isomorphic to

$$H^2_{\mathrm{mot}}(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1)(\epsilon)(1) \otimes \mathfrak{h}^{(a,b)}(A)(1)).$$

Clearly, $H^0_{\mathrm{mot}}(S_m, \delta_m^*(\mathrm{Anc}_G(V_{a,b})^\vee)) = (\delta_m^*(\mathrm{Anc}_G(V_{a,b})^\vee))^{S_m} = (\mathrm{Anc}_G(V_{a,b})^\vee)^{\delta_m(H)}$.

Recall that $e_m^{[a,b]}$ is a basis vector of $\delta_m^*(V_{a,b}^\vee)$. Also, recall that $\chi|_{\mathrm{Gal}(F_m/K)} = \sigma^{-a}\overline{\sigma}^{-b}$ and that $e_m^{[a,b]}$ transforms under $\delta_m(H)$ via $\sigma^{-a}\overline{\sigma}^{-b}$. Then $e_m^{[a,b]}$ is an element of $(\mathrm{Anc}_H(\delta_m^*(V_{a,b}^\vee)))^{S_m}$. By diagram 5.5, we have

$$\delta_m^* \circ \mathrm{Anc}_G = \mathrm{Anc}_H \circ \delta_m^*,$$

so

$$(\mathrm{Anc}_H(\delta_m^*(V_{a,b}^\vee)))^{S_m} = (\mathrm{Anc}_G(V_{a,b}^\vee))^{\delta_m(H)}.$$

By applying our pushforward map δ_{m*} to $e_m^{[a,b]}$, we get an element of

$$H^2_{\mathrm{mot}}(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathfrak{h}^{(a,b)}(A)(1)).$$

On the other hand, consider the cycle $\Delta_{\phi_m} := \epsilon_W \epsilon_A \mathrm{Graph}(\phi_m)^k$ from Section 5.4. Let

$$\rho^*\Delta_{\phi_m} \in H^2_{\mathrm{mot}}(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathrm{TSym}^k(\mathfrak{h}^1(A))(1))$$

be the cycle on $\epsilon^k \times A^k$ obtained by pullback. Take the projection of $\rho^*\Delta_{\phi_m}$ to

$$H^2_{\mathrm{mot}}(Y_1(N)_{F_m}, \mathrm{TSym}^k(\mathfrak{h}^1(\epsilon)(1)) \otimes \mathfrak{h}^{(a,b)}(A)(1)),$$

which is characterised by having a complex multiplication action of $\mathcal{O}_K$ given by $[x]^* = x^a\overline{x}^b$ on $\mathrm{TSym}^k(\mathfrak{h}^1(A))\otimes K$. We denote this projection by $z^{[a,b]}_{\mathrm{mot},m}$ and call it the **Heegner class**. Then $\delta_{m*}(e^{[a,b]}_m)$ is precisely the Heegner class.

5.6 Étale realisation of Heegner classes

We refer to the Appendix 7 for an introduction to étale cohomology.

Let L be a p-adic field. Consider a p-adic embedding $\sigma : K \hookrightarrow L$. The irreducible representations of $\mathrm{GL}_2(L)$ are of the form $\mathrm{Sym}^k(L^2)\otimes_L \det^m$, for $m\in\mathbb{Z}$. For positive integers a,b similarly as before, we can consider the representation $V_{a,b}$ of G over L (instead of E). The construction we had for our motivic cohomology groups in the previous sections then naturally extends to étale cohomology, now with coefficients in L.

Consider the p-adic realisation of $\mathcal{V}_{a,b}$,

$$\mathrm{Sym}^k(\mathfrak{h}^1(\epsilon))\otimes\tilde{\mathfrak{h}}^{(a,b)}(A),$$

where $(\tilde{\mathfrak{h}}^{(a,b)}(A))^\vee = \mathfrak{h}^{(a,b)}(A)$. Then the lisse étale L-sheaf associated to $V_{a,b} = \mathrm{Sym}^{a+b}(L^2)\boxtimes(\sigma^{-a}\otimes\overline{\sigma}^{-b})$ on the Shimura variety $\mathrm{Sh}_G(U)$ is precisely $\mathcal{V}_{a,b}$. We will write $\mathcal{V}_{a,b}$ for both the relative Chow motive and its p-adic realisation. As before, we get a projection from the étale realisation of relative Chow motives, to the ones with complex multiplication given by $[x]^* = x^a\overline{x}^b$. We will write

$$z^{[a,b]}_{\text{ét},m}\in H^2_{\text{ét},m}(Y_1(N)_{F_m},\mathcal{V}^\vee_{a,b}(1)) \tag{34}$$

for this projection. Since the realisation map is compatible with pushforward maps, we again obtain that $z^{[a,b]}_{\text{ét},m}$ is the pushforward via δ_m of $e^{[a,b]}_m$.

5.6.1 Deligne's Galois representations

Theorem 5.6.1. For all normalised Hecke newforms $f\in S_k(\Gamma_0(N),\chi)$, with Fourier coefficients in a number field K, there exists a compatible system $\{V_{f,\lambda}\}_{\lambda\in\mathrm{Spec}(\mathcal{O}_K)}\}$ os λ-adic K-rational representations of $G_\mathbb{Q}$ such that

$$L(\{V_{f,\lambda}\}_{\lambda\in\mathrm{Spec}(\mathcal{O}_K)},s\}) = L(f,s).$$

More precisely,

(i) For all primes $p\nmid N\cdot\mathrm{Norm}_{\mathbb{Q}^K}(K)$, $V_{f,\lambda}$ is unramified at p, and

$$\det((1-x\mathrm{Frob}_p)|_{V_{f,\lambda}}) = 1-a_p(f)x+\chi(p)p^{k-1}x^2;$$

(ii) For all primes $p\mid N$, $V_{f,\lambda}$ is ramified at p, and

$$\det((1-x\mathrm{Frob}_p)|_{V^{I_p}_{f,\lambda}}) = 1-a_p(f)x.$$

Proof. See [DarLec]. □

One can construct $\{V_{f,\lambda}\}$ from f. For f of weight 2, $\{V_{f,\lambda}\}$ are obtained from étale cohomology: the representaions are realised as the action of the absolute Galois group of $\mathbb{Q}$, $G_\mathbb{Q}$, on $H^1_{\text{ét}}(X_1(N)_{\overline{\mathbb{Q}}},\mathbb{Q}_l)$, for $l:=\mathrm{Norm}^K_\mathbb{Q}(\lambda)$. One can show that $H^1_t(X_1(N)_{\overline{\mathbb{Q}}_l},\mathbb{Q}_l) = (V_l(J_1(N))_{\overline{\mathbb{Q}}})^\vee$, for

$$V_l(J_1(N))_{\overline{\mathbb{Q}}} := \mathbb{Q}_l\otimes_{\mathbb{Z}_l}\varprojlim J_1(N)[l^n](\overline{\mathbb{Q}}),$$

where $J_1(N)$ is the Jacobian and $J_1(N)[l^n]$ is the group of l^n-torsion points of $J_1(N)$. This construction was first done by Eichler and Shimura, and completed by Igusa.

Now, assume f has weight greater than 2. Consider again the universal elliptic curve $\epsilon\longrightarrow Y_1(N)$ and then the fibred product $\epsilon^{k-2}\longrightarrow Y_1(N)$. Write $\mathcal{W}_{k-2}(N)$ for the compactification of ϵ^{k-2}. Then the representations $\{V_{f,\lambda}\}$ occur in the cohomology groups

$$H^{k-1}_{\text{ét}}(\mathcal{W}_{k-2}(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_l). \tag{35}$$

5.7 Exact sequences and projection to eigenspaces

Recall the theory of spectral sequences from the Appendix 7. In particular, consider the Hochschild-Serre exact sequence from 4.3.7 and 7.3.8.

Applying this to $G := \mathrm{Gal}(\overline{\mathbb{Q}}/K_m)$, $H := \mathrm{Gal}(\overline{\mathbb{Q}}/F_m)$ and $M := V_p(f)^*(\chi)$ one obtains the exact sequence:

$$0 \to H^1_{\text{ét}}\left(K_m/F_m, (V_p(f)^*(\chi))^{\mathrm{Gal}(F_m)}\right) \xrightarrow{\text{Inf}} H^1_{\text{ét}}\left(K_m, V_p(f)^*(\chi)\right) \xrightarrow{\text{Res}} H^1_{\text{ét}}\left(F_m, V_p(f)^*(\chi)\right)^{\mathrm{Gal}(K_m/F_m)} \to$$
$$\to H^2_{\text{ét}}\left(K_m/F_m, (V_p(f)^*(\chi))^{\mathrm{Gal}(F_m)}\right) \to \cdots$$

One can show that $(V_p(f)^*(\chi))^{\mathrm{Gal}(F_m)} = 0$, since the Galois representation is irreducible in this case (by a result in [Rib]). Thus, the exact sequence above yields an isomorphism

$$H^1_{\text{ét}}(K_m, V_p(f)^*(\chi)) \xrightarrow{\sim} H^1_{\text{ét}}(F_m, V_p(f)^*(\chi))^{\mathrm{Gal}(K_m/F_m)}. \tag{36}$$

Consider the representation $\mathrm{Sym}^k(\mathbb{Q}_p^2)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ and associate to it an étale $\mathbb{Q}_p$-sheaf, denoted $\mathcal{V}_K$. As before, we can interpret σ as a Galois character $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \overline{\mathbb{Q}}_p^{\times}$. Applying Proposition 7.3.7 to $H^1_{\text{ét}}(Y_1(N)_{F_m}, (\mathrm{Sym}^k(\mathfrak{h}^1(\epsilon)))^{\vee} \otimes \mathfrak{h}^{(a,b)}(A))$, where we take $\mathcal{F}$ to be $\mathcal{V}^{\vee}_{a,b}(1)$, X to be $Y_1(N)_{F_m}$, we obtain a sequence

$$\begin{aligned} H^2_{\text{ét}}\left(Y_1(N)_{F_m}, \mathcal{V}^{\vee}_{a,b}(1)\right) &\longrightarrow H^1\left(F_m, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{V}^{\vee}_k(1)) \otimes \sigma^a\overline{\sigma}^b\right) \\ &= H^1\left(K, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{V}^{\vee}_k(1)) \otimes \mathrm{Ind}^K_{F_m}\sigma^a\overline{\sigma}^b\right). \end{aligned}$$

The last equality above is obtained by applying Shapiro's lemma 4.3.3.

Let f be an eigenform of level N, weight $k+2$ and character ε. Consider the projector $z^{[k-j,j]}_{\text{ét},m} \in H^2_{\text{ét}}(Y_1(N)_{F_m}, \mathcal{V}^{\vee}_{k-j,j}(1))$ defined in (34). By equation (35) for $m \geq 0$ and $0 \leq j \leq k$, one obtains projections

$$\mathrm{pr}_f : H^2_{\text{ét}}(Y_1(N)_{F_m}, \mathcal{V}^{\vee}_{k-j,j}(1)) \longrightarrow H^1(F_m, V_p(f)^*(\sigma^{k-j}\overline{\sigma}^j)). \tag{37}$$

Define $z^{[f,j]}_{\text{ét},m} := \mathrm{pr}_f\left(z^{[k-j,j]}_{\text{ét},m}\right)$.

Definition 5.7.1. A **Hecke polynomial** at p of a newform $f \in S_{k+2}(\Gamma_1(N), \varepsilon)$ is the quadratic polynomial $X^2 - a_p(f)X + p^{k+1}\varepsilon(p)$.

Definition 5.7.2. A ***Heegner pair*** of finite type $(c, \mathcal{N}, \epsilon)$ and ∞-type (a,b) is a pair (f, χ), where χ is an algebraic Grössencharacter of finite type $(c, \mathcal{N}, \epsilon)$ and ∞-type (a,b), with $a, b \geq 0$, and f if a normalised cuspidal modular newform of level $\Gamma_1(N)$, character ϵ, and weight $a+b+2$.

We say (f,χ) has ***finite type*** $(c, \mathcal{N})$ if it has finite type $(c, \mathcal{N}, \epsilon)$ modulo N.

Let (f,χ) be a Heegner pair of finite type $(p^m, \mathcal{N}, \varepsilon)$ and ∞-type (a,b). Recall that the Grössencharacter χ on $\mathrm{Gal}(K^{ab}/K_m)$ restricts to $x \mapsto \varepsilon(x \mod \mathcal{N})^{-1}\sigma(x_p)^{-a}\overline{\sigma}(x_p)^{-b}$. If (f,χ) is a Heegner pair, one obtains for the representation $V := V_p(f)^* \otimes \chi$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ that $V^{\tau} \simeq V^*(1)$, for $\langle 1, \tau\rangle = \mathrm{Gal}(K/\mathbb{Q})$. Note that χ gives an extension of $\sigma^a\overline{\sigma}^b$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Proposition 5.7.3. The class $z^{[f,j]}_{\text{ét},m}$ lies in $H^1(F_m, V_p(f)^*(\chi))^{\mathrm{Gal}(F_m/K_m)}$.

Proof. Consider the action of $\hat{\mathcal{O}}^{\times}_{p^m}/U_{\mathcal{N},p^m}$ on the variety S_m, where we note that $\hat{\mathcal{O}}^{\times}_{p^m}/U_{\mathcal{N},p^m} \simeq (\mathbb{Z}/N\mathbb{Z})^{\times}$. Elements of $(\mathbb{Z}/N\mathbb{Z})^{\times}$ give us diamond operators $\langle a \mod \mathcal{N}\rangle$ on modular forms in $\Gamma_1(N)$ and we have an action on $Y_1(N) \times S_m$ given by $(\langle a \mod \mathcal{N}\rangle, a)$. Note that the embedding δ_m intertwines the action given on S_m with the one given on $Y_1(N) \times S_m$. As noted above, we also know that χ restricts to $\varepsilon\sigma^a\overline{\sigma}^b$ on $\mathrm{Gal}(K^{ab}/K_m)$. Then $z^{[f,j]}_{\text{ét},m}$ is invariant under $\mathrm{Gal}(K^{ab}/K_m)$ when we twist the Galois action by ε. □

Corollary 5.7.4. We can consider $z^{[f,j]}_{\text{ét},m}$ as an element in $H^1(K_m, V_p(f)^*(\chi))$.

Proof. This follows from equation (36). □

Proposition 5.7.5. Let (f, χ) be a Heegner pair of finite type $(p^m, \mathcal{N}, \varepsilon)$ and ∞-type $(k-j, j)$. Define

$$z^{[f,\chi]}_{\text{ét}} := \mathrm{norm}^{K_m}_{K}\left(z^{[f,j]}_{\text{ét},m}\right) \in H^1(K, V_p(f)^*(\chi)).$$

Remark 5.7.6. Note that the definition above is well-defined by Corollary 5.7.4.

Example 5.7.7 (The case $k = 0$). Assume that $k = 0$ (and $a = b = 0$). We will show how in this case the classes $z^{[0,0]}_{\text{mot},m}$ precisely correspond to Heegner points constructed by Gross in Chapter 3.

Recall that we have the map to divisors:

$$\begin{cases} X_0(N) \hookrightarrow J\left(X_0(N)\right), \\ \quad x \mapsto (x) - (0). \end{cases}$$

Also, note that $J(X_1(N)_{F_m})_{\mathbb{Q}} = \mathrm{CH}_1(X_1(N)_{F_m})_{\mathbb{Q}}$. We start with the pair $(\phi_m, A_m) = (A \to A_m, \mathbb{C}/(\tau_m\mathbb{Z} + \mathbb{Z}))$, which gives us a point $X_m = [(A_m, \phi_m)] \in Y_1(N)(K_m) \hookrightarrow X_1(N)(K_m)$. Let $f \in S_2(\Gamma_0(N), \chi)$ be a normalised Hecke newform with coefficients in a number field K. By modularity (Eichler-Shimura), we know that there exists a compatible system $\{V_{\lambda,f}\}_{\lambda\in\mathrm{Spec}(\mathcal{O}_K)}$ of λ-adic K-rational representations of $G_{\mathbb{Q}}$. Those are obtained from étale cohomology:

$$H^1_{\text{ét}}(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) = \left(V_p(J_1(N)_{\overline{\mathbb{Q}}})\right)^*,$$

where

$$V_p(J_1(N)_{\overline{\mathbb{Q}}}) = \varprojlim_n J_1(N)[p^n](\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

and $J_1(N)$ is the Jacobian of the modular curve $X_1(N)$, $J_1(N)[p^n]$ is the group of p^n-torsion points of $J_1(N)$ and where we take the inverse limit with respect to p-power maps. Note that $\mathrm{TSym}^0(\mathfrak{h}^1(\varepsilon)(1)) = \mathrm{TSym}^0(\mathfrak{h}^1(A)) = \mathbb{Q}$. Thus we consider

$$H^2_{\text{mot}}(Y_1(N)(F_m), \mathbb{Q}_p) = \mathrm{CH}^1(Y_1(N)(F_m)).$$

Remark 5.7.8. Note that in our construction of Heegner points from Chapter 3 we considered the modular parametrisation, which was a map:

$$X_0(N) \hookrightarrow \mathrm{Jac}(J_0(N)) \longrightarrow E.$$

Also, note that $\mathrm{Jac}(X_1(N)) \simeq \mathrm{Pic}^0(X_1(N))$, which is simply the 1-dimensional Chow group of $X_1(N)$. Thus we can understand in this case the 2-motivic cohomology group from our contruction of Heegner classes as a quotient of divisors, which is precisely the space our Heegner points "lived" in in Chapter 3.

Next, we are taking a point

$$z^{[0,0]}_{\text{mot},m} \in H^2_{\text{mot}}(Y_1(N)_{F_m}, \mathbb{Q}_p).$$

For $a = b = 0$, we also have $\mathcal{V}^{\vee}_{0,0}(1) = \mathbb{Q}_p$, so that for the étale realisation of $z^{[0,0]}_{\text{mot},m}$ we have

$$z^{[0,0]}_{\text{ét},m} \in H^2_{\text{ét}}(Y_1(N)_{F_m}, \mathbb{Q}_p).$$

The Hochschild-Serre spectral sequence gives us the map:

$$\begin{aligned} H^2_{\text{ét}}\left(Y_1(N)_{F_m}, \mathbb{Q}_p\right) &\longrightarrow H^1\left(F_m, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)\right) \\ &= H^1\left(K, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) \otimes \mathrm{Ind}^K_{F_m}\right), \\ &= H^1\left(K, \left(V_p(J_1(N)_{\overline{\mathbb{Q}}})\right)^* \otimes \mathrm{Ind}^K_{F_m}\right). \end{aligned}$$

Projecting our newform f as in (37), we obtain a class

$$z^{[f,0]}_{\text{ét},m} = \mathrm{pr}_f(z^{[0.0]}_{\text{ét},m}) \in H^1(F_m, V_p(f)^*(1)).$$

If (f, χ) is a Heegner pair of finite type $(p^m, \mathcal{N}, \epsilon)$ and infinite type $(0,0)$, we obtain a class in $H^1(K, V_p(f)^*(\chi))$, by taking the trace of K_m over K of $z^{[f,0]}_{\text{ét},m}$.

Recall that our Heegner points werre initially defined in 4.4.1 as classes in the cohomology group $H^1(K, E[p])$, which were then sent via the Kummer map to $H^1(K_n, E[p])^{\mathcal{G}_n}$ (using the Hochschild-Serre exact sequence). This corresponds to sending our generalised Heegner classes via the p-adic regulator map from $H^2_{\text{ét}}(Y_1(N)_{F_m}, \mathcal{V}^\vee_{a,b}(1))$ to $H^1(F_m, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{V}^\vee_k(1)) \otimes \sigma^a\overline{\sigma}^b)$, and then to considering edge maps

$$H^1(K_m, V_p(f)^*(\chi)) \xrightarrow{\sim} H^1(F_m, V_p(f)^*(\chi))^{\mathrm{Gal}(F_m/K_m)},$$

obtained similarly from the Hochschild-Serre spectral sequence.

6 Norm relations

The goal of this chapter is to prove *norm relations* for a set of generalised Heegner cycles defined in cohomology classes on the modular curve $Y_1(N(p^n))$ of level $\Gamma_1(N) \cap \Gamma_0(p^n)$ (for any $n \geq 1$).

6.1 Hecke operators

We start this section by introducing Hecke operators for modular curves as it was done in [KLZ17] §2.3 and §2.4 and [Kato] §2.8 and §2.9.

Let $M, N \in \mathbb{N}$ with $M + N \geq 5$. Define $Y(M, N)$ to be the $\mathbb{Z}[1/MN]$-scheme representing the functor

$$S \mapsto \{\text{isomorphism classes } (E, e_1, e_2)\},$$

where S is a $\mathbb{Z}[1/MN]$-scheme, E/S is an elliptic curve, $e_1, e_2 \in E(S)$ and

$$\beta : \begin{cases} (\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) & \longrightarrow E \\ \qquad (m, n) & \mapsto (me_1 + ne_2), \end{cases}$$

is an injection.

Definition 6.1.1. Let $M, N, A \in \mathbb{N}$ with $M + N \geq 5$. Define the following maps:

$$\begin{cases} Y(M, NA) & \longrightarrow Y(M, N), \\ (E, e_1, e_2) & \xmapsto{\mathrm{pr}_1} (E, e_1, Ae_2); \\ (E, e_1, e_2) & \xmapsto{\mathrm{pr}_2} (E/\langle Ae_2 \rangle, e_1 \pmod{Ae_2}, e_2 \pmod{Ae_2}); \end{cases}$$

$$\begin{cases} Y(AM, N) & \longrightarrow Y(M, N), \\ (E, e_1, e_2) & \xmapsto{\widehat{\mathrm{pr}}_1} (E, Ae_1, e_2); \\ (E, e_1, e_2) & \xmapsto{\widehat{\mathrm{pr}}_2} (E/\langle Ae_1 \rangle, e_1 \pmod{Ae_1}, e_2 \pmod{Ae_1}). \end{cases}$$

For $A \geq 1$ and M, N as above, we define $Y(M, N(A))$ as a $\mathbb{Z}[1/AM, N]$-scheme representing the functor

$$S \mapsto \{\text{isomorphism classes } (E, e_1, e_2, C)\},$$

with $(E, e_1, e_2) \in Y(M, N)(S)$ and C a cyclic subgroup of order AN such that the map

$$\begin{cases} (\mathbb{Z}/M\mathbb{Z}) \times C & \longrightarrow E \\ \qquad (x, y) & \mapsto xe_1 + y, \end{cases}$$

is injective. If such an injection exists, we say that C *contains* e_2 *and is complementary to* e_1. Similarly, $Y(M(A), N)$ classifies (E, e_1, e_2, C), with C a cyclic subgroup scheme of order AM containing e_1 and comlementary to e_2. Then one obtains $Y_1(N)$ for $Y(1, N)$, and similarly, we will write $Y_1(N(p^n))$ for $Y(1, N(p^n))$. Then there exist (natural) degeneracy maps

$$\mathrm{pr} : Y(M, N(A)) \longrightarrow Y(M, N), \qquad \mathrm{pr}' : Y(M, AN) \longrightarrow Y(M, N(A)),$$

such that $\mathrm{pr}_1 = \mathrm{pr} \circ \mathrm{pr}'$. Similarly we can define $\widehat{\mathrm{pr}}$ and $\widehat{\mathrm{pr}}'$ such that $\widehat{\mathrm{pr}}_1 = \widehat{\mathrm{pr}} \circ \widehat{\mathrm{pr}}'$.

Remark 6.1.2. Let $L \geq 3$ be such that $M \mid L$ and $AN \mid L$ (resp. $N \mid L$ and $AM \mid L$). Then we can consider $Y(M, N(A))$ (resp. $Y(M(A), N)$) as the quotient of $Y(L)$ by the action of the subgroup of $\mathrm{GL}_2(\mathbb{Z}/L)$ consisting of the matrices satisfying:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\begin{pmatrix} M & M \\ NA & N \end{pmatrix}}, \tag{38}$$

(resp. (mod $\begin{pmatrix} M & MA \\ N & N \end{pmatrix}$)).

Consider the map $\tilde{\varphi} : E \longrightarrow E/NC$ and for $e \in E(S)$, write e' for the image of e under $\tilde{\varphi}$. There exists an isomorphism

$$\varphi_A : \begin{cases} Y(M, N(A)) & \xrightarrow{\sim} Y(M(A), N), \\ (E, e_1, e_2, C) & \mapsto (E/NC, e_1', ([A]^{-1}(e_2) \cap C)', ([A]^{-1}\mathbb{Z}e_1)'). \end{cases}$$

Analogously we can define a map in the opposite direction

$$\varphi_{A^{-1}} : Y(M(A), N) \longrightarrow Y(M, N(A)).$$

The map φ_A (respectively $\varphi_{A^{-1}}$) corresponds to multiplication by A (respectively A^{-1}) on $\mathcal{H}$, and moreover

$$\mathrm{pr}_2 = \widehat{\mathrm{pr}} \circ \varphi_A \circ \mathrm{pr}', \qquad \widehat{\mathrm{pr}}_2 = \mathrm{pr} \circ \varphi_A^{-1} \circ \widehat{\mathrm{pr}}'.$$

Let ε_1 and ε_2 be the universal elliptic curves over $Y(M, N(A))$ and $Y(M(A), N)$ respectively. Then there exist isogenies

$$\lambda : E_1 \longrightarrow \varphi_A^*(\varepsilon_2), \qquad \lambda' : E_2 \longrightarrow \varphi_{A^{-1}}^*(\varepsilon_1).$$

Write

$$\Phi_A^* := \lambda^* \circ \varphi_A^*, \qquad (\Phi_{A^{-1}})_* := (\varphi_A)_* \circ (\lambda')_*.$$

Definition 6.1.3. Let l be a prime, $i, j, k \in \mathbb{Z}$. Let $\mathscr{H}_{\mathbb{Z}_p}$ denote the étale $\mathbb{Z}_p$-sheaf on $Y_1(N)$ given by the Tate module of the uiversal elliptic curve $\varepsilon/Y_1(N)$. For $l \mid MN$ the Hecke operator U_l' acting on $H^i_{\text{ét}}\left(Y(M, N), \mathrm{TSym}^k(\mathscr{H}_{\mathbb{Z}_p}(j))\right)$ is defined as

$$U_l' := (\mathrm{pr})_* \circ (\Phi_l)^* \circ (\widehat{\mathrm{pr}})^* = (\mathrm{pr})_* \circ (\Phi_{l^{-1}})_* \circ (\widehat{\mathrm{pr}})^*.$$

Note that the operator U_p' is the dual of the (universal) Hecke operator $U_p = (\widehat{\mathrm{pr}})_* \circ (\Phi_l)^* \circ (\mathrm{pr})^*$ under Poincaré duality.

Remark 6.1.4. Above we have defined the Hecke operator U_l only for $l \mid MN$, but one can define it completely analogously for the case $l \nmid MN$ (see [KLZ17]).

6.2 Cohomology classes on $Y_1(N(p^n))$

Let $\mathfrak{f}$ be a non-zero ideal of $\mathcal{O}_K$ such that $\mathcal{O}_K \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times$ is injective. Let K' be a field over K. A *CM-pair with modulus* $\mathfrak{f}$ *over* K' is a pair (E, α), where E is an elliptic curve over K' endowed with an isomorphism $\mathcal{O}_K \xrightarrow{\sim} \mathrm{End}(E)$ such that the composite map

$$\mathcal{O}_K \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{End}_{K'}(\mathrm{Lie}(E)) = K'$$

coincides with the inclusion map, and α is a torsion point in $E(K')$ such that the annihilator of α in $\mathcal{O}_K$ coincides with $\mathfrak{f}$. Write $K(\mathfrak{f})$ for the ray class field of $K \mod \mathfrak{f}$.

Theorem 6.2.1. There exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ which is isomorphic to $(\mathbb{C}/\mathfrak{f}, 1 \mod \mathfrak{f})$ over $\mathbb{C}$. This CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ is unique up to unique isomorphism.

Proof. See [Kato] §15.3.1. □

Now we return to our Heegner classes in étale cohomology

$$z^{[a,b]}_{\text{ét},m} \in H^2_{\text{ét},m}(Y_1(N)_{F_m}, \mathcal{V}^\vee_{a,b}(1)),$$

from (34). We can extend this definition to modular curves $Y_1(N(p^n))$ for any $n \geq 0$. Consider again our elliptic curve $A = \mathbb{C}/\mathcal{O}_K$ together with a generator t of $\mathcal{O}_K/\mathcal{N}\mathcal{O}_K$. By Theorem 6.2.1 the point

(A,t) (of $Y_1(N)(\mathbb{C})$) is defined over the ray class field F of K. Let $\tau \in \mathcal{H} \cap K$, as in 5.1, be such that its $\Gamma_1(N)$-orbit represents the point (A,t). For $m \geq n$ the point on $Y_1(N(p^n))$ corresponding to τ_m is defined over F_m. Consider the construction of $z^{[a,b]}_{\text{ét},m}$ we had for the curve $Y_1(N)$; replacing $Y_1(N)$ by $Y_1(N(p^n))$ we obtain classes

$$Z^{[a,b]}_{\text{ét},m,n} \in H^2_{\text{ét},m}(Y_1(N(p^n))_{F_m}, \mathcal{V}^\vee_{a,b}(1)). \tag{39}$$

Remark 6.2.2. From the definition of $Y_1(N(p^n))$ for $n=0$ it follows that $Z^{[a,b]}_{\text{ét},m,0} = z^{[a,b]}_{\text{ét},m}$.

Consider again $\tau_m = p^{-m}\tau$, $A_m = \mathbb{C}/\mathbb{Z}\tau_m + \mathbb{Z}$, $\phi_m : A \longrightarrow A_m$ the canonical cyclic p^m-isogeny, and t_m the torsion point of A_m, so that $\Gamma_1(N) \cdot \tau_m$ represents the pair (A_m, t_m). Also note that the modular curves $Y_1(N(p^n))$ are of level $\Gamma_1(N) \cap \Gamma_0(p^n)$.

Proposition 6.2.3. For $n \geq 1$ the points $Z^{[a,b]}_{\text{ét},m,n}$ defined in (39) satisfy

$$\operatorname{norm}^{F_{m+1}}_{F_m}\left(Z^{[a,b]}_{\text{ét},m+1,n}\right) = U'_p \cdot Z^{[a,b]}_{\text{ét},m,n}. \tag{40}$$

Proof. Let $\widehat{Y}$ be the modular curve of level $\widehat{\Gamma}(p^n) := \{g \in \Gamma_1(N) : g \equiv 0 \pmod{\begin{pmatrix} * & p \\ p^n & * \end{pmatrix}}\}$. We have the following diagram of modular curves:

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\varphi_{p^{-1}}} & Y(1, N(p^{n+1})) \\ \big\downarrow{\scriptstyle \widehat{\mathrm{pr}}} & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ Y(1, N(p^n)) & & Y(1, N(p^n)). \end{array}$$

Also, recall that we had:

$$\begin{array}{ccc} \widehat{Y} & \xrightarrow{\varphi_{p^{-1}}} & Y(1, N(p^{n+1})) \\ {\scriptstyle \widehat{\mathrm{pr}}'}\big\uparrow & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ Y(p, N(p^n)) & \xrightarrow{\widehat{\mathrm{pr}}_2} & Y(1, N(p^n)). \end{array}$$

By 6.2.1, the CM-point $\widehat{\tau}_m$ at level $\widehat{Y}$ corresponding to τ_m is defined over the field F_{m+1}. Write $\widehat{Z}^{[a,b]}_{\text{ét},m,n}$ for the corresponding class in cohomology, i.e.

$$\widehat{Z}^{[a,b]}_{\text{ét},m,n} \in H^2_{\text{ét}}(\widehat{Y}_{F_{m+1}}, \mathcal{V}^\vee_{a,b}(1)).$$

Then from the diagrams above we obtain:

$$(\widehat{\mathrm{pr}}')_*\left(\widehat{Z}^{[a,b]}_{\text{ét},m,n}\right) = (\widehat{\mathrm{pr}})^*\left(Z^{[a,b]}_{\text{ét},m,n}\right),$$

where $(\cdot)_*$ denotes the norm homomorphism and $(\cdot)^*$ denotes pullback. Thus

$$\operatorname{norm}^{F_{m+1}}_{F_m}\left(\widehat{Z}^{[a,b]}_{\text{ét},m,n}\right) = (\widehat{\mathrm{pr}})^*\left(Z^{[a,b]}_{\text{ét},m,n}\right), \tag{41}$$

and the orbit of $\widehat{\tau}_m$ under the action of $\mathrm{Gal}(F_{m+1}/F_m)$ is the preimage under $\widehat{\mathrm{pr}}$ of the $\Gamma(1, N(p^n))$-orbit of τ_m. Applying $(\mathrm{pr})_* \circ (\Phi_{p^{-1}})^*$ on both sides of (41), together with Definition 6.1.3 of U'_p, leaves us with

$$\operatorname{norm}^{F_{m+1}}_{F_m}\left((\mathrm{pr})_*(\Phi_{p^{-1}})^*\widehat{Z}^{[a,b]}_{\text{ét},m,n})\right) = U'_p \cdot Z^{[a,b]}_{\text{ét},m,n}.$$

Since $\Phi_{p^{-1}}$ is given by the action of the matrix $\begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}$ on the upper-half plane, it sends $\tau_m \mapsto \frac{\tau_m}{p} =$

τ_{m+1}. Thus, we conclude:

$$(\mathrm{pr})_*(\Phi_{p^{-1}})^*\left(\widehat{Z}^{[a,b]}_{\text{ét},m,n}\right) = Z^{[a,b]}_{\text{ét},m+1,n},$$

which gives us the desired result.

□

7 Appendix

7.1 Étale cohomology

We start by following [MilEt] and [StagMatt] for a short introduction to étale cohomology (for more details we refer to these two references).

Definition 7.1.1. A *site*, denoted $\mathrm{Cov}(\mathcal{C})$ is defined as the data of a category $\mathcal{C}$ and a collection of families $\{U_i \longrightarrow U\}_{i\in I}$ of morphisms in $\mathcal{C}$, satisfying:

(i) Given $U_i \longrightarrow U$ in some covering family and any morphism $V \longrightarrow U$, the fiber product $U_i \times_U V$ exists in $\mathcal{C}$,

(ii) Given a covering family $\{U_i \longrightarrow U\}_{i\in I}$ and any morphism $V \longrightarrow U$, the collection $\{U_i \times_U V \longrightarrow V\}_{i\in I}$ is again a covering family,

(iii) Given a covering family $\{U_i \longrightarrow U\}_{i\in I}$ and for each $i \in I$, another covering family $\{U_{ij} \longrightarrow U_i\}_{j\in J_i}$, the family of composites $\{U_{ij} \longrightarrow U\}_{i\in I, j\in J_i}$ is also a covering family,

(iv) If $f : V \longrightarrow U$ is an isomorphism, then $\{f\}$ is a covering family.

Example 7.1.2. Let X be a topological space. Consider the category of open sets of X, with morphisms given by inclusions. Covering families are given by tolopogical coverings $\{U_i \subset U\}_{i\in I}$.

Then one can define morphisms between sites as functors "respecting" the structure of the sites. Similarly, we can define isomorphisms between them.

Definition 7.1.3. Let $S = (\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site and let $\mathcal{D}$ be a category that admits arbitrary products. A functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{D}$ is called a *presheaf on the site S with values in $\mathcal{D}$.*

Definition 7.1.4. A *sheaf on the site S with values in $\mathcal{D}$* is a presheaf $\mathcal{F}$ such that for each covering family $\{U_i \longrightarrow U\}_{i\in I}$, the diagram

$$\mathcal{F}(U) \hookrightarrow \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I^2} \mathcal{F}(U_i \times_U U_j)$$

is exact.

We will consider the case of $\mathcal{D}$ being the category of abelian groups, Ab. Let $\mathcal{T}$ be a site, and $U \in \mathcal{T}$. Then there exists a section functor $\Gamma_U : \mathrm{Ab}(\mathcal{T}) \longrightarrow \mathrm{Ab}$, that sends F to $F(U)$. Also, note that any left exact aditive functor $f : \mathrm{Ab}(\mathcal{T}) \longrightarrow \mathcal{C}$, for $\mathcal{C}$ some abelian category, has right derived functors $R^q f$.

Definition 7.1.5. Let $F \in \mathrm{Ab}(\mathcal{T})$. Define the *$q$-th cohomology group of U with values in F* by $H^q(U, F) = (R^q\Gamma_U)(F)$.

Let X be a variety. Write $X_{\text{ét}}$ for the étale site of X. A ***sheaf of $\mathbb{Z}_p$-modules*** or a ***p-adic sheaf*** is a family $(\mathcal{M}_n : f_{n+1} : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n)$ such that

(i) for each n, $\mathcal{M}_n$ is a constructible sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules,

(ii) for each n, the map $f_{n+1} : \mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$ induces an isomorphism $\mathcal{M}_{n+1}/p^n\mathcal{M}_{n+1} \longrightarrow \mathcal{M}_n$.

For a sheaf $\mathcal{M} = (\mathcal{M}_n)$ of $\mathbb{Z}_p$-modules, define

$$H^r(X_{\text{ét}}, \mathcal{M}) := \varprojlim_n H^r(X_{\text{ét}}, \mathcal{M}_n).$$

Definition 7.1.6. A sheaf $\mathcal{M} = (\mathcal{M}_n)$ of $\mathbb{Z}_p$-modules is said to be *locally constant* if each $\mathcal{M}_n$ is locally constant. A locally constant sheaf of $\mathbb{Z}_p$-modules is often also called *lisse*.

Definition 7.1.7 (*Sheaves of $\mathbb{Q}_p$-modules*)**.** A sheaf of $\mathbb{Q}_p$-vector spaces is a $\mathbb{Z}_p$-sheaf $\mathcal{M} = (\mathcal{M}_n)$ such that we define

$$H^r(X_{\text{ét}}, \mathcal{M}) := \left(\varprojlim_n H^r(X_{\text{ét}}, \mathcal{M}_n)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Theorem 7.1.8. There is an equivalence of categories between $\mathrm{Ab}((\mathrm{Spec}(k))_{\text{ét}})$ and the category of of discrete G_k-modules preserving cohomology.

More precisely, if M is a discrete G_k-module, then it is an abelian group with an additive action of G_k such that $G_k \times M \longrightarrow M$ is abelian. By preserving cohomology, we mean that for every $\mathcal{F}$ associated to M,

$$H^1(G_k, M) = H^1((\mathrm{Spec}(k))_{\text{ét}}, \mathcal{F}),$$

where the left-hand side denotes group cohomology.

Remark 7.1.9. Theorem 7.1.8 allows us to work with group cohomology instead of étale cohomology (which is often much easier).

7.2 Realisations of relative motives

In this Section we mainly rely on [Kings].

Let $F/\mathbb{Q}$ be a real quadratic field. Let $\mathcal{A}_K^k(\mathbb{C})$ be the Shimura variety associated to $(\mathrm{Res}_{F/\mathbb{Q}} V_{2,F})^{\oplus k} \rtimes \mathrm{GL}$, where $V_2 := \mathbb{G}_{a,\mathbb{Q}}^{\oplus 2}$. Write $S_k := \mathcal{A}_k^0$. Then [Pink] shows that $\mathcal{A}_K^k(\mathbb{C})$ has a canonical structure of a normal quasiprojective variety and there exists a scheme $\mathcal{A}_K^k$ over $\mathbb{Q}$ whose $\mathbb{C}$-valued points are $\mathcal{A}_K^k(\mathbb{C})$.

Definition 7.2.1. With notationn as above, define

$$V_{K'}^{p,q} := \mathrm{Sym}^p(\mathcal{A}_{K'}, {}^t\Gamma_{p_1} \circ \pi_1, 0) \otimes \mathrm{Sym}^q(\mathcal{A}_{K'}, {}^t\Gamma_{p_2} \circ \pi_1, 0). \tag{42}$$

Then $\alpha \in \mathrm{Corr}^r(X, Y)$ acts on the motivic and étale cohomology group via

$$\begin{aligned} \alpha : H_?^i(X, j) \otimes_{\mathbb{Z}} \overline{Q} &\longrightarrow H_?^{i+2r}(Y, j+r) \otimes_{\mathbb{Z}} \overline{Q} \\ \zeta &\mapsto \mathrm{pr}_{2*}(\mathrm{cl}(\alpha) \cup \mathrm{pr}_1^* \zeta), \end{aligned}$$

with $\mathrm{cl}(\alpha) \in H_?^{2r}(X \times_S Y, r) \otimes_{\mathbb{Z}} \overline{Q}$ is the cohomology class of α; $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections and ? represents either étale or motivic cohomology. Define

$$H_{\text{ét}}^i((X, p, n), j) := \mathrm{image}(p^*(H_{\text{ét}}^{i+2n}(X, j+n) \otimes \overline{Q})),$$

Write pr for the projection $\mathcal{A}_{K'} \longrightarrow S_K$.

Next, define $G_{\mathbb{Q}}$-representations

$$V^{p,q} := \mathrm{Sym}^p V_{2,\overline{\mathbb{Q}}}^{\vee} \otimes \mathrm{Sym}^q V_{2,\overline{\mathbb{Q}}}^{\vee}.$$

Every G representation V defines a local system on S (cf. [Kings] pg. 70), which we denote $\tilde{V}$. Also, for an integer l, we obtain étale $\mathbb{Q}_l$-sheaves $\tilde{V}_l$ (cf. [Kings] pg. 70). Thus, $\tilde{V}^{p,q}$ and $\tilde{V}_l^{p,q}$ are local systems of $\overline{\mathbb{Q}}$-vector spaces (resp. étale $\mathbb{Q}_l \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$-sheaves on S).

Proposition 7.2.2. Let $\mathcal{V}_{\text{ét}}^{p,q}$ denote the vector bundle associated to the local system $\tilde{V}^{p,q}$. Then there exists an isomorphism:

$$H_{\text{ét}}^{i+p+q}(\mathcal{V}_{K'}^{p,q}, \mathbb{Q}_l(*)) \simeq H_{\text{ét}}^i(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{V}_l^{p,q}(*)).$$

Proof. See [Kings] Corollary 2.3.4. □

7.3 The Leray spectral sequence

Here we mainly rely on [MilSh] Chapter I, §12 and [SpecWol]. We also partially rely on the terminology introduced by [MilSh].

Let X, Y be varieties or schemes and $\pi : Y \longrightarrow X$ a morphism of varieties. For a sheaf $\mathcal{F}$ on $Y_{\text{ét}}$ define $\pi_* \mathcal{F}$ to be the sheaf on $X_{\text{ét}}$ with

$$\Gamma(U, \pi_* \mathcal{F}) = \Gamma(U_Y, \mathcal{F}),$$

where $U_Y := U \times_X Y$. Then the functor $\pi_* : \mathrm{Sh}(Y_{\text{ét}}) \longrightarrow \mathrm{Sh}(X_{\text{ét}})$ is left exact, so one can consider its rigth derived functors $R^r\pi_*$.

Proposition 7.3.1. Let $\pi : Y \longrightarrow X$ as above and $\mathcal{F}$ be a sheaf on $Y_{\text{ét}}$. Then $R^r\pi_*\mathcal{F}$ is the sheaf on $X_{\text{ét}}$ associated with the presheaf $U \mapsto H^r(U_Y, \mathcal{F})$.

Proof. See [MilSh] Prop. 12.1. ☐

Corollary 7.3.2. The stalk of $R^r\pi_*\mathcal{F}$ at $\overline{x} \longrightarrow X$ is

$$\varinjlim H^r(U_Y, \mathcal{F}),$$

where the limit is taken over all étale neighbourhoods (U, u) of $\overline{x}$.

Proof. See [MilSh] Cor. 12.2. ☐

Definition 7.3.3 (*First quadrant cohomological spectral sequence*)**.** Let $\{E_r^{p,q}\}$, for $p, q, r \in \mathbb{N}$, $r > a$, for some $a \in \mathbb{N}$, be a collection of abelian groups. For each $E_r^{p,q}$, $E_r^{p+r,q-r+1}$ let

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

be *boundary maps*, i.e. such that we have $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$. Also let

$$E_{r+1}^{p,q} \simeq \ker(d_r^{p,q})/\mathrm{im}(d_r^{p+r,q-r+1}).$$

A ***cohomological spectral sequence*** is defined to be the collection of $\{E_r^{p,q}\}$'s together with boundary maps. we say that a spectral sequence lives in the *first quadrant* if $p < 0$ or $q < 0$ implies $E_r^{p,q}$.

For given p, q, one can show that the group $E_r^{p,q}$ eventually stabilizes to a value $E_\infty^{p,q}$. This allows us to consider convergence of a spectral sequence. If the spectral sequence $E_r^{p,q}$ converges to groups H^n, we write

$$E_r^{p,q} \Rightarrow H^{p+q}.$$

This is the case if there is a filtration

$$0 = H_{n+1}^n \subseteq H_n^n \subseteq \cdots \subseteq H_0^n = H^n,$$

such that

$$E^{p,n-p} \simeq H_p^n / H_{p+1}^n.$$

Remark 7.3.4. Roughly speaking, a spectral sequence is a whole collection of data which "keeps track" of exact sequences that have maps between them.

Theorem 7.3.5 (*Leray spectral sequence*)**.** Let $\pi : Y \longrightarrow X$ be a morphism of varieties or schemes. For any sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, there exists a spectral sequence

$$H^r(X_{\text{ét}}, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y_{\text{ét}}, \mathcal{F}).$$

Proof. See [MilSh] Theorem 12.7. ☐

Remark 7.3.6. The Leray spectral sequence was initially defined for maps $\pi : Y \longrightarrow X$ between topological spaces: for $\mathcal{F}$ a sheaf on Y, there exists a sequence

$$H^r(X, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y, \mathcal{F}).$$

Corollary 7.3.7. Let X be a smooth variety over a field K of characteristic 0 and $\mathcal{F}$ a lisse étale sheaf on X. Then there exists a Leray spectral sequence

$$H^i_{\text{ét}}(\mathrm{Spec}(K), H^j(X_{\overline{K}}, \mathcal{F})) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathcal{F}).$$

Proof. See [KLZ15] pg. 6. □

Recall the Hochschild-Serre exact sequence from 4.3.7; more generally, we have a spectral sequence:

Theorem 7.3.8 (*Hochschild-Serre spectral sequence*)**.** Let X be a variety or a scheme. Let $\pi : Y \longrightarrow X$ be a Galois covering with Galois group G. For any sheaf $\mathcal{F}$ on $X_{\text{ét}}$, there exists a spectral sequence

$$E_2^{r,s} = H^r(G, H^s(Y_{\text{ét}}), \mathcal{F}(Y)) \Rightarrow H^{r+s}(X_{\text{ét}}, \mathcal{F}).$$

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