AUTUMN 2021: STUDENT SEMINAR IN SYMPLECTIC VS. CONTACT MANIFOLDS

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Conventions and Notation

Manifolds and submanifolds: All our manifolds are smooth, i.e. C^∞, and have no boundary unless otherwise indicated.

Symplectic manifolds are denoted (M, ω) and their points are usually p.

The special case of a cotangent bundle is $(T^*Q, -d\lambda)$ where $\lambda = \sum p_i dq_i$ is the tautological (or Liouville) 1-form.

Contact manifolds are cooriented and denoted (Y, α) and associated contact structure will be called by $\xi =$ ker α.

All our submanifolds are assumed to be *properly embedded*, i.e., if L is a submanifold of the manifold M, then L is a manifold and there is a proper injective immersion $L \to M$.

In particular, all our submanifolds admit adapted charts, under which the submanifold looks like a subspace of euclidean space.

A submanifold is called *closed* when it is compact and without boundary.

Lagrangian and Legendrian submanifolds are denoted L and K, respectively. Although Legendrian submanifolds are often called by L in the literature, we will stick to our convention for the sake of avoiding notational inconsistencies.

Tori: A *torus* T is an abelian compact connected Lie group, its *Lie algebra* is denoted by $\mathfrak{t} = \mathrm{Lie}(T)$, the *exponential map* by

$$\exp_T : \mathfrak{t} \longrightarrow T$$

and the *integral lattice* is $\mathfrak{t}_\mathbb{Z} := \ker(\exp_T)$. The *standard torus* of rank $n \geq 1$, is the product of n copies of S^1:

$$\mathbb{T}^n := (S^1)^n$$

and S^1 is the Lie group of complex numbers of absolute value 1, so elements of $\mathbb{T}^n$ are n-tuples

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right).$$

This identifies $\mathbb{T}^n$ with the quotient

$$\mathbb{T}^n \simeq \mathbb{R}^n/(2\pi\mathbb{Z})^n \simeq (\mathbb{R}/2\pi\mathbb{Z})^n$$

and we view this as a special case of the identification of a torus T with its Lie algebra modulo the integral lattice, $(2\pi\mathbb{Z})^n$ in this case, via the *exponential map*.

$$\exp_{\mathbb{T}^n} : \mathbb{R}^n \longrightarrow \mathbb{T}^n , \qquad \exp_{\mathbb{T}^n}(\theta_1, \ldots, \theta_n) = \left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) .$$

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Implicitly, we use the standard basis of $\mathbb{R}^n$ as the chosen basis $X_1, \ldots, X_n$ of the Lie algebra. This also yields global coordinates (mod 2π) θ_k on $\mathbb{T}^n$. The element

$$[\theta] := [\theta_1, \ldots, \theta_n] = \left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \in \mathbb{T}^n$$

can also be viewed as the element achieved from the *identity* element

$$\mathbb{1} := [0, \ldots, 0] = (1, \ldots, 1) \in \mathbb{T}^n$$

by flowing along X_1 for time θ_1, along X_2 for time θ_2, ..., and along X_n for time θ_n.

An abstract torus T gets identified with the standard torus $\mathbb{T}^n$ once we choose a basis $X_1, \ldots, X_n$ of its Lie algebra $\mathfrak{t}$ (i.e., identify $\mathfrak{t}$ with $\mathbb{R}^n$), under which the integral lattice gets identified with $(2\pi\mathbb{Z})^n$. We thus get a Lie group isomorphism given by the exponential maps

$$T \xrightarrow{\simeq} \mathbb{T}^n , \qquad \exp_T(\theta_1, \ldots, \theta_n) \longmapsto \left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) .$$

Such an isomorphism is called a *splitting* of T.

The dual vector space of the Lie algebra of a torus T is $\mathfrak{t}^* := \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$, with natural pairing $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}, \langle \xi, X \rangle := \xi(X)$. The *weight lattice* of T is the dual of the integral lattice $\mathfrak{t}_\mathbb{Z}$, that is, $\mathfrak{t}^*_\mathbb{Z} := \mathrm{Hom}_\mathbb{Z}(\mathfrak{t}_\mathbb{Z}, \mathbb{Z})$. Under a splitting $T \simeq \mathbb{T}^n$, which yields an isomorphism of the integral lattice $\mathfrak{t}_\mathbb{Z} \simeq (2\pi\mathbb{Z})^n$, and under the isomorphism of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$ via the euclidean inner product, the weight lattice gets identified with $\left(\frac{1}{2\pi}\mathbb{Z}\right)^n$.

Actions: We denote the diffeomorphism given by the action of $g \in T$ on a manifold M in terms of multiplication as

$$p \longmapsto g \cdot p \qquad (p \in M) .$$

In particular, the *standard (diagonal) action* of $\mathbb{T}^n$ on $\mathbb{C}^n$ is given by

$$[\theta] \cdot (z_1, \ldots, z_n) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

and the *standard action* of $\mathbb{T}^n$ on $\mathbb{CP}^n$ by

$$[\theta] \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{i\theta_1} z_1 : \ldots : e^{i\theta_n} z_n] .$$

Infinitesimal actions: Once we have an action of a torus T on a manifold M, we have for each $X \in \mathfrak{t}$ a corresponding vector field on M, denoted $X^\sharp \in \Gamma(TM)$ and defined at any $p \in M$ as the tangent vector to the curve through p produced by the action of the one-parameter subgroup of T generated by X, namely:

$$X^\sharp(p) := \left.\tfrac{d}{dt}\right|_{t=0} (\exp_T(tX) \cdot p) .$$

In particular, for the *standard (diagonal) action* of $\mathbb{T}^n$ on $\mathbb{C}^n$ and the k-th standard basis vector $X_k \in \mathbb{R}^n$ we have

$$X_k^\sharp = \tfrac{\partial}{\partial \theta_k} = i \left(z_k \tfrac{\partial}{\partial z_k} - \overline{z}_k \tfrac{\partial}{\partial \overline{z}_k} \right) .$$

Moment maps: With the above identifications, a *moment map* for an action of a torus T on a symplectic manifold (M, ω) is defined to be a map

$$\mu : M \longrightarrow \mathfrak{t}^* ,$$

such that

- μ is T-invariant: $\mu(g \cdot p) = \mu(p)$ for all $g \in T$, $p \in M$, and

- for each $X \in \mathfrak{t}$, the *coordinate function* $\langle \mu, X \rangle : M \to \mathbb{R}$ is a *hamiltonian function* for the vector field $X^\sharp$ on M induced by X, that is,

$$d\langle \mu, X \rangle = -\imath_{X^\sharp}\omega \ . \qquad \text{Notice our sign convention!}$$

An action of T admitting a moment map is called a *hamiltonian action*.

Local models: To represent the local geometry of symplectic toric manifolds, we consider the *Darboux* $\mathbb{T}^n$*-model*,

$$(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu_0) \ ,$$

where we ω_0 is the usual symplectic form

$$\omega_0 = \tfrac{i}{2} \sum_{k=1}^n dz_k \wedge d\overline{z}_k \ ,$$

the standard torus $\mathbb{T}^n$ acts diagonally by

$$[\theta] \cdot (z_1, \dots, z_n) = (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \ ,$$

the vector field on $\mathbb{C}^n$ induced by the k-th standard basis vector $X_k \in \mathbb{R}^n$ is

$$X_k^\sharp = \tfrac{\partial}{\partial \theta_k} = i \left(z_k \tfrac{\partial}{\partial z_k} - \overline{z}_k \tfrac{\partial}{\partial \overline{z}_k} \right) \ .$$

and the moment map $\mu_0 : \mathbb{C}^n \to \mathbb{R}^n$ is the map

$$\mu_0(z_1, \dots, z_n) := \tfrac{1}{2}(|z_1|^2, \dots, |z_n|^2) \ .$$

The image of this moment map is the positive octant, $\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n | \text{ all } x_i \geq 0\}$.

Fubini-Study: We choose a scaling factor giving the *Fubini-Study form* on $\mathbb{CP}^n$ as

$$\omega_{\text{FS}} = \tfrac{i}{2} \partial\bar{\partial} \ln(1 + |z|^2)$$

with respect to standard charts with n coordinates z_j, $0 \leq j \leq n$, $j \neq k$, on each open set

$$\mathcal{U}_k = \{[z_0 : \ldots : z_{k-1} : 1 : z_{k+1} : \ldots : z_n] \in \mathbb{CP}^n\} \quad \longrightarrow \quad \mathbb{C}^n \ .$$

In particular for $n = 1$, we have that the sphere $\mathbb{CP}^1$ has $\omega_{\mathsf{FS}} = \frac{1}{4}\omega_{\text{eucl}}$ and total area π with respect to ω_{FS}, whereas the euclidean area of a unit sphere in $\mathbb{R}^3$ is 4π.

Symplectic reduction of $(\mathbb{C}^{n+1}, \omega_0, \mathbb{T}^{n+1}, \mu_0)$ at level $\frac{1}{2}$ (that is, at the unit sphere) produces as reduced space $(\mathbb{CP}^n, \omega_{\text{FS}})$.

Baby case: On $\mathbb{R}^2 \simeq \mathbb{C}$ we have points

$$x + iy = z = re^{i\theta}$$

and standard symplectic form

$$dx \wedge dy = \tfrac{i}{2} dz \wedge d\overline{z} = r dr \wedge d\theta \ .$$

The Lie group S^1 (unit circle in $\mathbb{C}$) acts by multiplication, $e^{i\alpha} \cdot \left(re^{i\theta}\right) = re^{i(\theta+\alpha)}$, so that the Lie algebra element X generating $e^{it}, t \in \mathbb{R}$, corresponds to the vector field $X^\sharp = \frac{\partial}{\partial \theta}$ and has moment map (or hamiltonian function) $\frac{r^2}{2}$. The image of this moment map is the positive axis.

Symplectic toric manifolds: A *symplectic toric manifold*, denoted

$$(M, \omega, T, \mu) \ ,$$

is a compact connected symplectic manifold (M, ω) equipped with an effective hamiltonian action of a torus T of dimension equal to half the dimension of the manifold,

$$n = \dim T = \frac{1}{2} \dim M \ ,$$

and with a choice of a corresponding moment map $\mu : M \to \mathfrak{t}^*$. We thus restrict to *compact* symplectic toric manifolds and the Darboux $\mathbb{T}^n$-model above is not a symplectic toric manifold in our sense. However, much of the theory extends to *noncompact* symplectic toric manifolds where the moment map is a proper map.

Isomorphism: Two symplectic toric manifolds, $(M_k, \omega_k, T, \mu_k)$, $k = 1, 2$, are *isomorphic* if there exists an equivariant symplectomorphism $\varphi : M_1 \to M_2$. Equivariance means

$$\varphi(g \cdot p) = g \cdot \varphi(p), \qquad \text{for all } g \in T \text{ and } p \in M_1 \ .$$

Note that the torus is fixed and that the moment maps necessarily differ by a constant, in the sense that

$$\mu_1 = \mu_2 \circ \varphi + c \quad \text{ for some } c \in \mathfrak{t}^*.$$

(for general hamiltonian torus actions, moment maps are unique up to a constant).

Elementary geometry: A *convex polytope* $\Delta \subset \mathfrak{t}^*$ is the convex hull of a finite set of points in $\mathfrak{t}^*$. A *face* of a convex polytope Δ is a nonempty intersection of Δ with a closed halfspace whose boundary is disjoint from the interior of Δ. In particular, the whole polytope is a face of itself. The dimension of a face is the dimension of its affine hull. A *vertex* is a 0-dimensional face, an *edge* is a 1-dimensional face and a *facet* is a face of codimension 1 with respect to the dimension of the polytope.

A polytope $\Delta \subset \mathfrak{t}^*$ is called **unimodular** if, for each vertex v, there is a $\mathbb{Z}$-basis $\eta_1, \ldots, \eta_n$ of the *weight lattice* $\mathfrak{t}^*_{\mathbb{Z}}$ and a neighborhood $\mathcal{U}$ of v, such that

$$\Delta \cap \mathcal{U} = \{ v + t_1 \eta_1 + \ldots + t_n \eta_n \mid t_i \in [0, \epsilon) \} \ .$$

Notice that then this basis is unique for each vertex (up to vector order).

Classification: *** Spoiler Alert *** By Delzant's theorem, $2n$-dimensional symplectic toric manifolds (up to isomorphism) are classified by *unimodular polytopes* in $\mathfrak{t}^*$ up to translation. This bijective correspondence is given by the image of the moment map.

There is a weaker notion of equivalence allowing for a group isomorphism $\lambda : T \to T$ and the symplectomorphism $\varphi : M_1 \to M_2$ to be equivariant w.r.t. λ, i.e.,

$$\varphi(g \cdot p) = (\lambda(g)) \cdot \varphi(p), \quad \text{ for all } g \in T, p \in M_1 \ .$$

In this case, the classification is given by unimodular polytopes in $\mathfrak{t}^*$ up to translation and the action of the general linear group $\mathrm{GL}(n, \mathbb{Z})$.

Minor Terminology Controversies

Symplectic toric manifolds (short **STM**) vs. *toric symplectic manifolds*: toric manifolds are viewed as fundamental mathematical objects and here they are treated from the symplectic viewpoint.

Moment map vs. *momentum map*: although directly related to linear and angular momenta, the concept of a moment map has gained independence in geometry. The word *moment* was preferred by Atiyah, Donaldson, Fulton, Guillemin, Sternberg, etc. and is easier to say.

hamiltonian vs. *Hamiltonian*, **lagrangian** vs. *Lagrangian*: the ultimate honor is to have a word based on one's name enter the language uncapitalized, like *abelian, euclidean, pasteurized,* etc.

i is the imaginary unit (and neither $\sqrt{-1}$ nor j).

n is the dimension of the torus and $2n$ the dimension of a corresponding STM.

This leaves k, m, j,... for other indices.

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