Open Books and Lefschetz Fibrations

Joël Beimler

May 31, 2021

Why Open Books?

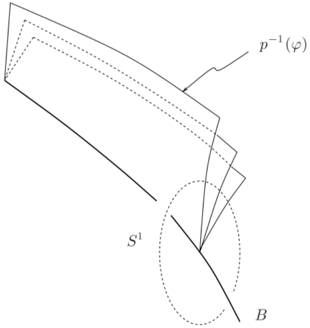
- Factor manifolds into lower dimensional ones
- Contact invariant
- Can read off some topological properties of a contact manifold from the associated open book decomposition

Definition

An **open book decomposition** of a manifold M^n is a pair (B, π) for

- B^{n-2} a codimension two submanifold with trivial normal bundle;
- $\pi : M \setminus B \to S^1$ a fibration such that on a tubular neighbourhood $B \times \mathbb{D}^2$, we have $\pi(x, re^i\varphi) = \varphi$.

- B is the **binding**
- $F := \pi^{-1}(\varphi)$ is the **page**
- $\partial F = B$



An open book decomposition near the binding[1]

[1]Image credit: H. Geiges, *An Introduction to Contact Topology*

An Easy Example

$$M = \mathbb{C}$$

$$B = \{0\}$$

$$\pi(z) = \frac{z}{|z|}$$

Easy Ex

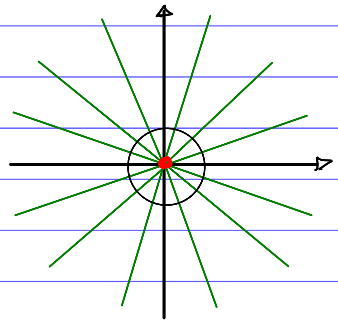
$M = \mathbb{C}$

$B = \{0\}$

$\pi : \mathbb{C} \setminus \{0\} \longrightarrow S^1$

$z = re^{i\varphi} \longmapsto \frac{z}{|z|} = e^{i\varphi}$

pages: $\pi^{-1}(\varphi) = \{ re^{i\varphi} \mid r > 0 \}$



How to get more examples

Let $f : M \to \mathbb{C}$ be a map transverse to the pages of the standard open book on $\mathbb{C}$ so that 0 is in the image of f. Then setting

$$B = f^{-1}(0), \qquad \pi(x) = \frac{f(x)}{|f(x)|}$$

defines an open book decomposition of M.

A slightly more interesting example

$$M = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

$$f : S^3 \to \mathbb{C}, \quad f(z_1, z_2) = z_1$$

$$f^{-1}(0) = \{(0, z_2) \mid |z_2| = 1\} = S^1 = B$$

Ex 2

$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$

$f: S^3 \longrightarrow \mathbb{C}$

$(z_1, z_2) \longmapsto z_1$

$\Rightarrow$ • $B = f^{-1}(0) = \{(0, z_2) \in S^3\} \cong S^1$

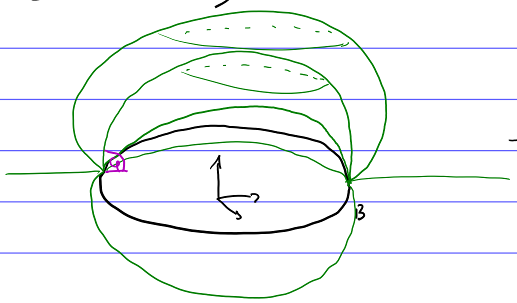
• $\pi : S^3 \setminus B \longrightarrow S^1$

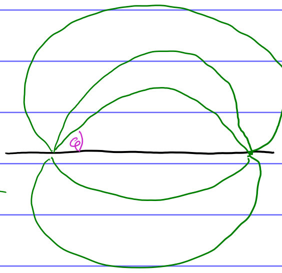
$(z_1, z_2) \longmapsto \frac{z_1}{|z_1|}$

Pages: $\pi^{-1}(\theta) = \{(\sqrt{1-|z_2|^2}\, e^{i\theta}, z_2) \mid |z_2| \leq 1\}$

$= D^2$

$S^3 = \mathbb{R}^3 \cup \{\infty\}$





Abstract Open Books

We can reconstruct (up to diffeomorphism) an open book decomposition from the pages and information about how they glue together as they move around the binding:

Definition

An **abstract open book** (F, ψ) consists of a manifold F^n with nonempty boundary and $\psi \in \mathrm{Diff}(F, \partial F)$ equal to id near ∂F. ψ is called the **monodromy**.

Definition

The **mapping torus** associated to F and $\psi \in \mathrm{Diff}(F)$ is

$$F(\psi) = F \times [0, 2\pi]/((x, 2\pi) \sim (\psi(x), 0)).$$

Abstract Open Books

Theorem

Given an abstract open book (F^n, ψ), the $(n+1)$-manifold

$$OB(F, \psi) = F(\psi) \cup_{\partial F \times S^1} (\partial F \times \mathbb{D}^2)$$

carries an open book decomposition with binding $B \cong \partial F$, pages $\cong F$.

- Open book on S^3 from before: page D^2, monodromy id_{D^2}.

We'll abuse notation from here on and let $OB(F, \psi)$ refer to both the glued manifold and the open book decomposition induced on it.

Contact Geometry

"Odd-dimensional cousin of symplectic geometry"

Definition

A **contact structure** ξ on a manifold M^{2n+1} is a codimension-one distribution of TM such that for any $\alpha \in \Omega^1(M)$ locally satisfying $\xi = \ker \alpha$, we have

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Any such α is called a **contact form**.

Contact Geometry

- (W, ω) a symplectic manifold such that $\omega = d\lambda$ near ∂W
- Vector field $V_\lambda \in \mathfrak{X}(W)$ defined by

  $$\imath_{V_\lambda}\omega = \lambda$$

  is **transverse** to ∂W

Then λ is a contact form on ∂W.
We say W has **contact-type** boundary.

Contact Open Books

Definition

A contact manifold $(M, \xi = \ker \alpha)$ is **supported by** an open book decomposition (B, π) of M if

- $d\alpha$ is symplectic on the pages $\pi^{-1}(\varphi)$;
- α is contact on B.

Contact Open Books

- $(W^{2n}, d\lambda, V_\lambda)$ **Liouville domain**: compact exact symplectic manifold such that V_λ as in $\imath_{V_\lambda} d\lambda = \lambda$ is outward pointing on ∂W;
- $\psi \in \mathrm{Symp}(W, d\lambda; \partial W)$ equal to id near ∂W;

Theorem (Giroux)

Then $OB(W, d\lambda; \psi)$ carries a contact structure supported by the open book decomposition.

$OB(W, d\lambda; \psi)$ is thus a $(2n+1)$-manifold with

- Liouville domains $(W, d\lambda)$ as pages
- so that λ is contact on the binding $B = \partial W$.

Some Theorems

- (Giroux) Every compact contact manifold (M^{2n+1}, ξ) admits a supporting open book with ($2n$-dimensional) Liouville domains as pages.
- Contact structures supported by the same open book are isotopic.

Giroux Correspondence

Let M^3 be a closed oriented 3-manifold. There is a bijective correspondence between

$$\{\text{Oriented contact structures on } M \text{ up to isotopy}\}$$

and

$$\{\text{Open book decompositions of } M \text{ up to “positive stabilization”}\}.$$

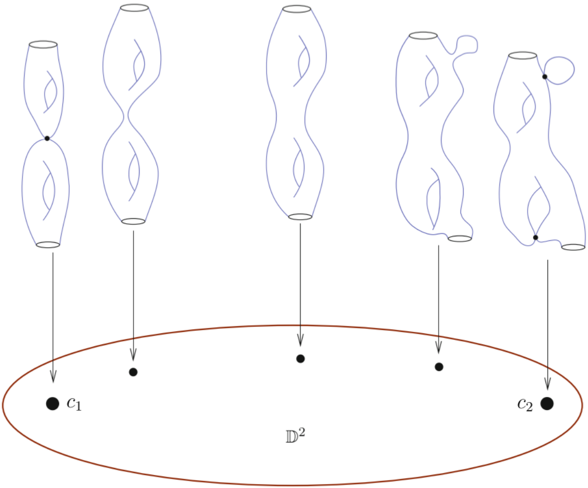
Lefschetz Fibrations

Definition

A **Lefschetz fibration** $f : W^{2n} \to \mathbb{D}^2$ is a smooth map with finitely many critical points near which there are complex charts in which f can be written as

$$f(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2.$$

- A Lefschetz fibration is **(exact) symplectic** if W carries an exact symplectic form $d\lambda$ making the fibres into Liouville domains.



A Lefschetz fibration over $\mathbb{D}^2$ with two critical values c_1 and c_2[2]

[2] Image credit: C. Wendl, *Holomorphic Curves in Low Dimensions*

Open Books through Lefschetz Fibrations

We consider Lefschetz fibrations whose fibres $F_z = f^{-1}(z)$ have boundary; then W is a manifold with **corners**. ∂W consists of

- $\partial_v W := f^{-1}(\partial \mathbb{D}^2)$; fibre bundle over S^1, so of the form $\partial_v W \cong F(\psi)$.
- $\partial_h W := \bigcup_{z \in \mathbb{D}^2} \partial F_z$; $\mathbb{D}^2$ contractible, so $\partial_h W \cong \partial F \times \mathbb{D}^2$.

These components meet in $\partial(\partial_v W) = \partial(\partial_h W) = \bigcup_{z \in \partial \mathbb{D}^2} \partial F_z \cong S^1 \times \partial F$.

$$\Longrightarrow \qquad \begin{aligned} \partial W &= \partial_v W \cup_{S^1 \times \partial F} \partial_h W \\ &= F(\psi) \cup_{S^1 \times \partial F} (\partial F \times \mathbb{D}^2). \end{aligned}$$

ψ is in fact the **monodromy** of f (depends on critical points of f).

Fillability

To sum up:

- $f : (W^{2n+2}, \omega = d\lambda) \to \mathbb{D}^2$ symplectic Lefschetz fibration, monodromy ψ, fibre F^{2n} with $\partial F \neq \emptyset$

$\implies$ $\partial W \cong OB(F, d\lambda; \psi)$ is a contact manifold supported by the induced open book.

Definition

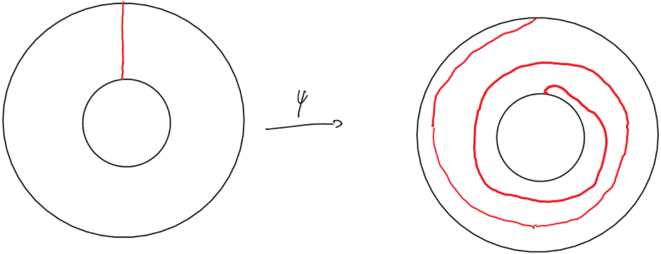
A contact manifold (M, ξ) is **fillable** (mod. variations) if there exists (W, ω) such that $\partial W = M$ and ω induces ξ.

$\implies$ A symplectic Lefschetz fibration induces a certain kind of filling of the boundary of its total space.

Example

- $V = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid z_0^2 + z_1^2 + z_2^2 = 1\}$
- $f : V \to \mathbb{C}$ given by $f(z_0, z_1, z_2) = z_0$ is a Lefschetz fibration
- Restrict to $W = V \cap \mathbb{D}^6 \cong V \cap (\mathbb{D}^2 \times \mathbb{D}^4)$, so that $f : W \to \mathbb{D}^2$.

$\implies$ there is an open book of ∂W.
pages: Annuli
monodromy: "Square of a (right-handed) Dehn twist"
$\partial W = V \cap S^5$ is called a **Brieskorn manifold**.



The monodromy of f

Final Ex

$V = \{(z_0, z_1, z_2) \mid z_0^2 + z_1^2 + z_2^2 = 1\}$

$f : V \longrightarrow \mathbb{C}$

$(z_0, z_1, z_2) \longmapsto z_0$

This is Lefschetz:

Critical points: $(\pm 1, 0, 0)$

Consider a nbhd of $(1,0,0)$ in which we can describe V with

$\tilde{v} : B_\varepsilon(0) \longrightarrow V$

$(z_1, z_2) \longmapsto \left(\sqrt{1 - z_1^2 - z_2^2}, z_1, z_2\right)$

$f \circ \tilde{v} : B_\varepsilon(0) \longrightarrow B_\delta(1)$

For δ small, $\sigma : B_\delta(1) \longrightarrow \mathbb{C}$ will be a complex chart, and

$w \longmapsto 1 - w^2$

$\sigma \circ f \circ \tilde{v}(z_1, z_2) = z_1^2 + z_2^2.$