Notes: b-Contact Structures on Symplectic Hyperboloids

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The McGehee Transformation

If we have a vector field on $M := T^*\mathbb{R}^+ \times T^*\mathbb{S}^{n-1}$, then we can look at the push-forward via τ_{McG}^{-1} of this to the two components of $X \setminus Z$ and try to extend it to Z to get a b^3-vector field.

Q: Does this push forward always extend to a b^3-vector field on X?
A: No. (for example the radial vector field)
Q: Does the push forward of a Liouville vector field extend to a b^3-Liouville vector field on X?
A: Yes, assuming it extends to a b^3-vector field in the first place.

If we have a hypersurface $S \subset M$ which is closed as a subset. Then we can define:
$S_{McG} := \overline{\tau_{\mathrm{McG}}^{-1}(\mathrm{S})} \subset X$
(Be aware that $\tau_{\mathrm{McG}}^{-1}(\mathrm{S})$ is by definition a subset of X\Z, but we take it's closure in X.)

Q: Is this extension always a hypersurface in X?
A: No.
Q: Is the extension of a contact type hypersurface b^3-contact type?
A: No, even if it is a hypersurface in X the b^3-contact type property can still fail.

Theorem

From now on we assume that S is a hypersurface in $T^*\mathbb{R}^n$ given by $S = H^{-1}(0)$ and H is given by: $H(q,p) = q^t Bp - 1$ where B is a (non-singular) $n \times n$-matrix.

Then S is of contact type, since all non-degenerate quadratics are transverse to the radial vector field.
Next, we can look at it as a hypersurface in M and thus at S_{McG} as we've seen before.

Thm: If $B + B^t$ is positive definite, then S_{McG} is a (smooth) hypersurface in X and of b^3-contact type.

Proof:

Fix $\phi: O \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n \to U \subset \mathbb{R}^{n-1}$ an arbitrary chart and we denote its inverse by ψ.
Then we can write the splitting as $q = r \cdot \psi$.
Since we want to preserve the structure of the cotangent bundle, we get an induced map on the coordinates on the cotangent fiber. Thus we have:

$$p = \frac{\partial r}{\partial q} P_r + \frac{\partial \phi}{\partial q} \eta$$

Now one can simply calculate: $\frac{\partial r}{\partial q_i} = \frac{q_i}{r} = \psi_i$. The matrix $\frac{\partial \phi}{\partial q}$ we can of course not calculate explicitly, because ϕ is arbitrary, but we can always say that it is $\frac{1}{r}\Psi$ for some matrix Ψ, which is independent of r. Thus we have the formula:

$$p = \psi P_r + \frac{1}{r} \Psi \eta$$

Now we can express H in terms of the n-spherical coordinates, this gives us:

$H(r, P_r, \phi, \eta) = (r\psi)^t B\left(\psi P_r + \frac{1}{r}\Psi\eta\right) - 1 = rP_r\psi^t B\psi + \psi B\Psi\eta - 1$

Thus, S seen as a hypersurface in M is given by:

$\{rP_r\psi^t B\psi + \psi B\Psi\eta - 1 = 0\}$

Now we can substitute $r = \frac{2}{x^2}$ and get that $\tau^{-1}_{\mathrm{McG}}(S) \subset X\backslash Z$ is given by:

$$\left\{\frac{2}{x^2}P_r\psi^t B\psi + \psi B\Psi\eta - 1 = 0\right\} = \{2P_r\psi^t B\psi + x^2\psi B\Psi\eta - x^2 = 0\}$$

We claim that the last expression also exactly defines S_{McG} as a subset of X. To show this we define the map:

$G: X \to \mathbb{R}: (x, P_r, \phi, \eta) \mapsto 2P_r\psi^t B\psi + x^2\psi B\Psi\eta - x^2$

From what we've seen above, $\tau^{-1}_{\mathrm{McG}}(S)$ is contained in $G^{-1}(0)$ and thus so is S_{McG} (in fact intersected with $X \setminus Z$ these are all the same). It remains now only to be shown that we can reach all the points in $G^{-1}(0) \cap Z$ with sequences in $\tau^{-1}_{\mathrm{McG}}(S)$ (which we skip).

Now we show that S_{McG} is of b^3-contact type. For this we define the vector field:

$V \coloneqq (P_r + 1)\partial_{P_r} + \eta\partial_\eta$

Which one can verify to be a Liouville vector field with respect to ω.

Finally, we claim that this is transverse to S_{McG} which we prove by calculating:

$\iota_V dG = 2(P_r{+}1)\psi^t B\psi + x^2\psi B\Psi\eta = G + x^2 + 2\psi^t B\psi$

On S_{McG} we have $G = 0$ so this is $x^2 + 2\psi^t B\psi$ which is always strictly positive.

Since x^2 is non-negative and we have: $2\psi^t B\psi = \psi^t(B + B^t)\psi$ which is positive, because $B + B^t$ is positive definite by assumption.

This also implies that S_{McG} is a hypersurface in X. Since $\iota_V dG$ doesn't vanish for G=0 and so dG doesn't vanish either and hence 0 is a regular value for G.

All in all we have that S_{McG} is a b^3-contact type hypersurface in X.

Q.E.D.

If we now have a Hörmander representative with a single block from case 1 or 2, then this looks like either:

$$H(q,p) = q^{\mathsf{T}}\begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix} p$$

For some $\lambda > 0$

Or:

$$H(q,p) = q^{\mathsf{T}}\begin{pmatrix} \begin{matrix}\lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1\end{matrix} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \begin{matrix}\lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1\end{matrix} \end{pmatrix} p$$

For some $\lambda_1, \lambda_2 > 0$

Thus $B + B^t$ looks like:

$$\begin{pmatrix} 2\lambda & 1 & \\ 1 & \ddots & \ddots \\ \end{pmatrix} \qquad \begin{pmatrix} 2\lambda_1 & 0 & 1 & \\ 0 & \ddots & \ddots & \ddots \\ 1 & & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 2\lambda & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\lambda \end{pmatrix} \quad \text{Or} \quad \begin{pmatrix} 2\lambda_1 & 0 & 1 & & & \\ 0 & \ddots & \ddots & \ddots & & \\ 1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 2\lambda_1 \end{pmatrix}$$

For $n = 1$ this is always positive definite, since we have $\lambda > 0$ (and $\lambda_1 > 0$ respectively).
For $n = 2$ this is positive definite if we have $\lambda > \frac{1}{2}$ (or $\lambda_1 > \frac{1}{2}$ respectively).
For bigger n this lower bound grows, but $\lambda > 1$ (or $\lambda_1 > 1$ respectively) is always enough, because then the matrix is strictly diagonally dominant. (I.e. the diagonal entry is in absolute value larger than the sum of the absolute values of all the other entries in the same row.) Combined with all the diagonal entries being positive, this implies positive definiteness.

Naturally, we can also stick multiple blocks of this together. Then the whole quadratic will satisfy the requirements if and only if all the individual blocks do.