Convexity of the Contact Moment Map Image

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Abstract

The first goal of this talk is to introduce the necessary language (integral lattice, rational polyhedral sets) to state a convexity theorem about the image of the contact moment map. The second part of the talk will then focus on the hypothesis side of this convexity theorem and the goal will be to show that these can be loosened if a given dimension requirement is met.

1 From the Exponential Map to Integral Lattices

Recall. Let G be a Lie Group and let $\mathfrak{g} = T_eG$ be its Lie algebra.

1. Given an element $X \in \mathfrak{g}$, we get a unique left-invariant vector field on G and an associated one-parameter subgroup $\alpha^X : \mathbb{R} \to G$ such that $\alpha^X(0) = e$ and $\dot{\alpha}^X(0) = X$. The exponential map is the given as the time-one map of α^X, that is, the map defined by

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\mapsto \alpha^X(1) \end{aligned}$$

2. The exponential map is natural in the sense that for any Lie group homomorphism $\varphi : G \to H$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\varphi_e} & \mathfrak{h} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow{\varphi} & H \end{array} \qquad \begin{array}{ccc} X & \xmapsto{\qquad d\varphi_e \qquad} & d_e\varphi(X) \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ \exp_G(X) & \xmapsto{\varphi} & \varphi(\exp_G(X)) = \exp_H(d_e\varphi(X)) \end{array}$$

3. The exponential map is a local diffeomorphism, but in general it is not surjective (even if the Lie group is connected).

Recall. Let G be a Lie group and let $m : G \times G \to G$ denote the multiplication. The differential of the multiplication at the origin is given by

$$\begin{aligned} dm_{(e,e)} : \mathfrak{g} \oplus \mathfrak{g} &\to \mathfrak{g} \\ (X, Z) &\mapsto X + Z \end{aligned}$$

Proposition 1. *Let G be a connected Lie group. The exponential map* $\exp : \mathfrak{g} \to G$ *is a homomorphism if and only if G is abelian.*

Proof. $\underline{\Longrightarrow}$: G is a connected topological group and therefore generated by any neighbourhood of the identity. Since $\exp : \mathfrak{g} \to G$ is a bijection on a neighbourhood of the identity, it follows that any $g \in G$ can be written as a product of elements in the image of exp i.e.

$$g = \prod_i \exp(X_i) \qquad \text{for some } X_i \in \mathfrak{g}$$

Doing the same for $h \in G$ we can write $h = \prod_j \exp(Z_j)$ for some $Z_j \in \mathfrak{g}$. Finally, using that exp is supposed to be a homomorphism we get

$$\begin{aligned} gh &= \prod_i \exp(X_i) \prod_j \exp(Z_j) \\ &= \exp\left(\sum_i X_i + \sum_j Z_j\right) \\ &= \exp\left(\sum_j Z_j + \sum_i X_i\right) \\ &= \prod_j \exp(Z_j) \prod_i \exp(X_i) \\ &= hg \end{aligned}$$

$\underline{\Longleftarrow}$: If G is abelian, then the multiplication $m : G \times G \to G$ is a homomorphism since

$$\begin{aligned} m((g_1, h_1) \cdot (g_2, h_2)) &= m(g_1 g_2, h_1 h_2) \\ &= g_1 g_2 h_1 h_2 \\ &= g_1 h_1 g_2 h_2 \\ &= m(g_1, h_1) \cdot m(g_2, h_2), \end{aligned}$$

where for readibility we used both m and $\cdot$ to denote multiplication. The statement now follows from the naturality of the exponential map since the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} \oplus \mathfrak{g} & \xrightarrow{dm_{(e.e)}} & \mathfrak{g} \\ {\scriptstyle \exp_G \times \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_G} \\ G \times G & \xrightarrow{m} & G \end{array} \qquad \begin{array}{ccc} (X, Z) & \xmapsto{dm_{(e.e)}} & X + Y \\ {\scriptstyle \exp_G \times \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_G} \\ (\exp_G(X), \exp_G(Z)) & \xmapsto{m} & \exp_G(X)\exp_G(Z) = \exp_G(X + Z) \end{array}$$

■

Corollary 2. *For connected Abelian Lie Groups the exponential map is surjective.*

Proof. This is the same argument as in the first part of the previous proof together with the obvious fact that $\mathfrak{g}$ is closed under addition. ■

Proposition 3. *For connected Abelian Lie groups we have*

$$G \cong \mathfrak{g}/\mathfrak{g}_{\mathbb{Z}}$$

with $\mathfrak{g}_{\mathbb{Z}} = \ker(\exp)$ *a discrete subgroup of* $\mathfrak{g}$.

Proof. The kernel of the exponential map is discrete because it is a local diffeomorphism. The rest is then just the first Isomorphism theorem together with the two previous results. ■

For the talk of today (and other talks yet to come), the discrete subgroup $\mathfrak{g}_{\mathbb{Z}}$ is very important. We will therefore give it a name and try to understand it a bit better.

Definition. The discrete subgroup $\mathfrak{g}_{\mathbb{Z}} = \ker(\exp)$ of $\mathfrak{g}$ is called the **integral lattice**.

Proposition 4. *A discrete subgroup* Γ *($\mathfrak{g}_{\mathbb{Z}}$ in our case) of a finite dimensional vector space* V *($\mathfrak{g}$ in our case) is generated over* $\mathbb{Z}$ *by linearly independent vectors* $v_1, ..., v_k$ *i.e.*

$$\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k.$$

Proof. We proceed by induction over $n = \dim(V)$ and assume that $\Gamma \neq 0$ since otherwise the statement is trivially true.

n=1: We may assume that $V \cong \mathbb{R}$ by choosing a basis. Then Γ is generated by its smallest positive Element v_1.

Ind.: Choose a euclidean metric on V and an element $v_1 \in \Gamma$ of minimal norm. We get an orthonormal splitting

$$V = \mathbb{R}v_1 \oplus W \quad \text{with} \quad W = (\mathbb{R}v_1)^\perp$$

and consider the associated projection

$$\pi : \Gamma \leq \mathbb{R}v_1 \oplus W \to W.$$

Note that this is a homomorphism: write $\Gamma \ni g_i = c_i v_1 + w_i$ for $c_i \in \mathbb{R}$, $w_i \in W$ and note that

$$\begin{aligned} \pi(g_1 + g_2) &= \pi((c_1 + c_2)v_1 + w_1 + w_2) \\ &= w_1 + w_2 \\ &= \pi(g_1) + \pi(g_2) \end{aligned}$$

and that hence $\pi(\Gamma)$ is a subgroup of W i.e. $\pi(W) \leq W$.

Claim: $\pi(\Gamma)$ does not contain a nonzero element of norm smaller than $\|v_1\|/2$.

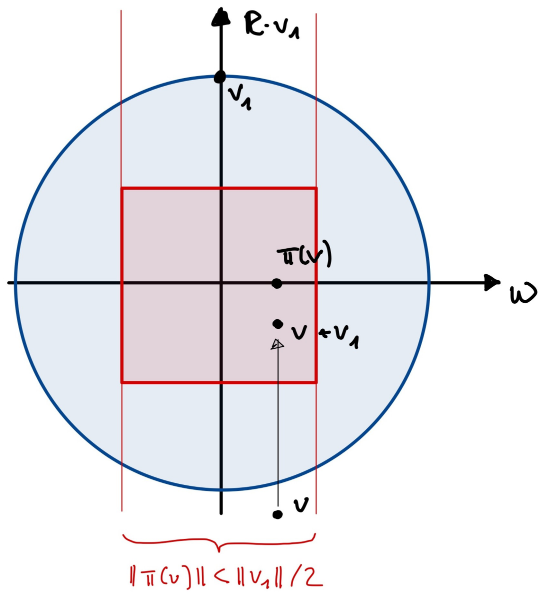

Figure 1: Sketch for the proof of the Claim.

Proof of Claim: (See figure 1) For a given $0 < \|\pi(v)\| < \|v_1\|/2$, there is a $n \in \mathbb{Z}$ such that the projection of $v + nv_1$ onto $\mathbb{R}v_1$ has norm at most $\|v_1\|/2$. But this means that $v + nv_1$ has norm at most $\|v_1\|/\sqrt{2}$ which contradicts the minimality of v_1 and hence proves the claim.

Consequence: It follows that $\pi(\Gamma)$ is discrete and by the induction hypothesis generated in W by linearly independent vectors $w_2, ..., w_k$ with $k \leq n$ i.e. $\pi(\Gamma) = \langle w_2, ..., w_k \rangle_{\mathbb{Z}}$.

The kernel of π is generated (over $\mathbb{Z}$) by v_1 so we get a short exact sequence

$$0 \to \langle v_1 \rangle_{\mathbb{Z}} \hookrightarrow \Gamma \xrightarrow{\pi} \langle w_2, ..., w_k \rangle_{\mathbb{Z}} \to 0.$$

This short exact sequence is split: We can construct a section s for π by choosing $n_2, ..., n_k \in \mathbb{Z}$, setting

$$\begin{aligned} s : \pi(\Gamma) &\to \Gamma \\ w_i &\mapsto n_i v_1 + w_i \end{aligned}$$

and extending linearly. Taking $v_i = n_i v_i + w_i$ we get that

$$\Gamma = \langle v_1 \rangle_{\mathbb{Z}} \oplus \langle v_2, ..., v_k \rangle_{\mathbb{Z}} = \langle v_1, ..., v_k \rangle_{\mathbb{Z}}$$

The v_i are linearly independent since $\pi(v_2), ..., \pi(v_k)$ are linearly independent and $\pi(v_1) = 0$:

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \lambda_i v_i \\ \implies 0 &= \sum_{i=1}^{k} \lambda_i \pi(v_i) = \sum_{i=2}^{k} \lambda_i w_i \\ \implies \lambda_i &= 0 \quad \text{for} \quad i = 2, ..., k \\ \implies \lambda_1 &= 0. \end{aligned}$$

■

Corollary 5. • *Any connected Abelian Lie group G is isomorphic product of the standard torus G^k and the standard vector space $\mathbb{R}^s$: $G \cong G^k \times \mathbb{R}^s$.*

- *Any simply connected Abelian Lie group G is isomorphic to $\mathbb{R}^s$: $G \cong \mathbb{R}^s$.*
- *Any compact connected Abelian Lie group G is isomorphic to the standard torus G^k: $G \cong G^k$.*

Proof. By the above we already have

1. $\exp : \mathfrak{g} \to G$ is a surjective homomorphism,
2. $\ker(\exp)$ is a discrete subgroup of $\mathfrak{g}$ and hence
3. $\mathfrak{g}_{\mathbb{Z}}$ is generated by linearly independent vectors $v_1, ..., v_k \in \mathfrak{g}$.

We can then find $v_{k+1}, ..., v_n \in \mathfrak{g}$ (where $n = \dim(G)$) so that $v_1, ..., v_n$ form a basis of $\mathfrak{g}$. This determines an isomorphism $\mathfrak{g} \cong \mathbb{R}^n$ such that

$$\mathfrak{g}_{\mathbb{Z}} \cong \mathbb{Z}^k \times 0 \trianglelefteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n.$$

This immediately gives

$$G \cong \mathfrak{g}/\mathfrak{g}_{\mathbb{Z}} \cong \mathbb{R}^n/\mathbb{Z}^k \times 0 = \mathbb{R}^k/\mathbb{Z}^k \times \mathbb{R}^{n-k} = G^k \times \mathbb{R}^{n-k}.$$

The other two statements follow immediately. ■

2 Convex Rational Polyhedral Cones

Convention. For the rest of these notes, G is assumed to be a compact connected Abelian Lie group, that is, G is isomorphic to $\mathbb{T}^k$ for some $k \in \mathbb{N}$. We will write $\mathfrak{g}$ for its Lie algebra and $\mathfrak{g}_{\mathbb{Z}}$ for the integral lattice.

Definition. A **polyhedral set** in $\mathfrak{g}^*$ is the intersection of finitely many closed half spaces.

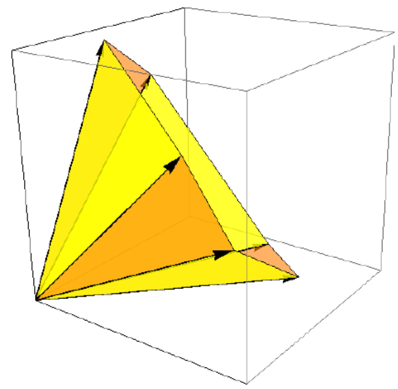

Figure 2: Polyhedral cone with 6 faces in three dimensions. [BMT14]

Remark. The boundary $\partial\mathbb{H}$ of a half-space $\mathbb{H}$ in $\mathfrak{g}^*$ is a hyperplane and its annihilator $(\partial\mathbb{H})^\circ$ is thus a one-dimensional subspace of $\mathfrak{g}$. Hence $(\partial\mathbb{H})^\circ\backslash\{0\}$ has two connected components and we choose one of them and denote it by $(\partial\mathbb{H})^\circ_+$. We can use any element $v \in (\partial\mathbb{H})^\circ_+$ to describe the half-space by

$$\mathbb{H} = \{\varphi \in \mathfrak{g}^* \mid \langle \varphi, v\rangle \geq 0\}\,.$$

Reversely, for every non-zero vector $v \in \mathfrak{g}$ we get a half-space $\mathbb{H}$ such that $\langle v\rangle_{\mathbb{R}_{>0}} = (\partial\mathbb{H})^\circ_+$ by imposing the above condition.

Notation. We will often characterise half-spaces using elements of their annihilators and denote them by $\mathbb{H}_v$.

Remark. A polyhedral set Δ is thus described by

$$\Delta = \bigcap_{i=1}^{k} \mathbb{H}_{v_i} = \{\varphi \in \mathfrak{g}^* \mid \langle \varphi, v_i\rangle \geq 0,\ i = 1, ..., k\}$$

with $k \in \mathbb{N}$ is finite.

Definition.

- A vector $v \in \mathfrak{g}$ is called **rational** if the one-dimensional subspace $\langle v\rangle_{\mathbb{R}}$ it spans has non-trivial intersection with $\mathfrak{g}_{\mathbb{Z}}$, that is, if there exists $t \in \mathbb{R}$ such that $tv \in \mathfrak{g}_{\mathbb{Z}}\backslash\{0\}$.
- An element $v \in \mathfrak{g}_{\mathbb{Z}}$ of the integral lattice is called **primitive** if there do not exist $k \in \mathbb{Z}$ with $|k| > 1$ and $u \in \mathfrak{g}_{\mathbb{Z}}$ such that $v = ku$.

Remark. A vector $v \in \mathfrak{g}$ is rational if and only if the one-parameter subgroup $t \mapsto \exp(tv),\ t \in \mathbb{R}$ it generates is isomorphic to S^1.

Indeed, if v is not rational, then $tv \notin \mathfrak{g}_{\mathbb{Z}} = \ker(\exp)$ for all $t \in \mathbb{R}\backslash\{0\}$. Hence the homomorphism $t \mapsto \exp(tv)$ has trivial kernel and its image is isomorphic to $\mathbb{R}$. Reversely, if v is rational, then (by rescaling) we may assume that $v \in \mathfrak{g}$ is primitive. The kernel of the homomorphism $t \mapsto \exp(tv)$ is then given by $\mathbb{Z}$ and hence the image is isomorphic to $\mathbb{R}/\mathbb{Z} \cong S^1$.

Definition. A polyhedral set is **rational** when the annihilators of all the half-spaces are spanned by rational vectors (which again by rescaling might be assumed to be primitive) i.e.

$$\Delta = \bigcap_i \mathbb{H}_{v_i}$$

for some finite collection of primitive vectors $v_i \in \mathfrak{g}_{\mathbb{Z}}$.

3 Convexity Theorems

Recall.

1. Let $(Y_1, \xi_1 = \ker(\alpha_1)$ and $(Y_2, \xi_2 = \ker(\alpha_2))$ be two co-oriented contact manifolds. A diffeomorphism $\varphi : Y_1 \to Y_2$ is a **contactomorphism** if the differential $\mathrm{d}\varphi$ maps ξ_1 to ξ_2 preserving the co-orientations. That is, $\varphi^*\alpha_2 = f\alpha_1$ for some positive function f.

2. An action $\psi : G \to \mathrm{Diff}(Y)$ of a Lie group G on a manifold Y **preserves a contact structure** ξ if for every $g \in G$ the corresponding diffeomorphism $\psi_g : Y \to Y$ is a contactomorphism. We call such an action of G on (Y, ξ) a **contact action**.

Lemma 6. *Suppose a compact Lie group H acts on (Y, ξ) preserving a co-oriented contact structure $\xi = \ker(\alpha)$ for some 1-form α. Then there esixts a H-invariant 1-form $\tilde{\alpha}$ with $\ker(\tilde{\alpha}) = \xi$.*

Proof. For every $h \in H$, the corresponding diffeomorphism $\psi_h : Y \to Y$ is a contactomorphism, hence $(\psi_h)^*\alpha = f_h\alpha$ for some positive function f_h depending smoothly on h. Define then a new contact form $\tilde{\alpha}$ by averaging over H for all $p \in Y$ using a bi-invariant measure $\mathrm{d}h$:

$$\begin{aligned}
\tilde{\alpha}_p &= \int_H ((\psi_h)^*\alpha)_p \mathrm{d}h \\
&= \int_H (f_h(p)\alpha_p)\mathrm{d}h \\
&= \left(\int_H f_h(p)\mathrm{d}h\right)\alpha_p.
\end{aligned}$$

Since $f_h > 0$ for all $h \in H$, the integral $\int_H f_h(p)\mathrm{d}h$ is positive and $\tilde{\alpha}$ is indeed nowhere zero. ∎

Convention.

1. For the remainder of these notes, we will write (Y, ξ) for a compact and connected co-oriented contact manifold. We will denote the chosen component of $\xi^\circ \backslash 0$ by ξ°_+.

2. We will write $\psi : G \to \mathrm{Diff}(Y)$ for a contact action of G on the contact manifold (Y, ξ).

Recall.

1. $\mu : \xi^\circ_+ \to \mathfrak{g}^*$ is the moment map defined by the contact distribution. Its image $\mu(\xi^\circ_+)$ depends only on the action and the contact distribution.

2. $\mu_\alpha : Y \to \mathfrak{g}^*$ is the moment map defined by the contact form α (such that $\alpha(Y) \subset \xi^\circ_+$ and then set $\mu_\alpha = \mu \circ \alpha$). Its image $\mu_\alpha(Y)$ depends on the action and the chosen contact form.

The two images are related by

$$\mu(\xi^\circ_+) = \mathbb{R}_+\mu_\alpha(Y)$$

where the $\mathbb{R}_+$ stems from the possible rescaling of α by a positive function.

Definition. The **moment cone** $C(\mu)$ of $\mu : \xi_+^\circ \to \mathfrak{g}^*$ is the set

$$C(\mu) := \mu(\xi_+^\circ) \cup \{0\}$$

Remark. If there is an invariant contact form α such that μ_α is well-defined, then we have that

$$C(\mu) = \{t\varphi \mid \varphi \in \mu_\alpha(Y),\ t \in [0, \infty)\}\,.$$

Unlike the image of μ_α, the moment cone $C(\mu)$ is invariant under the action of G on the contact manifold (Y, ξ).

Recall. Let M be a smooth manifold. A Lie group action $\psi : G \to \mathrm{Diff}(M)$ is called **effective** if the only element fixing all of M is the neutral element i.e.

$$\psi_g = \mathrm{id}_M \qquad \Longrightarrow \qquad g = e.$$

Theorem 7 (Lerman's Convexity Theorem)**.** *Let (Y, ξ) be a contact manifold of dimension at least 3 with an effective contact action of a torus G. If 0 is not in the image of μ_α i.e. $0 \notin \mu_\alpha(M)$, then the fibres $\mu^{-1}(\varphi)$ are connected and the moment cone $C(\mu)$ is a convex rational polyhedral cone.*

This theorem is a generalisation of another convexity theorem of Banyaga and Molino. This theorem holds for contact toric manifolds:

Definition. A **contact toric G-manifold** is a contact manifold (Y, ξ) with an effective contact action of a torus G such that $2\dim(G) = \dim(Y) + 1$.

Theorem 8 (Banyaga and Molino's Convexity Theorem)**.** *Let (Y, ξ) be a contact toric G-manifold. Then the moment cone $C(\mu)$ is a convex rational polyhedral cone.*

That this theorem follows directly from the above, that is, for contact toric G-manifolds we have $0 \notin \mu_\alpha(Y)$, follows from the following lemma:

Lemma 9. *Let (Y, ξ) be a contact toric G-manifold.*

1. *No G-orbit $G \cdot p$ is tangent to the contact structure ξ, that is, $T_p(G \cdot p) \not\subset \xi_p$ for all $p \in Y$.*
2. *All isotropy groups G_p are connected.*

Proof. Take $p \in Y$ and let G_p be its stabiliser i.e. its isotropy group. G_p acts on the tangent space T_pY since for $g \in G_p$ we have

$$\mathrm{d}(\psi_g)_p : T_pY \to T_{\psi_g(p)}Y = T_pY$$

Since the contact form α is G-invariant ($(\psi_g)^*\alpha = \alpha$), its kernel at p corresponding to the hyperplane ξ_p is a G_p-invariant subspace of T_pY. Indeed, for $X \in \xi_p = \ker(\alpha_p)$ and $g \in G_p$ we have

$$\alpha_p(\mathrm{d}(\psi_g)_p X) = ((\psi_g)^*\alpha)_p(X) = \alpha_p(X) = 0$$

showing that $\mathrm{d}(\psi_g)\,\xi_p \subset \xi_p$. Actually, recalling that $(\xi_p, \omega_p = \mathrm{d}\alpha_p\big|_{\xi_p})$ is a symplectic vector space, the Lie group homomorphism

$$\begin{aligned} \rho : G_p &\to Sp(\xi_p, \omega_p) \\ g &\mapsto \mathrm{d}(\psi_g)_p \end{aligned}$$

is a symplectic representation. Suppose $u, v \in \xi_p$ and write

$$\begin{aligned} \omega_p(\mathrm{d}(\psi_g)_p\, u, \mathrm{d}(\psi_g)_p\, v) &= ((\psi_g)^*\omega)_p(u, v) \\ &= ((\psi_g)^*\mathrm{d}\alpha)_p(u, v) \\ &= \mathrm{d}((\psi_g)^*\alpha)_p\,(u, v) \\ &= \mathrm{d}\alpha_p(u, v) \\ &= \omega_p(u, v) \end{aligned}$$

showing that indeed $\mathrm{d}(\psi_g)_p \in Sp(\xi_p, \omega_p)$

Fact: It is a (far from trivial) fact that since the G-action is effective, this representation is faithful i.e. $\ker(\rho) = \{e\}$.

Proof: See e.g. [ADSL03], Lemma III.4.

Set then

$$V = T_p(G \cdot p) \cap \xi_p$$

and note that G_p acts trivially on V since it acts trivially on $T_p(G \cdot p)$. Take $X \in T_p(G \cdot p)$ and note that we can write

$$X = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{g(t)}(p)$$

for some smooth curve $g : \mathbb{R} \to G$ such that $g(0) \in G_p$. Take then $h \in G_p$ and note that

$$\begin{aligned} \mathrm{d}(\psi_h)_p\, X &= \mathrm{d}(\psi_h)_p \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{g(t)}(p) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{hg(t)}(p) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{g(t)h}(p) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \psi_{g(t)}(p) \\ &= X \end{aligned}$$

where we used the product rule, that ψ is a left action and finally that G is Abelian.

Claim: V is an isotropic subspace of the symplectic vector space $(\xi_p, \omega_p = \mathrm{d}\alpha_p|_{\xi_p})$, that is, $V^\omega \subset V$ or equivalently $\omega_p\big|_{V \times V} = 0$.

Proof of claim: For this we need the following two facts about the fundamental vector fields:

1. The tangent vectors of the form $X_p^\sharp$ for $X \in \mathfrak{g}$ span $T_p(G \cdot p)$ [Wie21] and

2. the map

$$\begin{aligned} \sharp : \mathfrak{g} &\to \mathfrak{X}(Y) \\ X &\mapsto X^\sharp \end{aligned}$$

is a Lie algebra homomorphism [Lee03], that is, for all $X, Z \in \mathfrak{g}$ we have $([X,Z]_{\mathfrak{g}})^\sharp = \left[X^\sharp, Z^\sharp\right]_{\mathfrak{X}(Y)}$

In our case, $\mathfrak{g}$ is abelian, i.e. $[X, Z] = 0$ for all $X, Z \in \mathfrak{g}$, so it also follows that $\left[X^\sharp, Z^\sharp\right] = 0$ for all $X, Z \in \mathfrak{g}$. Consider then the function $p \mapsto \alpha_p(X_p^\sharp)$ for a given $X \in \mathfrak{g}$ and note that it is G-invariant. Indeed, by using G-invariance of μ_α we get

$$\begin{aligned} \psi_g(p) \mapsto \alpha_{\psi_g(p)}(X^\sharp_{\psi_g(p)}) &= \langle \mu_\alpha(\psi_g(p)), X \rangle \\ &= \langle \mu_\alpha(p), X \rangle \\ &= \alpha_p(X_p^\sharp). \end{aligned}$$

It follows then that for any $Z \in \mathfrak{g}$ we have $Z^\sharp(\alpha(X^\sharp)) = 0$. By the first property of fundamental vector fields, there exist $X, Z \in \mathfrak{g}$ such that $x = X_p^\sharp$, $z = Z_p^\sharp$ and hence

$$\begin{aligned} \omega_p(x, z) &= \mathrm{d}\alpha_p(X_p^\sharp, Z_p^\sharp) \\ &= X_p^\sharp(\alpha(Z^\sharp)) - Z_p^\sharp(\alpha(X^\sharp)) - \mathrm{d}\alpha_p(\left[X^\sharp, Z^\sharp\right]) \\ &= 0 - 0 + 0 \end{aligned}$$

showing that V is indeed isotropic.

As G_p is compact, there is a G_p-invariant almost complex structure J on ξ_p which is compatible with ω, that is, $\omega(\cdot, J\cdot)$ is an inner product. Note that since J is G_p-invariant and G_p acts trivially on V, it also acts trivially on JV.

Claim: $V + JV = V \oplus JV$ is a symplectic subspace of (ξ_p, ω).

Proof of Claim: Take $v \in V \cap JV$, that is $V \ni v = J\tilde{v}$ for some $\tilde{v} \in V$. We have

$$\omega(v, Jv) = \omega(v, J^2\tilde{v}) = -\omega(v, \tilde{v}) = 0.$$

since V is a isotropic subspace i.e. $\omega\big|_{V\times V} = 0$. Since $\omega(\cdot, J\cdot)$ is an inner product, it follows that $v = 0$. Thus $V \cap JV = \{0\}$ and $V + JV = V \oplus JV$. ω is non-degenerate on $V \oplus JV$ since for every $v \in V\backslash\{0\}$, we have $\omega(v, Jv) > 0$ proving the claim.

Set now

$$W = (V \oplus JV)^\omega$$

and note that since $V \oplus JV$ is symplectic we have $\xi_p = V \oplus JV \oplus W$.

Claim: W is a (symplectic) subrepresentation of $\rho : G_p \to Sp(\xi_p, \omega_p)$.

Proof of claim: We must show that for $u \in W$, we have $\mathrm{d}(\psi_g)_p u \in W = (V \oplus JV)^\omega$ for all $g \in G_p$. Take thus $v \in V \oplus JV$ and compute using that G_p acts trivially on $V \oplus JV$

$$\begin{aligned}\omega_p(\mathrm{d}(\psi_g)_p u, v) &= \omega_p(\mathrm{d}(\psi_g)_p u, \mathrm{d}(\psi_g)_p v)\\ &= \omega_p(u, v)\\ &= 0\end{aligned}$$

Conclusion: There is a decomposition of (symplectic) representations given by

$$\xi_p = V \oplus JV \oplus W$$

and the subrepresentation W is faithful since the whole representation is faithful but $V \oplus JV$ is the trivial representation.

To conclude we need the following lemma from representation theory ([ADSL03], Lemma III.6):

Lemma 10. *Suppose $\rho : H \to Sp(W, \omega)$ is a symplectic representation of a compact Abelian Lie group H on a symplectic vector space (W, ω) which is faithful. Then*

$$\dim(W) \geq 2\dim(H)$$

and if equality holds, then H is connected.

Using $\dim(Y) = 2\dim(G) - 1$ we get

$$\begin{aligned}\dim(W) &= \dim(\xi_p) - \dim(V) - \dim(JV)\\ &= \dim(Y) - 1 - 2\dim(V)\\ &= 2\dim(G) - 2 - 2\dim(V).\end{aligned}$$

There are two cases to distinguish:

1. $\underline{T_p(G \cdot p) \subset \xi_p}$: In this case we have $\dim(V) = \dim(G \cdot p) = \dim(G) - \dim(G_p)$ which yields

   $$\dim(W) = 2\dim(G_p) - 2$$

   contradicting the Lemma above.

2. $\underline{T_p(G \cdot p) \not\subset \xi_p}$: In this case, $\dim(V) = \dim(G \cdot p) - 1$ since ξ_p has codimension one and we get

   $$\dim(W) = 2\dim(G_p).$$

   Thus by the lemma G_p is connected.

■

Corollary 11. *Let (Y, ξ) be a contact toric G-manifold. Then*

1. *the G-action has no fixed points and*

2. *the moment map μ_α does not vanish at any point for any G-invariant contact form α, or equivalently, $\mu(\xi_+^\circ) \neq 0$.*

Proof. 1. If $p \in Y$ were a fixed point, then $T_p(G \cdot p) = 0$ would be tangent to ξ_p.

2. Since $T_p(G \cdot p) \not\subset \xi_p$ and the fundamental vector fields of the form $X^\sharp$ for $X \in \mathfrak{g}$ span $T_p(G \cdot p)$, there exists a (non-zero) $Z \in \mathfrak{g}$ such that $Z_p^\sharp \notin \xi_p = \ker(\alpha_p)$. By definition of the α-moment map we then have

$$\langle \mu_\alpha(p), Z \rangle = \left\langle \alpha_p, Z_p^\sharp \right\rangle \neq 0$$

■

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