Lusternik–Schnirelmann theory

Ana Žegarac

10.05.2021

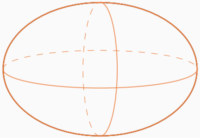
Literature:

- W. Klingenberg: "Lectures on closed geodesics" (1978);
- V. Ginzburg: lecture notes for the course Morse theory (2021).

Closed geodesics?

How many closed geodesics are there on a Riemannian manifold?

- On a **closed surface of negative curvature** every curve that is not null-homotopic can be deformed into a closed geodesic (Hadamard, 1898).
- On a **simply connected** compact surface there exist at least three closed geodesics without self-intersections (Lusternik and Schnirelmann, 1929).



- On a **compact Riemannian manifold** there exists at least one closed geodesic (Lusternik and Fet, 1951).

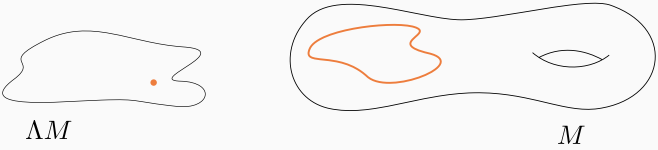
Background

Let (M, g) be a compact Riemannian manifold. We denote by

$$\Lambda M := H^1(S^1, M).$$

In other words, $c \in \Lambda M$ are maps $c : S^1 \to M$ that are absolutely continuous and

$$\int_0^1 g(\dot{c}(t), \dot{c}(t))\, \mathrm{d}t < \infty.$$



Energy integral

We define the **energy integral** by

$$E : \Lambda M \to \mathbb{R}, \qquad E(c) := \frac{1}{2}\int_0^1 g(\dot{c}(t), \dot{c}(t))\,\mathrm{d}t.$$

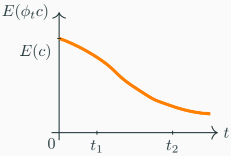
Theorem
A curve $c \in \Lambda M$ is a closed geodesic or a constant map if and only if it is a critical point of E.

We denote by $\phi_t : \Lambda M \to \Lambda M$ the negative gradient flow of E for time t.

Properties of the negative gradient flow of E

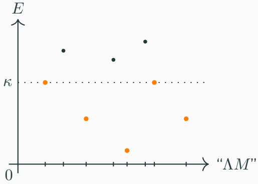
Important property of the negative gradient flow ϕ of E:

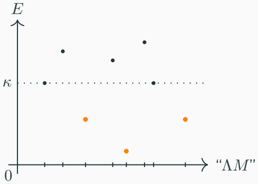
$$E(\phi_{t_1} c) \geq E(\phi_{t_2} c) \qquad \text{for} \qquad t_1 \leq t_2.$$



We denote

$\Lambda^\kappa M := \{c \in \Lambda M \mid E(c) \leq \kappa\}$ $\qquad$ $\Lambda^{\kappa-} M := \{c \in \Lambda M \mid E(c) < \kappa\}$





Lusternik–Schnirelmann theory on ΛM

A ϕ-**family** is non-empty set $\mathscr{A}$ of subsets $A \subset \Lambda M$ such that

1. $A \neq \emptyset$;
2. $E|_A$ is bounded;
3. if $A \in \mathscr{A}$, then $\phi_s(A) \in \mathscr{A}$ for all $s \geq 0$.

Examples of $\mathscr{A}$:

(a) $\mathscr{A} = \{\{\phi_t c\} \mid t \geq 0\}$;

Lusternik–Schnirelmann theory on ΛM

A ϕ-**family** is non-empty set $\mathscr{A}$ of subsets $A \subset \Lambda M$ such that

1. $A \neq \emptyset$;
2. $E|_A$ is bounded;
3. if $A \in \mathscr{A}$, then $\phi_s(A) \in \mathscr{A}$ for all $s \geq 0$.

Examples of $\mathscr{A}$:

(b) $\mathscr{A} = \{$set of all elements in a connected component of $\Lambda M\}$;

Lusternik–Schnirelmann theory on ΛM

A ϕ-**family** is non-empty set $\mathscr{A}$ of subsets $A \subset \Lambda M$ such that

1. $A \neq \emptyset$;
2. $E|_A$ is bounded;
3. if $A \in \mathscr{A}$, then $\phi_s(A) \in \mathscr{A}$ for all $s \geq 0$.

Examples of $\mathscr{A}$:

(c) $\mathscr{A} = \{|u| \mid u \in w\}$.

non-trivial singular homology class of ΛM

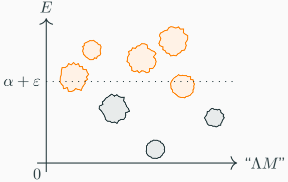
union of images of singular simplices of u

Lusternik–Schnirelmann theory on ΛM

Let $\alpha \in \mathbb{R}$ and choose ε such that there are no critical values of E in $(\alpha, \alpha + \varepsilon]$.

A ϕ-**family mod** $\Lambda^\alpha M$ is non-empty set $\mathscr{A}$ of subsets $A \subset \Lambda M$ such that

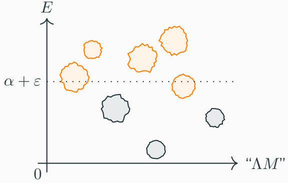
1. $A \neq \emptyset$;
2. $E|_A$ is bounded;
3. if $A \in \mathscr{A}$, then $\phi_s(A) \in \mathscr{A}$ for all $s \geq 0$;
4. if $A \in \mathscr{A}$, then $A \not\subset \Lambda^{\alpha+\varepsilon} M$.



The main theorem of the talk

We define the **critical value of a ϕ-family $\mathscr{A}$ of ΛM mod $\Lambda^\alpha M$** by

$$\kappa_{\mathscr{A}} \coloneqq \inf_{A \in \mathscr{A}} \sup E|_A.$$



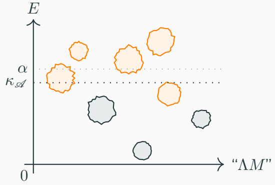
Theorem
It holds $\kappa_{\mathscr{A}} > \alpha$ and there exists a critical point c of E with $E(c) = \kappa_{\mathscr{A}}$.

To show existence of a closed geodesic on M, we need this critical value $\kappa_{\mathscr{A}}$ to be greater than zero.

Proof of the theorem

Proof: showing $\kappa_{\mathscr{A}} > \alpha$

- Recall if $A \in \mathscr{A}$, then $A \not\subset \Lambda^{\alpha+\varepsilon} M$.
- Assume $\alpha \geq \kappa_{\mathscr{A}} := \inf_A \sup E|_A$.
- Then $\exists A \in \mathscr{A}$ such that $\sup E|_A \leq \alpha$.
- Thus $A \subset \Lambda^\alpha M \subset \Lambda^{\alpha+\varepsilon} M$.
- Contradiction.



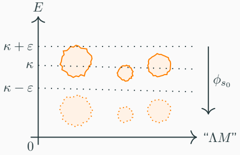
Proof of the theorem

Proof: showing existence of critical point c of E with $E(c) = \kappa_{\mathscr{A}}$.

- Assume c is **not** a critical point of E with $E(c) = \kappa_{\mathscr{A}}$.
- Recall
  1. $\kappa_{\mathscr{A}} := \inf_{A \in \mathscr{A}} \sup E|_A$;
  2. if $A \in \mathscr{A}$, then $\phi_s(A) \in \mathscr{A}$ for all $s \geq 0$.
- By definition of $\kappa_{\mathscr{A}}$, for every $\varepsilon > 0$ there exists $A \subset \Lambda^{\kappa_{\mathscr{A}}+\varepsilon}$.

Lemma. If κ is not a critical value of E, then $\kappa > 0$ and there exist $\varepsilon > 0$ and $s_0 \geq 0$ such that

$$\phi_{s_0}\left(\Lambda^{(\kappa+\varepsilon)} M\right) \subset \Lambda^{(\kappa-\varepsilon)-} M$$



- Thus by Lemma, $\exists A \in \mathscr{A}$ such that $\sup E|_{\phi_{s_0} A} \leq \kappa_{\mathscr{A}} - \varepsilon < \kappa_{\mathscr{A}}$.
- But then $\kappa_{\mathscr{A}} := \inf_A \sup E|_A < \kappa_{\mathscr{A}}$.
- Contradiction.

□

Recap

We saw basics of Lusternik–Schnierelmann theory on ΛM:

- Definition of ϕ-family $\mathscr{A}$ of ΛM (mod $\Lambda^\alpha M$);
- Critical value of $\mathscr{A}$:
$$\kappa = \inf_{A \in \mathscr{A}} \sup E|_A;$$
- **Theorem**: κ is an actual critical value of E.
- Possible application: existence of a closed geodesic on a compact Riemannian manifold.

Thank you!