Planes in four space and four associated CM points

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Discriminants

For any two-dimensional rational subspace $L \subset \mathbb{R}^4$ we define the *discriminant* of L to be

$$\mathrm{disc}(L) = \mathrm{covol}(L \cap \mathbb{Z}^4)^2 \in \mathbb{Z}_{>0}$$

For any integer $D > 0$ set

$$\mathcal{R}_D = \{L \subset \mathbb{R}^4 \text{ rational plane} : \mathrm{disc}(L) = D\} \subset \mathrm{Gr}_{2,4}(\mathbb{R}).$$

This is a finite set and non-empty if and only if $D \not\equiv 0, 7, 12, 15 \bmod 16$. In this case, one can show that

$$|\mathcal{R}_D| = D^{1+o(1)}.$$

The first two CM points

By rotating L to a fixed copy of $\mathbb{R}^2$ in $\mathbb{R}^4$ the lattice $L \cap \mathbb{Z}^4 \subset L$ yields a lattice with complex multiplication in $\mathbb{R}^2$. It is well-defined up to (possibly not orientation preserving) isometries and so we obtain a point (the *shape* of $L \cap \mathbb{Z}^4$)

$$z_1(L) = [L \cap \mathbb{Z}^4] \in \mathrm{PGL}_2(\mathbb{Z}) \Big\backslash \mathrm{PGL}_2(\mathbb{R}) \Big/ \mathrm{O}(2) =: \mathcal{X}_2$$

Similarly, one defines the shape $z_2(L) = [L^\perp \cap \mathbb{Z}^4]$.

The Klein map

To define the second pair of CM points attached to L we make use of a quaternionic accident identifying $\mathrm{Gr}_{2,4}(\mathbb{R})$ with $\mathbb{S}^2 \times \mathbb{S}^2$ up to index two.

We identify $\mathbb{R}^4$ with the algebra $\mathbf{D}$ of Hamiltionian quaternions given by

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \ \mathrm{ij} = -\mathrm{ji} = \mathrm{k}.$$

For $x = x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k}$ we denote

$$\overline{x} = x_0 - x_1\mathrm{i} - x_2\mathrm{j} - x_3\mathrm{k},$$
$$\mathrm{Nr}(x) = x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2,$$
$$\mathrm{Tr}(x) = x + \overline{x} = 2x_0.$$

For $L \in \mathcal{R}_D$ with integral basis v_1, v_2 we set

$$a_1(L) = v_1\overline{v_2} - \tfrac{1}{2}\operatorname{Tr}(v_1\overline{v_2}),$$
$$a_2(L) = \overline{v_2}v_1 - \tfrac{1}{2}\operatorname{Tr}(\overline{v_2}v_1).$$

These are two integer vectors in $\mathbb{Z}^3$ that are independent of the choice of basis up to changing the sign in both of them simultaneously. Also,

$$L = \{x \in \mathbf{D} : a_1(L)x = xa_2(L)\}.$$

Using these vectors we may define the shapes

$$z_3(L) = [a_1(L)^\perp \cap \mathbb{Z}^3],\ z_4(L) = [a_2(L)^\perp \cap \mathbb{Z}^3] \in \mathcal{X}_2.$$

Equidistribution theorem

Theorem (Aka, Einsiedler, W. 2019)

Let p, q be two odd primes and consider the set $\mathcal{J}_D$ of tuples

$$\big(L, z_1(L), z_2(L), z_3(L), z_4(L)\big) \in \mathrm{Gr}_{2,4}(\mathbb{R}) \times \mathcal{X}_2^4$$

for $L \in \mathcal{R}_D$. When $D \to \infty$ with $|\mathcal{R}_D| \neq \emptyset$ and $-D$ is a non-zero square modulo p as well as modulo q, the sets $\mathcal{J}_D$ are equidistributed.

Note:

- Maass (50's) and Schmidt (90's) show equidistribution of the tuples $(L, z_1(L))$ on average.
- Equidistribution of the individual factors follows from Duke's theorem and its refinements (here by Harcos and Michel).

Action of SU_2^2

Note that $\mathrm{SU}_2^2 = \{x \in \mathbf{D} : \mathrm{Nr}(x) = 1\}^2$ acts on $\mathbf{D}$ via

$$(g_1, g_2).x = g_1 x g_2^{-1}$$

preserving the norm.
The map $L \mapsto (a_1(L), a_2(L))$ is equivariant in the sense that

$$a_1\big((g_1, g_2).L\big) = g_1 a_1(L) g_1^{-1}, \quad a_2\big((g_1, g_2).L\big) = g_2 a_2(L) g_2^{-1}.$$

We consider the stabilizer subgroup

$$\mathbb{H}_L = \{(g_1, g_2) \in \mathrm{SU}_2^2 : (g_1, g_2).L = L\}^\circ$$

One can naturally embed $\mathbb{H}_L$ into

$$\mathbb{G} = \mathrm{SU}_2^2 \times \mathrm{SL}_2^4 .$$

We deduce the main theorem from an equidistribution result for the orbits

$$\mathbb{G}(\mathbb{Q})\mathbb{H}_L(\mathbb{A}) \subset \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}).$$

By a joinings classification of Einsiedler and Lindenstrauss, we essentially need to understand if $\mathbb{H}_L$ is contained in any subgroup of $\mathbb{G}$ (that projects onto each factor).

Example of a stabilizer

We consider $\mathbf{D} = \mathrm{Mat}_2$ and the subspace

$$L = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

Then

$$a_1(L) = a_2(L) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One checks

$$\mathbb{H}_L(\mathbb{R}) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(a_1(L)) \times \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(a_2(L)) = A \times A$$

where

$$A = \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}^\times \right\}.$$

We would now like to understand the way in which $A \times A$ acts on the subspaces $L, L^\perp, a_1(L)^\perp, a_2(L)^\perp$.

- In the basis E_{11}, E_{22}, the matrix $a = (a_s, a_t) \in A \times A$ acts on L as $a_{st^{-1}}$.
- In the basis E_{12}, E_{21}, the matrix a acts on $L^\perp$ as a_{st}.
- By equivariance, a acts on $a_1(L)^\perp$ by conjugating with a_s. In the basis E_{12}, E_{21} this amounts to a_{s^2}.
- Similarly, for $a_2(L)^\perp$ we get a_{t^2}.

$$\rightsquigarrow \quad \left\{(a_s, a_t, a_{st^{-1}}, a_{st}, a_{s^2}, a_{t^2}) : s, t > 0\right\} \subset \mathrm{SL}_2(\mathbb{R})^6.$$

$$\rightsquigarrow \quad \left\{(a_s, a_t, a_{st^{-1}}, a_{st}, a_{s^2}, a_{t^2}) : s, t > 0\right\} \subset \mathrm{SL}_2(\mathbb{R})^6.$$

Fixing any two factors, say the 4th and the 5th, one sees that there is no non-trivial subgroup $M < \mathrm{SL}_2(\mathbb{R})^2$ that projects onto each component and that contains

$$\left\{(a_{st}, a_{s^2}) : s, t > 0\right\}.$$