Group actions on contact manifolds and the contact moment map

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Abstract

This 30min talk will introduce the concept of a contact moment map, and provide an example of explicitly working one out.

1 Reminder of concepts

Definition 1.1.

1. (Y^{2n+1}, α) is a *contact manifold* if the 1-form α is such that $\alpha \wedge d\alpha^n \neq 0$.
2. The co-dimension 1 distribution, $\xi = ker(\alpha)$, is the *contact structure.*
3. ξ is *co-orientable* if its annihilator $\xi^0 \subset T^*Y$ is an orientable line bundle. Note this is always true in our case (with global contact form).
4. A *co-orientation* of ξ is an orientation on ξ^0, i.e. a unit length section of ξ^0.
5. Then $\xi^0 \setminus T^*Y_0$, T^*Y_0 being the image of the zero section, is a symplectic submanifold of (T^*Y, ω_{can}).
6. Choosing a component ξ^0_+ of $\xi^0 \setminus T^*Y_0$ is a *symplectisation* of (M, ξ).

Definition 1.2. Let G be a Lie group, and $\sigma : G \times Y \to Y$ be a G-action on Y.

1. σ is *free* if σ_g has fixed points only if $g = e$.
2. σ is *proper* if $(\sigma, id) : G \times Y \to Y \times Y$ is a proper map, i.e. the pre-image of a compact set is compact.

2 The contact moment map

Definition 2.1. The action of a Lie group G on a contact manifold $(Y, \xi = ker\alpha)$ is a *contact action* if it preserves ξ and its co-orientation, i.e. the lifted actions of G are such that $T^*\sigma_g(\xi^0_+) = \xi^0_+$ and $T\sigma_g(\xi) = \xi$ for every $g \in G$. Equivalently σ is a contact group action if σ_g is a contactomorphism for every $g \in G$.

Lemma 2.2. *Suppose G is a compact Lie group, σ a contact action on $(Y, \xi = ker\alpha)$, then there's a G-invariant 1-form β such that $\xi = ker(\beta)$ and $\beta(Y) \subset \xi^0_+$, i.e. a contact form for co-oriented ξ.*

Proof. First choose α on Y with $ker\alpha = \xi$ and $\alpha(Y) \subset \xi^0_+$. We let $\beta := \int_G (\sigma_g)^* \alpha \, d\lambda$, where λ is the unique *right* Haar measure. By the preserving properties of σ, β is a contact form for co-oriented ξ and is G-invariant by

$$\sigma_h^* \beta = \int_G (\sigma_g)^* (\sigma_h)^* \alpha \, d\lambda = \int_G (\sigma_{g \circ h})^* \alpha \, d\lambda = \int_G (\sigma_g)^* \alpha \, d\lambda = \beta$$

where we have used the right invariance of the right Haar measure (considering $(\sigma_{g\circ h})^* \alpha(X) = f(g \circ h)$ real-valued). □

Remark. This is also true when G is not compact, the proof is in [2], and also [1] . So we can always assume a contact form β is G-invariant.

Now we temporarily turn to the general case when M is *any* manifold.

Definition 2.3. Suppose $\sigma : G \times M \to M$ is a group action on a manifold M preserving a 1-form β, the corresponding *β-moment map* $\mu_\beta : M \to \mathfrak{g}^*$ is defined by

$$\mu_\beta(x)(X) := \beta_x(X^\sharp(x))$$

for all $x \in M$ and all vectors X in the Lie algebra $\mathfrak{g}$ of G, where $X^\sharp$ is the induced vector field on M, i.e. $X^\sharp(x) = \frac{d}{dt}_{|t=0} \sigma_{\exp(tX)}(x)$.

Proposition 2.4. *Defined as above, μ is G-equivariant with respect to the given action of G on M and the coadjoint of G on $\mathfrak{g}^*$, i.e. $\mu_\beta(\sigma_g(x)) = Ad^*(g)\mu_\beta(x)$.*

Proof. We use the inner product notation:

$$\begin{aligned}
\langle \mu_\beta(\sigma_g(x), X \rangle &= \beta_{\sigma_g(x)}(X^\sharp(\sigma_g(x))) \\
&= \beta_{\sigma_g(x)}(\frac{d}{dt}_{|t=0} \exp(tX) \cdot (\sigma_g(x))) \\
&= \beta_x(D\sigma_{g^{-1}}(\sigma_g(x)) \frac{d}{dt}_{|t=0} \exp(tX) \cdot (\sigma_g(x)) \\
&= \beta_x(\frac{d}{dt}_{|t=0} \sigma_{g^{-1}\exp(tX)g}(x))
\end{aligned}$$

Now we claim $g^{-1}(\exp tX)g = \exp(tAd_{g^{-1}}X)$ and this is just by showing for $c_g(h) := ghg^{-1}$ conjugation, $c_{g^{-1}}\gamma^X(t) = \gamma^{Ad_{g^{-1}}X}(t)$ since the left hand side solves the defining ODE.

So now

$$\begin{aligned}
\langle\mu_\beta(\sigma_g(x), X\rangle &= \beta_x(\frac{d}{dt}_{|t=0}\sigma_{g^{-1}\exp(tX)g}(x))) \\
&= \beta_x(\frac{d}{dt}_{|t=0}\exp(tAd_{g^{-1}}X)\cdot x)) \\
&= \beta_x((Ad_{g^{-1}}X)^\sharp(x)) \\
&= \langle\mu_\beta(x), Ad_{g^{-1}}X\rangle \\
&= \langle Ad_g^*\mu_\beta(x), X\rangle.
\end{aligned}$$

as required. □

Remark. If we have a contact action of G on $(Y, \xi = ker\beta)$, then the above moment map depends on the choice of contact form β. Indeed, if $f: Y \to (0,\infty)$ is smooth, and G-invariant then $f\beta$ is also a preserved contact form for co-oriented ξ, while $\mu_{f\beta} = f\mu_\beta$. In general, $\mu : \Gamma(T^*Y) \to \Gamma(\mathfrak{g}^*)$ is $C^\infty(Y)$-linear and hence a point operator.

Definition 2.5. Suppose G is a Lie group acting on Y preserving the co-oriented contact structure ξ. We can define $\phi : T^*Y \to \mathfrak{g}^*$ by

$$\phi(q,p)(X) = q(X^\sharp(p)).$$

The restriction $\mu = \phi_{|\xi_+^0}$ depends only on the action of the group and on the contact structure, and we define it as *the contact moment map.*

Proposition 2.6. *Let (M,ξ) be a co-oriented contact manifold with an action of a Lie group G preserving the contact distribution and its co-orientation. Suppose there exists an invariant 1-form α with $ker\alpha = \xi$ and $\alpha(M) \subset \xi_+^0$. Then the α-moment map μ for the action of G on the symplectisation ξ_+^0 are related by*

$$\mu \circ \alpha = \mu_\alpha,$$

where ξ_+^0 is the component of $\xi \setminus 0$ containing the image of $\alpha : M \to \xi$.

Proof. If α is any invariant contact form with $ker\alpha = \xi$ and $\alpha(M) \subset \xi_+^0$ then $\mu\circ\alpha(p)(X) = \phi\circ\alpha(p)(X) = \alpha_p(X^\sharp(p)) = \mu_\alpha(p)(X)$. Hence $\mu\circ\alpha = \mu_\alpha$, so μ is 'universal'. □

Remark. If G is an action preserving the contact distribution ξ, the vector fields in $\mathfrak{g}$ are contact, i.e. the induced flows preserve ξ. The space of contact vector fields is isomorphic to the space of sections of the bundle

$TY \setminus \xi \to Y$ (a choice of a contact form identifies $TY \setminus \xi$ with $Y \times \mathbb{R}$ and contact vector fields with functions). Thus a contact group action gives rise to a linear map

$$\mathfrak{g} \to \Gamma(TY \setminus \xi), \qquad X \mapsto (X^\sharp \bmod \xi).$$

The moment map should be the transpose of this map. The total space of the bundle $(TY \setminus \xi)^*$ naturally maps into the space dual to the space of sections $\Gamma(TY \setminus \xi)$:

$$(TY \setminus \xi)^* \ni \eta \mapsto (s \mapsto \eta(s(\pi(\eta)))),$$

where $\pi : (TY \setminus \xi)^* \to Y$ is the projection. In other words, the transpose $\mu : (TY \setminus \xi)^* \to \mathfrak{g}^*$ should be given by

$$\langle \mu(\eta), X \rangle = \langle \eta, X^\sharp(\pi(\eta)) \bmod \xi \rangle.$$

Under the identification $\xi^0 \cong (TY \setminus \xi)^*$, the equation above becomes

$$\langle \mu(p, q), X \rangle = \langle q, X^\sharp(p) \rangle$$

for all $p \in Y, q \in \xi^0_p$ and $X \in \mathfrak{g}$, which is exactly the definition of μ given earlier as the restriction of the moment map for the lifted action of G on the cotangent bundle T^*Y.

Proposition 2.7. *Let $\mu : \xi^0_+ \to \mathfrak{g}^*$ be the contact moment map for the contact action of a Lie group G on a co-oriented contact manifold $(M, \xi = ker\alpha)$. Then μ is G-equivariant with respect to the given action of G on M and the coadjoint of G on $\mathfrak{g}^*$.*

Proof. Simply apply Prop. 2.4 and 2.6. □

3 Example on $\mathbb{R}^3$

Taking cylindrical coordinates, i.e. $(r, \theta, z) \cong (r\cos\theta, r\sin\theta, z)$ in the x, y, z coordinates, then $(\mathbb{R}^3, \alpha = dz + r^2 d\theta)$ is a contact manifold. At (r, θ, z) the contact plane is $\xi := \text{span}\{\frac{\partial}{\partial r}, r^2\frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\}$. We fix a co-orientation $\xi^0_+ = \{s\alpha \,|\, s > 0\}$.

Take $G = S^1 = [0, 2\pi]/\sim$ acting on $\mathbb{R}^3$ by rotation around the z-axis, and ϕ to be the coordinate such that $\frac{\partial}{\partial \phi}$ is unit length. Consider $\mathfrak{g} := T_0 S^1 = \text{span}\{\frac{\partial}{\partial \phi}_{|0}\}$, and by unit length $\exp(t\frac{\partial}{\partial \phi}_{|0}) = t$. Identify $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$. We have simply $\sigma_\phi(r, \theta, z) = (r, \theta + \phi, z)$, so $D\sigma_\phi(v_{(r,\theta,z)}) = v_{(r,\theta+\phi,z)}$, under the identification, and so it preserves the contact form α

and the co-orientation.

Then the α-moment map $\mu_\alpha : \mathbb{R}^3 \to \mathfrak{g}^* = \text{span}\{d\phi_{|0}\}$ is just

$$\begin{aligned}
\mu_\alpha(r,\theta,z)\Big(\frac{\partial}{\partial\phi}_{|0}\Big) &= \alpha_{(r,\theta,z)}\Big(\frac{\partial}{\partial\phi}^{\sharp}\Big) \\
&= \alpha_{(r,\theta,z)}\Big(\frac{d}{dt}_{|t=0}\exp(t\frac{\partial}{\partial\phi})\cdot(r,\theta,z)\Big) \\
&= \alpha_{(r,\theta,z)}\Big(\frac{d}{dt}_{|t=0}(r,\theta+t,z)\Big) \\
&= \alpha\Big(\frac{\partial}{\partial\theta}\Big)_{|(r,\theta,z)} \\
&= r^2,
\end{aligned}$$

so $\mu_\alpha(r,\theta,z) = r^2 d\phi_{|0}$.

The contact moment map $\mu : \xi^0_+ \to \mathfrak{g}^*$ is given by

$$\mu(s\alpha(r,\theta,z))\Big(\frac{\partial}{\partial\phi}_{|0}\Big) = \mu_{s\alpha}(r,\theta,z)\Big(\frac{\partial}{\partial\phi}_{|0}\Big) = sr^2.$$

References

[1] Michèle Audin, Ana Cannas da Silva, and Eugene Lerman. Symplectic geometry of integrable hamiltonian systems. 2003.

[2] Eugene Lerman. Contact toric manifolds. *Journal of Symplectic Geometry*, 1(4).