Symplectic Reduction

Elia Mazzucchelli

November 28, 2021

Abstract

After a motivation given by Noether's principle, the Marsden-Weinstein-Meyer Theorem concerning the symplectic reduction of a Hamiltonian G-space is proven, along with some examples. A discussion of alternative equivalent reductions, such as the one at coadjoint orbits and the shifting trick, follows. The procedure of reduction in stages is explored, together with a comment on the case where the reduction is done with weaker assumptions, which leads to the concept of symplectic orbifold. In conclusion, an infinite dimensional example of symplectic reduction concerning the moduli space of flat connections is introduced.

Notation

(M, ω, G, μ)	Hamiltonian G-space with moment map μ
$G_p := \{g \in G : g \cdot p = p\} \subset G$	stabiliser of $p \in M$
$\mathfrak{g}_p := \mathrm{Lie}(G_p)$	symmetry algebra of $p \in M$
$\mathfrak{g}_p^0 := \{\xi \in \mathfrak{g}^* : \langle \xi, X \rangle = 0 \ \ \forall X \in \mathfrak{g}_p\} \subset \mathfrak{g}^*$	annihilator of $\mathfrak{g}_p$
$\mathcal{O}_p := \{g \cdot p : g \in G\} \subset M$	orbit of $p \in M$ under the action of G
$\mathrm{Ad}_g \in \mathrm{End}(\mathfrak{g}), \quad \mathrm{Ad}_g^* \in \mathrm{End}(\mathfrak{g}^*)$	adjoint and coadjoint action for $g \in G$
$G_\xi := \{g \in G : \mathrm{Ad}_g^* \, \xi = \xi\} \subset G$	coadjoint stabiliser of $\xi \in \mathfrak{g}$
$\mathfrak{g}_\xi := \mathrm{Lie}(G_\xi) \subset \mathfrak{g}$	symmetry algebra of $\xi \in \mathfrak{g}^*$
$\mathcal{O}_p^\xi := \{g \cdot p : g \in G_\xi\} \subset M$	orbit of $p \in M$ under the action of G_ξ
$\mathcal{O}_\xi := \{\mathrm{Ad}_g^* \, \xi : g \in G\} \subset \mathfrak{g}^*$	coadjoint orbit of $\xi \in \mathfrak{g}^*$
$\sigma \colon G \times M \to M$	left action of G on M
$\sigma^p := \sigma(\cdot, p) \colon G \to M$	orbit map of $p \in M$
$\sigma_g := \sigma(g, \cdot) \in \mathrm{Diff}(M)$	diffeomorphism by the action of $g \in G$
$X^\# \in \Gamma(TM)$	fundamental vector field associated to $X \in \mathfrak{g}$

1 Motivation: Noether's Principle

The phase space of a system of n particles is the space parametrizing the position and momenta of the particles. The mathematical model for the phase space is a symplectic manifold (M, ω). Classical physicists realized that, whenever there is a symmetry group G of dimension k acting on a mechanical system, then the number of degrees of freedom for the position and momenta of the particles may be reduced by $2k$. Symplectic reduction formulates this feature mathematically.

Let (M, ω, G, μ) be a Hamiltonian G-space and assume that G is **compact and connected**.

Definition 1.1. Let $f \in C^\infty(M)$ and denote by v_f the corresponding Hamiltonian vector field and by Φ^f the flow of v_f. We call f an **integral of motion** of (M, ω, G, μ) if it is G-invariant, i.e. if $\forall\, g \in G : f \circ \sigma_g = f$. If μ is constant on the flow trajectories of v_f, i.e. if $\mu \circ \Phi_t^f = \mu$ then the corresponding one-parameter group of diffeomorphisms $t \mapsto \Phi_t^f$ is called a **symmetry** of (M, ω, G, μ). The Noether principle, Theorem 1.2, asserts that there is a one-to-one correspondence between symmetries and integrals of motion.

Theorem 1.2. *(Noether) A function $f \in C^\infty(M)$ is an integral of motion if and only if $t \mapsto \Phi_t^f$ is a symmetry.*

Proof. Since G is compact and connected, $\exp\colon \mathfrak{g} \to G$ is surjective, hence f is G-invariant if and only if $\forall\, X \in \mathfrak{g} : f \circ \sigma_{\exp(tX)} = f$, or equivalently if $\mathcal{L}_{X^\#} f = 0$. Let $X \in \mathfrak{g}$ and set $\mu^X := \langle \mu, X \rangle \in C^\infty(M)$. Then by Cartan's magic formula we have that

$$\mathcal{L}_{v_f} \mu^X = i_{v_f} d\mu^X = i_{v_f} i_{X^\#}\omega = -i_{X^\#} i_{v_f}\omega = -i_{X^\#} df = -\mathcal{L}_{X^\#} f.$$

□

Observe that a Hamiltonian system (M, ω, H) with complete Hamiltonian vector field v_H of H is a Hamiltonian $\mathbb{R}$-space. The $\mathbb{R}$-action in this case is the action by the flow of v_H, and the moment map is H itself. An $\mathbb{R}$-invariant function $f \in C^\infty(M)$ is then the same as a conserved quantity, and H being constant on the integral curves of v_f is the same as v_f being an **infinitesimal symmetry**, that is, v_f is symplectic and $\mathcal{L}_{v_f} H = v_f(H) = 0$. If $E \in \mathbb{R}$ is a regular value of H, then $H^{-1}(E)/\mathbb{R}$ is a manifold of dimension $\dim M - 2$ equipped with a canonical symplectic form induced by ω and it is called the **manifold of constant energy solutions**. More generally, for any integral of motion $f \in C^\infty(M)$ and $a \in \mathbb{R}$ a regular value of f, we can consider the $(\dim M - 2)$-dimensional manifold $M_{red} = f^{-1}(a)/\mathbb{R}$ equipped with a canonical symplectic form ω_{red} and Hamiltonian function H_{red} induced by ω and H respectively, see [CdS01, Section 24.2] for the local construction in coordinates. Such a reduction is called a **reduced phase space** (M_{red}, ω_{red}) with **reduced Hamiltonian** H_{red} and it turns out that understanding the trajectories of this reduced system is equivalent to understanding the trajectories of the starting system (M, ω, H) (note that the notation is ambiguous, since the reduction depends both on the constant of motion f and on its regular value a). This is probably the physically motivated simplest example of the so called symplectic reduction procedure, which comes under the name of Marsden-Weinstein Theorem and will be discussed now.

2 The Marsden-Weinstein-Meyer Theorem

We begin with the following simple piece of linear algebra.

Lemma 2.1. *Let $W \subset (V, \omega)$ be a subspace of a symplectic vector space. Then ω induces canonical symplectic forms on the quotients $W/(W^\omega \cap W)$ and $W^\omega/(W^\omega \cap W)$. Note that in the special case where W is coisotropic or isotropic, then the quotient is W/W^ω and W^ω/W respectively.*

Proof. We consider one case, the other one is analogous recalling that $(W^\omega)^\omega = W$. Let $u, v \in W$ and denote the respective equivalence classes by $[u], [v] \in \overline{W} := W/(W^\omega \cap W)$. We define a bilinear form $\bar{\omega}$ on $\overline{W}$ by $\bar{\omega}([u], [v]) := \omega(u, v)$. That $\bar{\omega}$ is a well-defined symplectic form on $\overline{W}$ follows from the universal property of the corresponding kernel $\ker \omega|_{W \otimes W}$, or alternatively, it is a straightforward computation. □

We will need the following standard result from Differential Geometry, see [Mer20, Theorem 15.7] and [Mer20, Proposition 16.19] for a proof with the Frobenius Theorem or [CdS01, Theorem 23.4] for a different proof in the case where G is compact.

Theorem 2.2. *Let a Lie group G act on a manifold M freely and properly. Then the orbit space M/G admits the structure of a topological manifold of dimension $\dim M - \dim G$. Moreover there exists a unique smooth structure on M/G such that the quotient map $\pi\colon M \to M/G$ is a smooth submersion. Then $\pi\colon M \to M/G$ is a principal G-bundle.*

Definition 2.3. An action σ of G on M is called **infinitesimally free at** $p \in M$ if $\mathfrak{g}_p = \{0\}$ and we say that it is **infinitesimally free**, if it is infinitesimally free at every point in M. We say that σ is **locally free at** $p \in M$ if there exists a neighborhood U of the identity in G such that $\sigma^p|_U$ is free, or equivalently, if G_p is a discrete subgroup of G. We say that σ is **locally free** if it is locally free at every point in M.

Remark 2.4. A free action is obviously locally free. The converse is not true because e.g. the action of any discrete group is locally free, but need not be globally free. The next proposition shows the equivalence between the definitions in Definition 2.3.

Proposition 2.5. *Let σ be an action of G on M and let $p \in M$. Then σ is locally free at p if and only if it is infinitesimally free at p. In particular, the action is locally free if and only if it is infinitesimally free.*

Proof. If σ is locally free at p, then σ_p is injective on a neighbourhood of $e \in G$ and since it has constant rank, it is an immersion. Hence, $d\sigma^p(e)$ is injective and $\mathfrak{g}_p = \{0\}$.

Conversely, assume that σ is infinitesimally free at p. Since σ^p has constant rank, it has maximal rank everywhere. In particular, σ^p is locally injective around the identity. □

From now on, we consider a Hamiltonian G-space (M, ω, G, μ). The following simple Lemma has been already introduced in the first presentation of week 9.

Lemma 2.6. *Let $d\mu_p\colon T_pM \to T_{\mu(p)}\mathfrak{g}^* \cong \mathfrak{g}^*$ denote the differential of μ at $p \in M$. Then*

$$\ker d\mu_p = (T_p\mathcal{O}_p)^{\omega_p} \quad \textit{and} \quad \operatorname{im} d\mu_p = \mathfrak{g}_p^0.$$

Corollary 2.7. The action of G is locally free at p if and only if p is a regular point of μ.

In general the action of G may not preserve the level sets of μ. Equivariance of μ implies that those elements of G which do preserve $\mu^{-1}(\xi)$ are precisely those that fix ξ under the coadjoint action. So the largest possible subgroup which acts on the level $\mu^{-1}(\xi)$ is the coadjoint stabilizer G_ξ. This means that we can at best remove the degenerate directions which are tangent to the orbit of G_ξ. Thankfully the following lemma ensures that this will be enough to recover a non-degenerate form from the pullback of ω.

Lemma 2.8 (Reduction Lemma). *Let $\xi \in \mathfrak{g}^*$ be a regular value of μ and let $p \in \mu^{-1}(\xi)$. Then*

1. $\mu^{-1}(\mathcal{O}_\xi) = G \cdot \mu^{-1}(\xi)$,
2. $\mu^{-1}(\xi) \cap \mathcal{O}_p = \mathcal{O}_p^\xi$,
3. $\mu^{-1}(\xi)$ *and* $\mathcal{O}_p$ ***intersects cleanly***, *i.e.* $T_p\mu^{-1}(\xi) \cap T_p\mathcal{O}_p = T_p\mathcal{O}_p^\xi$ *and*
4. $T_p\mu^{-1}(\xi) = (T_p\mathcal{O}_p)^{\omega_p}$.

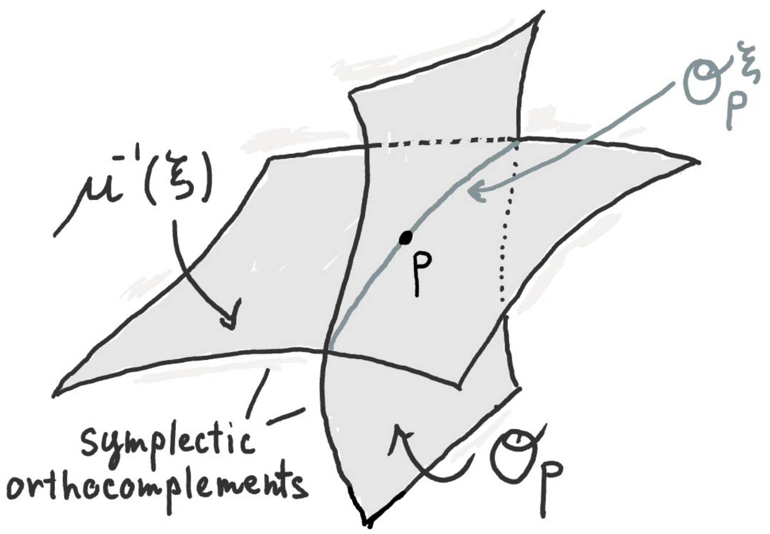

Figure 1: The two submanifolds $\mu^{-1}(\xi)$ and $\mathcal{O}_p$ intersect in the orbit of the isotropy group $\mathcal{O}_p^\xi$ and their tangent spaces $T_p\mu^{-1}(\xi)$ and $T_p\mathcal{O}_p$ are symplectic orthocomplements intersecting in the space $T_p\mathcal{O}_p^\xi$.

Proof. For claim 1 note that

$$p \in \mu^{-1}(\mathcal{O}_\xi) \iff \mu(p) = \mathrm{Ad}_g^*\,\xi \iff \mu(g^{-1}\cdot p) = \mathrm{Ad}_{g^{-1}}^*\,\mu(p) = \xi \iff p \in G\cdot\mu^{-1}(\xi).$$

For claim 2 note that

$$g\cdot p \in \mu^{-1}(\xi) \iff \xi = \mu(g\cdot p) = \mathrm{Ad}_g^*\,\mu(p) = \mathrm{Ad}_g^*\,\xi \iff g \in G_\xi.$$

For claim 3 note that if $X \in \mathfrak{g}$ then

$$d\mu_p(X_p^\#) = \frac{d}{dt}\bigg|_0 \mu(\exp(tX) \cdot p) = \frac{d}{dt}\bigg|_0 \operatorname{Ad}^*_{\exp(tX)} \mu(p) = \operatorname{ad}^*_X \xi.$$

It follows that if $d_p\mu_p(X_p^\#) = 0$ then $X \in \mathfrak{g}_\xi$, which implies that $X_p^\# \in T_p\mathcal{O}_p^\xi$. The other inclusion follows from claim 2.

Claim 4 follows from $T_p\mu^{-1}(\xi) = \ker d\mu_p$ and Lemma 2.6. □

Assumptions 2.9. We introduce three non-equivalent assumptions for the Marsden-Weinstein-Meyer Theorem. Only **one** of the following assumptions is sufficient.

1. Assume G acts freely and properly on M.
2. Assume $\xi \in \mathfrak{g}^*$ is a regular value of μ and G_ξ acts freely and properly on $\mu^{-1}(\xi)$.
3. Assume $\xi = 0$ and G acts freely and properly on $\mu^{-1}(0)$.

By Corollary 2.7 it follows that 1 implies 2. In the special case where $\xi = 0$, we have that $G_\xi = G$ and $0 \in \mathfrak{g}^*$ is a regular value since G acts locally freely on $\mu^{-1}(0)$. Note that if G is compact then the action of G and G_ξ are both proper. If G is abelian, then $G_\xi = G$ and thus if G acts freely on $\mu^{-1}(\xi)$, then ξ is a regular value.

Theorem 2.10 (Marsden-Weinstein-Meyer)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space. Let $\iota_\xi \colon \mu^{-1}(\xi) \hookrightarrow M$ be the embedding given by the inclusion. Assume that either of the Assumptions 2.9 hold. Then the orbit space $M^\xi := \mu^{-1}(\xi)/G_\xi$ is a smooth manifold and $\pi_\xi \colon \mu^{-1}(\xi) \to M^\xi$ is a principal G_ξ-bundle. Moreover, there exists a canonical symplectic form ω^ξ on M^ξ satisfying $\iota_\xi^*\omega = \pi_\xi^*\omega^\xi$.*

Definition 2.11. The symplectic manifold (M^ξ, ω^ξ) is called the **reduction of** (M, ω) **at** $\xi \in \mathfrak{g}^*$ with respect to G, μ, or the **reduced space**, or the **symplectic quotient**, or the **Marsden-Weinstein-Meyer quotient**. Many authors attribute such names to the reduction at level $\xi = 0$.

Proof. Since ξ is assumed to be a regular value, the level $\mu^{-1}(\xi)$ is a smooth submanifold of M of codimension $\dim G_\xi$ and since the action of G_ξ on $\mu^{-1}(\xi)$ is free and proper, Theorem 2.2 ensures that the quotient map $\pi_\xi \colon \mu^{-1}(\xi) \to M^\xi$ is a principal G_ξ-bundle. For every $p \in \mu^{-1}(\xi)$ we have the following short exact sequences of vector spaces:

$$0 \longrightarrow T_p\mathcal{O}_p^\xi \longrightarrow T_p\mu^{-1}(\xi) \xrightarrow{d\pi_\xi(p)} T_{\pi_\xi(p)}M^\xi \longrightarrow 0,$$

$$0 \longrightarrow T_p\mu^{-1}(\xi) \xrightarrow{d\iota_p} T_pM \xrightarrow{d\mu_p} T_{\mu(p)}\mathfrak{g}^* \cong \mathfrak{g}^* \longrightarrow 0.$$

By Corollary 2.7 and Lemma 2.8 we have that

$$T_p\mu^{-1}(\xi) = (T_p\mathcal{O}_p)^{\omega_p} \quad \text{and} \quad T_p\mu^{-1}(\xi) \cap T_p\mathcal{O}_p = T_p\mathcal{O}_p^\xi.$$

Then Lemma 2.1 gives a canonical symplectic form ω_p^ξ on the quotient

$$T_p\mu^{-1}(\xi)/(T_p\mathcal{O}_p \cap T_p\mu^{-1}(\xi)) = T_p\mu^{-1}(\xi)/T_p\mathcal{O}_p^\xi \cong T_{\pi_\xi(p)}M^\xi.$$

By construction, see proof of Lemma 2.1, we have that $\iota_\xi^*\omega = \pi_\xi^*\omega^\xi$. Hence,

$$\pi_\xi^* d\omega^\xi = d\pi_\xi^*\omega^\xi = d\iota_\xi^*\omega = \iota_\xi^* d\omega = 0$$

and thus the closedness of ω^ξ follows from injectivity of $d\pi_\xi(p)$ restricted to $T_p\mu^{-1}(\xi)/T_p\mathcal{O}_p^\xi$. $\square$

Example 2.12 (Fubini-Study)**.** See [DHJvdH19, Example 6.9]. Consider the Hamiltonian U(1)-space consisting of

$$(\mathbb{R}^{2n+2}, \omega_{st}) \cong (\mathbb{C}^{n+1}, \omega := \frac{i}{2}\sum_{j=0}^{n} dz_j \wedge d\bar{z}_j)$$

with the diagonal U(1)-action $e^{i\theta} \cdot (z_0, \ldots, z_n) := (e^{i\theta}z_0, \ldots, e^{i\theta}z_n)$ for $e^{i\theta} \in S^1 \cong \mathrm{U}(1)$ and a corresponding moment map

$$\mu\colon \mathbb{C}^{n+1} \to \mathfrak{u}(1) \cong \mathbb{R}, \quad z \mapsto \frac{1}{2}\|z\|^2.$$

Then $\mu^{-1}(\frac{1}{2}) = S^{2n+1} \subset \mathbb{C}^{n+1}$ and the symplectic quotient is the projective space $M^{\frac{1}{2}} = S^{2n+1}/\mathrm{U}(1) = \mathbb{C}P^n$ equipped with the **Fubini-Study** symplectic form $\omega^{\frac{1}{2}} = \omega_{FS}$, see second presentation of week 9. One can see this by considering the the vector field associated to the action which is

$$\partial_\theta = i\sum_{j=0}^{n} z_j\partial_{z_j} - \bar{z}_j\bar{\partial}_{z_j}$$

and noticing that the 2-form $d\mu\wedge d\theta - \omega$ is a closed U(1)-invariant horizontal form on the principal U(1)-bundle

$$\pi := \pi_{\frac{1}{2}}\colon S^{2n+1} \to \mathbb{C}P^n$$

and we can therefore push it forward to a 2-form ω_{FS} on $\mathbb{C}P^n$ through π. Then, ω_{FS} is closed and symplectic since by definition we have that $i_{\pi_* X}\omega_{FS} = i_X(d\mu \wedge d\theta - \omega)$ and the latter vanishes if and only if $X \in \Gamma(TS^{2n+1})$ is a multiple of ∂_θ, in which case $\pi_* X = 0$. One could take this as the definition of ω_{FS}, or alternatively, one can compute the expression in local coordinates of $\pi^*\omega_{FS}$ to see that this is indeed the Fubini-Study form we introduced earlier in the seminar.

Example 2.13 (Grassmannians)**.** Consider the right multiplication action of $\mathrm{U}(k)$ on the space $\mathbb{C}^{n\times k}$ of complex $n \times k$-matrices with the standard symplectic structure defined as in the example above. Identifying the Lie algebra $\mathfrak{u}(k)$ with its dual via the inner product $(A, B) := \mathrm{tr}(A^*B)$, the moment map of the action is given by

$$\mu\colon \mathbb{C}^{n\times k} \to \mathfrak{u}(k), \quad A \mapsto \frac{1}{2i}AA^*.$$

Then $\mu^{-1}(\frac{\mathbb{1}}{2i})$ is the space of unitary k-frames $A \in \mathbb{C}^{n\times k}$ with $AA^* = \mathbb{1}$ and the reduction $\mu^{-1}(\frac{\mathbb{1}}{2i})/\mathrm{U}(k) = \mathrm{G}(k, n)$ is the **Grassmannian**.

3 Reduction at Coadjoint Orbits and the Shifting Trick

Let (M, ω, G, μ) be a Hamiltonian G-space. Equivariance of the moment map led us to consider the action of the coadjoint stabilizer on a regular level instead of the action of the entire group G. On the other hand, it follows from the equivariance of μ that we can consider an action of the entire group G if we are willing to enlarge the level ξ to the preimage of its entire coadjoint orbit $\mathcal{O}_\xi$. The following Lemma shows us that Assumptions 2.9 are enough to ensure that the action of G on $\mu^{-1}(\mathcal{O}_\xi)$ defines a principal G-bundle $\pi_{\mathcal{O}_\xi}\colon \mu^{-1}(\mathcal{O}_\xi) \to M^{\mathcal{O}_\xi}$, see [DHJvdH19, Lemma 6.15].

Lemma 3.1. *If the coadjoint orbit $\mathcal{O}_\xi$ contains a regular value of μ, then every point in $\mathcal{O}_\xi$ is a regular value of μ. In this case, the moment map is transverse and regular to $\mathcal{O}_\xi$ and hence $\mu^{-1}(\mathcal{O}_\xi)$ is a smooth submanifold of M of codimension equal to the codimension of G_ξ in G.*

Proof. Assume that $\xi \in \mathcal{O}_\xi$ is a regular value of μ, then by Corollary 2.7 it follows that $G_p \subset G$ is discrete for every $p \in \mu^{-1}(\xi)$. Let $\eta \in \mathcal{O}_\xi$ and $g \in G$ such that $\eta = \mathrm{Ad}_g^* \xi$. Then for every $q \in \mu^{-1}(\eta)$ we have that G_q is conjugate to $G_{g^{-1}.p}$ which is discrete. Therefore, also G_q is discrete and thus its Lie algebra is trivial in $\mathfrak{g}$. By Lemma 2.6 it follows that $d\mu_q$ is surjective. ☐

Lemma 3.2. *The action of G_ξ on $\mu^{-1}(\xi)$ is free if and only if the action of G on $\mu^{-1}(\mathcal{O}_\xi)$ is free.*

Proof. Given $p \in \mu^{-1}(\mathcal{O}_\xi)$ let $\eta = \mu(p)$ and $g \in G$ such that $\eta = \mathrm{Ad}_g^*\, \xi$. Then for $h \in G_p$ we have that

$$\eta = \mu(p) = \mu(h \cdot p) = \mathrm{Ad}_h^* \eta \implies h \in G_\eta = g\, G_\xi\, g^{-1},$$

so we can write $h = g h_0 g^{-1}$ for some $h_0 \in G_\xi$ which must fix $g^{-1} \cdot p$. Now from the equivariance of μ we see that $g^{-1} \cdot p \in \mu^{-1}(\xi)$ and because the action of G_ξ was assumed to be free it follows that h_0 and therefore also h must be the identity in G. The other direction is clear. ☐

Under Assumptions 2.9, the action of G on the preimage of the coadjoint orbit defines a principal G-bundle which we denote by $\pi_{\mathcal{O}_\xi}\colon \mu^{-1}(\mathcal{O}_\xi) \to M^{\mathcal{O}_\xi}$ with the inclusion $\iota_{\mathcal{O}_\xi}\colon \mu^{-1}(\mathcal{O}_\xi) \to M$. We might hope that we can push $\iota_{\mathcal{O}_\xi}^* \omega$ forward to a symplectic form on $M^{\mathcal{O}_\xi}$; however, this is not the case since it fails to vanish on the vertical bundle, see [DHJvdH19, p.38]. Instead, there exists a unique canonical symplectic form $\omega^{\mathcal{O}_\xi}$ on $M^{\mathcal{O}_\xi}$ satisfying

$$\pi_{\mathcal{O}_\xi}^* \omega^{\mathcal{O}_\xi} = \iota_{\mathcal{O}_\xi}^* (\omega - \mu^* \omega^{KKS}),$$

where $\omega^{KKS} \in \Omega^2(\mathfrak{g}^*)$ is the so called **Kostant-Kirillov symplectic form** or the **Lie-Poisson symplectic form** and is defined by the equation $\omega_\eta^{KKS}(\mathrm{ad}_X^* \eta, \mathrm{ad}_Y^* \eta) := \eta([X, Y])$ for every $X, Y \in \mathfrak{g}$. We then denote by $(M^{\mathcal{O}_\xi}, \omega^{\mathcal{O}_\xi})$ the so obtained symplectic reduction. Note that there is a natural diffeomorphism $M^{\mathcal{O}_\xi} \cong M^\xi$.

We now look at an alternative construction. Consider $M \times \mathcal{O}_\xi$ equipped with the symplectic form $\omega \oplus (-\omega^{KKS})$ so that the component-wise action of G is Hamiltonian with moment map $\nu(p, \eta) := \mu(p) - \eta$. Then $\nu^{-1}(0)$ is equivariantly diffeomorphic to $\mu^{-1}(\mathcal{O}_\xi)$ and we refer to ν as the **shifted moment map** since it is effectively shifting μ so that it vanishes on $\mu^{-1}(\mathcal{O}_\xi)$. Assumptions 2.9 are sufficient to form the symplectic quotient at the zero level of ν as the following Lemma shows, see [DHJvdH19, Lemma Lemma 6.17].

Lemma 3.3. *(Shifting Trick) If $\xi \in \mathfrak{g}^*$ is a regular value of μ and G_ξ acts freely and properly on $\mu^{-1}(\xi)$, then zero is a regular value of ν and G acts freely and properly on $\nu^{-1}(0)$.*

Proof. If G_ξ acts freely on $\mu^{-1}(\xi)$, then G_η acts freely on $\mu^{-1}(\eta)$ for every $\eta \in \mathcal{O}_\xi$ since the stabilizers are conjugate. Let $p \in \nu^{-1}(0)$ and $g \in G_{(p,\eta)}$, then

$$(g \cdot p, \mathrm{Ad}_g^* \eta) = g \cdot (p, \eta) = p = (p, \eta).$$

It follows that $g \in G_\eta$ and hence $g = e$. We conclude that G acts freely on $\nu^{-1}(0)$ and thus zero is a regular value of ν by Corollary 2.7. □

It follows that under Assumptions 2.9, we have three candidates for a symplectic quotient and following result ensures that these are all naturally isomorphic to each other, see [DHJvdH19, Proposition 6.18].

Proposition 3.4. *There are canonical symplectomorphisms*

$$(M^\xi, \omega^\xi) \cong (M^{\mathcal{O}_\xi}, \omega^{\mathcal{O}_\xi}) \cong ((M \times \mathcal{O}_\xi)^0, \nu^0).$$

Corollary 3.5. For any two points ξ and η in the same coadjoint orbit, there is a natural symplectomorphism

$$(M^\xi, \omega^\xi) \cong (M^\eta, \omega^\eta).$$

Remark 3.6. Whenever we can form the symplectic quotient M^ξ, we may choose instead to work with the symplectic quotient at the zero level of the shifted moment map ν. We may always assume that our symplectic quotients are formed at the zero level of the moment map, and in particular that they are G-orbit spaces. Unless specified otherwise, the symplectic quotients in the rest of the notes are assumed to be taken at the zero level of the given moment map.

Example 3.7. In the same setting of Example 2.12, if we reduce at another level $\xi > 0$ then we obtain as a reduced space the same smooth manifold $\mu^{-1}(\xi)/S^1 \cong \mathbb{C}P^n$ but the symplectic form is scaled.

Example 3.8. Consider the action of a Lie group G on its cotangent bundle T^*G which is induced by the right translations $R_{g^{-1}}$. That is, the corresponding symplectomorphism $\sigma_g \in \mathrm{Diff}(T^*G)$ is given by $\sigma_g(h, \eta) = (R_{g^{-1}}(h), dR_g(hg^{-1})^*\eta)$, where $dR_g(hg^{-1})^* \colon T_h^*G \to T_{hg^{-1}}^*G$ denotes the adjoint map of $dR_g(hg^{-1})$. For every $X \in \mathfrak{g}$ the 1-parameter group $\Phi^X : t \mapsto R_{\exp(-tX)}$ of diffeomorphisms of G is generated by the vector field $X^\#(h) = dL_h(e)X$, and the flow $t \mapsto \sigma_{\exp(t\xi)}$ is generated by the Hamiltonian function

$$H_X(h, \eta) = -\eta(X^\#(h)) = -(dL_h(e)^*\eta)(X).$$

One can check that $H_X = i_{X^\#}\lambda_{can}$, where λ_{can} is the canonical Liouville form on T^*G, and that $\sigma_g^*\lambda_{can} = \lambda_{can}$. Hence it follows that the action is Hamiltonian. One can check that a corresponding moment map is

$$\mu \colon T^*G \to \mathfrak{g}^*, \quad (h, \eta) \mapsto -dL_h^*\eta.$$

It is easier to understand this action if we trivialize the bundle T^*G. The above formulas suggest that we should identify the cotangent space T^*_hG with $\mathfrak{g}^*$ by means of the left invariant forms on G. In other words, we shall use the diffeomorphism

$$T^*G \to G \times \mathfrak{g}^*, \quad (h, \eta) \mapsto dL_h(e)^*\eta.$$

Then the action of G on $G \times \mathfrak{g}^*$ is given by $\sigma_g = (R_{g^{-1}}, \mathrm{Ad}^*_{g^{-1}})$ and is generated by the Hamiltonian functions

$$H_X \colon G \times \mathfrak{g}^* \to \mathbb{R}, \quad (h, \xi) \mapsto -\xi(X).$$

Hence the moment map $\mu \colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is just minus the projection onto the second component. Then for every $\xi \in \mathfrak{g}^*$ the level set at the coadjoint orbit is $\mu^{-1}(\mathcal{O}_\xi) \cong G \times \mathcal{O}_\xi$ and G acts freely and properly on it. Thus the symplectic reduction is given by $\mathcal{O}_\xi$ and one can check that the corresponding reduced symplectic form is the Kostant-Kirillov form ω^{KKS}, see above.

4 Reduction in Stages

Let (M, ω, G, μ_G) be a Hamiltonian G-space and let $H \subset G$ be a **normal Lie subgroup**. Then there is an induced Hamiltonian action of H on M with moment map μ_H given by composing the projection $\mathfrak{g}^* \to \mathfrak{h}^* \subset \mathfrak{g}^*$ with μ_G. If we form the symplectic quotient at a regular level of μ_H, it may carry some residual symmetry from the original G-action. Denoting the inclusion by $j \colon H \to G$ we have that $\mu_H = J^* \circ \mu_G$, where $J := dj(e) \colon \mathfrak{h} \to \mathfrak{g}$ and $J^* \colon \mathfrak{g}^* \to \mathfrak{h}^*$ denotes the adjoint of J. Assume that H acts freely and properly on $\mu_H^{-1}(0)$. Then we can form the symplectic quotient $\mu_H^{-1}(0)/H$ which we denote by (M^H, ω^H). Under the identification $\mathfrak{h} \cong J(\mathfrak{h}) \subset \mathfrak{g}$ we have that $\ker J^* = \mathfrak{h}^0 \subset \mathfrak{g}^*$, where $\mathfrak{h}^0$denotes the annihilator of $\mathfrak{h}$. Since H is invariant under the conjugation action of G it follows that $\mathfrak{h}$ is invariant under the adjoint action of G. Hence G preserves $\mathfrak{h}^0$ and therefore also the zero level set of μ_H since

$$\mu_H^{-1}(0) = \mu_G^{-1}(\ker J^*) = \mu_G^{-1}(\mathfrak{h}^0).$$

Normality of H implies furthermore that G preserves the H-orbits and therefore acts on the orbit space M^H in such a way that $\pi_H \colon \mu_H^{-1}(0) \to M^H$ is equivariant. Finally, this action on M^H clearly reduces to an action of the quotient group G/H which is Hamiltonian by the following result, see [DHJvdH19, Proposition 6.21].

Proposition 4.1. *Let $\iota_H \colon \mu_H^{-1}(0) \hookrightarrow M$ denote the inclusion. There is a natural Hamiltonian action of the quotient group G/H on M^H with moment map $\mu_{G/H}$ satisfying*

$$P^* \circ (\pi_H^* \mu_{G/H}) = \iota_H^* \mu_G,$$

where $P^* \colon (\mathfrak{g}/\mathfrak{h})^* \to \mathfrak{g}^*$ denotes the adjoint of the quotient map $P \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ which is the derivative at the identity $P := dp(e)$ of the quotient map $p \colon G \to G/H$.

Reduction by a normal subgroup is simplified in the case of a product group $G = G_1 \times G_2$

with $H = G_1$. In this situation, the reduced moment map is essentially the moment map for the action of the G_2 component, see [DHJvdH19, Corollary 6.23].

Corollary 4.2. Let $(M, \omega, G = G_1 \times G_2, \mu = (\mu_1, \mu_2))$ be a Hamiltonian G-space and denote by M_1 the reduction of (M, ω) at zero with respect to G_1, μ_1. Then the moment map μ_{G/G_1} for the action of $G/G_1 \cong G_2$ on M_1 satisfies

$$\pi_1^* \mu_{G/G_1} = \iota_{G_1}^* \mu_2,$$

where $\pi_1 \colon \mu_1^{-1}(0) \to M_1$ is the G_1-bundle map and $\iota_1 \colon \mu_1^{-1}(0) \hookrightarrow M$ is the inclusion.

Going back to the more general case, if G acts freely and properly on $\mu_G^{-1}(0)$ and H acts freely and properly on $\mu_H^{-1}(0)$, then G/H acts freely and properly on $\mu_{G/H}^{-1}(0)$ and we can form the reduction $(M^H)^{G/H} := \mu_{G/H}^{-1}(0)/(G/H)$. This space is diffeomorphic to the reduction $M^G := \mu_G^{-1}(0)/G$ and we should hope that the symplectic structures $(\omega^H)^{G/H}$ and ω^G are also isomorphic. The following result ensures this and effectively allows us to perform symplectic reduction in multiple stages given a tower of normal subgroups, see [DHJvdH19, Theorem 6.24]. Such operation is usually called **reduction in stages** and it is of particular relevance in the reduction of toric actions, since tori are products of circles.

Theorem 4.3 (Reduction in Stages)**.** *There is a natural symplectomorphism*

$$((M^H)^{G/H}, (\omega^H)^{G/H}) \cong (M^G, \omega^G).$$

Corollary 4.4. Let $(M, \omega, G = G_1 \times G_2, \mu = (\mu_1, \mu_2))$ be a Hamiltonian G-space and denote by M_1 the reduction of (M, ω) at zero with respect to G_1, μ_1 as in Corollary 4.2. Then there is a canonical symplectomorphism

$$M_1/G_2 \cong M^0 := \mu^{-1}(0)/G.$$

Example 4.5. In the same setting of Example 2.12, let $\mathbb{T}^{n+1}$ also act on $(\mathbb{C}^{n+1}, \omega)$ by diagonal multiplication. This new Hamiltonian action commutes with the U(1)-action and hence it descends to the reduced space $\mathbb{C}P^n$. The reduced moment map is given by

$$\mathbb{C}P^n \to \mathbb{R}^{n+1}, \quad [z_0 : z_1 : \ldots : z_n] \mapsto \frac{1}{2}(|z_0|^2, |z_1|^2, \ldots, |z_n|^2).$$

where we choose $(z_0, z_1, \ldots, z_n) \in \mu^{-1}(\frac{1}{2})$.

5 Orbifolds

We now see that if we drop the assumption on G_ξ of acting freely on the level set $\mu^{-1}(\xi)$, then the reduced quotient may have singularities. However, if the level ξ is a regular value of μ, we know that then the action of G on $\mu^{-1}(\xi)$ is at least locally free, see Corollary 2.7. In this case, it turns out that the reduction $\mu^{-1}(\xi)/G$ is modelled on a Euclidean space quotiented by the action of a finite group, that is, it has an orbifold structure, see [Sat56] for an introduction to orbifolds. For this discussion we focus our attention to the case of tori.

Let (M, ω, G, μ) be a Hamiltonian G-space. Assume now that G is **connected, compact and abelian** and hence an n**-torus**. Note that since G is compact, every orbit $\mathcal{O}_p$ is embedded in M. Let then $S \subset M$ be a submanifold containing p and transverse to $\mathcal{O}_p$ at p, such S is called a **slice** for $\mathcal{O}_p$ at p. Choose a slice chart for $\mathcal{O}_p$ in M, that is, coordinates $x = (x^1, \dots, x^m) \colon U \to \mathbb{R}^m$ centered at p such that

$$\mathcal{O}_p : x^1 = \ldots = x^n = 0 \quad \text{and} \quad S : x^{n+1} = \ldots = x^m = 0, \quad m = \dim M.$$

Assume that $S \subset U$ and denote by $S_\varepsilon := x^{-1}(x(S) \cap B_\varepsilon(0))$, where $B_\varepsilon(0) \subset \mathbb{R}^m$ is the Euclidean ball. The following result can be found in [CdS01, Theorem 23.5].

Theorem 5.1. *(Slice Theorem) Let G be a compact Lie group acting on a manifold M such that G acts locally freely at $p \in M$. Then, for sufficiently small $\varepsilon > 0$ the map $\sigma \colon G \times S_\varepsilon \to M, (g, p) \mapsto g \cdot p$ is an embedding of $G \times S_\varepsilon$ onto a G-invariant neighbourhood $\mathcal{U}$ of $\mathcal{O}_p$.*

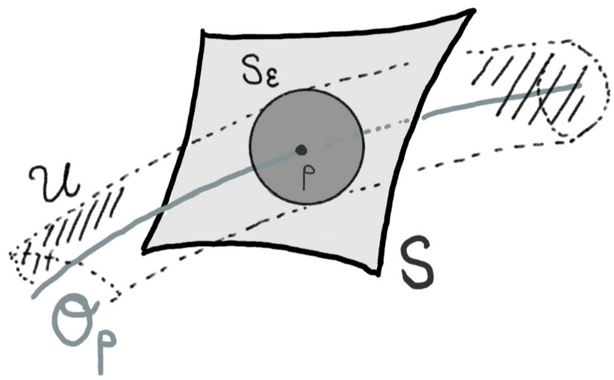

Figure 2: If the action is locally free at $p \in M$, then the orbit of a small disk S_ε in a slice S for $\mathcal{O}_p$ in M gives a G-invariant tubular neighbourhood of the orbit $\mathcal{O}_p$.

In the setting above, for $\xi \in \mathfrak{g}^*$ we have that $G_\xi = G$ and if ξ is a regular value of μ (note that the set of singular values has measure zero in $\mathfrak{g}^*$ by Sard's Theorem) then $\mu^{-1}(\xi) \subset M$ is a submanifold of codimension n. Note that since ξ is a regular value, by Corollary 2.7 we have that G acts locally freely on $\mu^{-1}(\xi)$ and since G is compact, for every $p \in \mu^{-1}(\xi)$ the discrete subgroup G_p is actually finite. Moreover, by Theorem 5.1 it follows that $\mu^{-1}(\xi)/G$ is modeled on S_ε/G_p, where S_ε is a G_p-invariant disk in $\mu^{-1}(\xi)$ through p and transverse to $\mathcal{O}_p$. Hence, locally $\mu^{-1}(\xi)/G$ looks like the quotient of $\mathbb{R}^{m-n}$ by a finite group, that is, $\mu^{-1}(\xi)/G$ is an **orbifold**. Note that the reason of this is that we did not require G to act freely on $\mu^{-1}(\xi)$.

Example 5.2. Consider the S^1-action on $\mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) := (e^{ik\theta} z_1, e^{i\theta} z_2)$ for some fixed integer $k \geq 2$. This is Hamiltonian with moment map

$$\mu \colon \mathbb{C}^2 \to \mathbb{R}, \quad (z_1, z_2) \mapsto -\frac{1}{2}(k|z_1|^2 + |z_2|^2).$$

Every $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid. Then for $z = (z_1, z_2) \in \mu^{-1}(\xi)$ we have that

$$S^1_z = \begin{cases} \{1\} & \text{if } z_2 \neq 0 \\ \{e^{\frac{2\pi i l}{k}} : 0 \leq l \leq k-1\} \cong \mathbb{Z}_k & \text{if } z_2 = 0. \end{cases}$$

The reduced space $\mu^{-1}(\xi)/S^1$ is called a **teardrop orbifold** or **conehead**; it has one **cone** (also known as a **dunce cap**) singularity of type k (with cone angle $\frac{2\pi}{k}$).

Example 5.3. Let S^1 act on $\mathbb{C}^2$ by $e^{i\theta} \cdot (z_1, z_2) := (e^{ik\theta} z_1, e^{il\theta} z_2)$ for some integers $k, l \geq 2$. Suppose that k and l are relatively prime. Then for $z = (z_1, z_2) \in \mu^{-1}(\xi)$ we have that

$$S^1_z = \begin{cases} \mathbb{Z}_k & \text{if } z_1 \neq 0 \\ \mathbb{Z}_l & \text{if } z_2 \neq 0 \\ \{1\} & \text{if } z_1, z_2 \neq 0. \end{cases}$$

The quotient $\mu^{-1}(\xi)/S^1$ is called a **football orbifold**. It has two cone singularities, one of type k and another of type l.

Example 5.4. More generally, the reduced spaces of S^1 acting on $\mathbb{C}^n$ by

$$e^{i\theta} \cdot (z_1, \ldots, z_n) := (e^{ik_1\theta} z_1, \ldots, e^{ik_n\theta} z_n),$$

are called **weighted** or **twisted projective spaces**.

6 The Moduli Space of Flat Connections

We investigate an infinite dimensional example, namely the space $\mathcal{A}$ of connections on a principal bundle with compact or semisimple structure group G, over a closed oriented 2-dimensional Riemannian manifold M. It turns out, that $\mathcal{A}$ has a symplectic structure and a Hamiltonian space structure with respect to the group $\mathcal{G}$ of gauge transformations acting on $\mathcal{A}$ via pullback and whose moment map is given by the curvature. The corresponding symplectic reduction is then the moduli space $\mathcal{M}$ of flat connections. The content of this section can be found in [CdS01, Chapter 25], [MS17, 5.3.18] and [AB83].

Let $\pi\colon P \to M$ be a principal G-bundle. For $g \in G$, we denote by $L_g \in \mathrm{Diff}(G)$ the left multiplication by g in G and by $\tau_g \in \mathrm{Diff}(P)$ the right action of G on P associated to the principal bundle structure, see [Mer20, Lecture 16] for an introduction to principal bundles. We denote by $\Omega^k(P, \mathfrak{g}) := \Gamma(\bigwedge_{i=1}^k T^*P \otimes \mathfrak{g})$ the vector space of $\mathfrak{g}$-valued k-forms on P. By $\Omega^k_G(P, \mathfrak{g})$ we denote the subspace of $\Omega^k(P, \mathfrak{g})$ consisting of G-equivariant horizontal forms, where a form $A \in \Omega^k(P, \mathfrak{g})$ is **equivariant** if $\tau_g^* A = \mathrm{Ad}_{g^{-1}} A$ for every $g \in G$ and is **horizontal** if it vanishes whenever any of its variables is a vertical vector, i.e. an element of the vertical bundle $VP := \ker d\pi$. Let $\mathrm{Ad}(P) := (P \times_{G,\mathrm{Ad}} \mathfrak{g}) \to M$ denote the associated vector bundle to $\pi\colon P \to M$ generated by the representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, see [Mer20, Lecture 18] for an introduction to associated bundles. There is a linear isomorphism $\Omega^k_G(P,\mathfrak{g}) \cong \Omega^k(M, \mathrm{Ad}(P))$, see [Mer20, Theorem 39.3].

Definition 6.1. A **connection form** A on $\pi\colon P \to M$ is an element of $\Omega^1(P,\mathfrak{g})$ with the following properties:

- $\forall g \in G : \tau_g^* A = \mathrm{Ad}_{g^{-1}} A$ and
- $\forall X \in \mathfrak{g} : A(X^\#) = X$, where $X^\# \in \Gamma(TP)$ denotes the fundamental vector field associated to X with respect to the action τ.

Such a form defines a connection on P given by the distribution $\ker A \subset TP$. We denote by $\mathcal{A}$ the space of connections defined on P. It is easy to see that this space is an affine space modelled on the linear space $\Omega^1_G(P,\mathfrak{g})$. Moreover, $\mathcal{A}$ can be turned into a Banach space by defining the Sobolev space of sections $W^{k,p}(M, \mathrm{Ad}(P))$ using higher order covariant derivatives, see [Weh04].

Definition 6.2. A **gauge transformation** of P is a principal bundle isomorphism, i.e. a G-equivariant diffeomorphism $\Phi\colon P \to P$ such that $\pi \circ \Phi = \pi$. We denote by $\mathcal{G}$ the space of gauge transformations of P. Note that $\mathcal{G}$ forms a group together with the composition of functions. Moreover, $\mathcal{G}$ can be turned into a Banach manifold modelled on $W^{k,p}(M, \mathrm{Ad}(P))$, see [Weh04].

Lemma 6.3. *We have the following linear isomorphisms*

$$\mathcal{G} \cong \{f\colon P \to G \text{ smooth} : f \circ \tau_g = c_{g^{-1}} \circ f \quad \forall g \in G\} \cong \Gamma(P \times_{G,c} G),$$

where $c_g \in \mathrm{Diff}(G)$ *denotes the conjugation by* $g \in G$ *and* $P \times_{G,c} G$ *denotes the associated bundle to* P *with respect to the conjugation action on* G*. One can then show that*

$$\mathrm{Lie}(\mathcal{G}) \cong \Gamma(P \times_{G,\mathrm{Ad}} \mathfrak{g}) = \Omega^0(M, \mathrm{Ad}(P)) \cong \Omega^0_G(P,\mathfrak{g}).$$

Proof. For what concerns $\mathcal{G}$, the first isomorphism is easy by recalling that τ acts freely on P and is fibre-preserving and the second isomorphism is proven analogously to [Mer20, Corollary 39.4]. □

Proposition 6.4. *There is an action of* $\mathcal{G}$ *on* $\mathcal{C}$ *given by*

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A}, \quad (\Phi, A) \mapsto \Phi^* A = dL_{f^{-1}} \circ df + \mathrm{Ad}_{f^{-1}} A,$$

where we identified $\Phi \in \mathcal{G}$ *with a* (τ, c)*-equivariant function* $f\colon P \to G$ *as in Lemma 6.3.*

Proof. By the identification of Lemma 6.3 we have that $\Phi(p) = \tau_{f(p)}(p)$ for every $p \in P$. Then by the chain rule we have that $d(\tau_f)(p) = d\tau^p(f(p)) \circ df_p + d\tau_{f(p)}$. The claim follows then by (τ, Ad)-equivariance of A and by the fact that $A(d\tau^p(f(p)) \circ df_p(Z)) = dL_{f(p)^{-1}}(f(p)) \circ df_p(Z)$ for every $Z \in T_pP$ which can be deduced by the second condition in Definition 6.1. □

Definition 6.5. Let $\pi\colon P \to M$ be a principal G-bundle with connection A. The **curvature form** $F_A \in \Omega^2(P, \mathfrak{g})$ of A is defined by

$$F_A(X, Y) := -A([X^h, Y^h]) \quad \forall X, Y \in \Gamma(TP),$$

where $X^h \in \Gamma(TP)$ denotes the horizontal component of X with respect to the connection distribution $\ker A$. We say that A is **flat** if $F_A = 0$, which is equivalent to $\ker A$ being an integrable distribution. The curvature form F_A belongs to $\Omega^2_G(P, \mathfrak{g})$ and it satisfies

- $F_A = dA + \frac{1}{2}[A, A]$ (**Cartan's Structure Equation**) and
- $dF_A = [F_A, A]$ (**Bianchi Identity**)

see [Mer20, Theorem 41.6].

From now on, we assume that M is a **closed oriented 2-dimensional Riemannian** manifold and that G is compact or semisimple. In this setting, Atiyah and Bott [AB83] noticed that $\mathcal{A}$ is an infinite-dimensional symplectic manifold. This will require choosing an Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, which under these assumptions exists, either by averaging any inner product if G is compact, or by using the Killing form if G is semisimple. Since $\mathcal{A}$ is an affine space modeled on $\Omega^1_G(P, \mathfrak{g})$, its tangent space at a point A can be identified with $\Omega^1_G(P, \mathfrak{g})$. We can thus define a symplectic form ω on $\mathcal{A}$ by

$$\omega\colon \Omega^1_G(P, \mathfrak{g}) \times \Omega^1_G(P, \mathfrak{g}) \to \mathbb{R}, \quad \omega(a, b) := \int_M \langle a \wedge b \rangle,$$

where for basis X_i of $\mathfrak{g}$ we have that $\langle a^i \otimes X_j \wedge b^j \otimes X_i \rangle := a^i \wedge b^j \langle X_i, X_j \rangle$ and then we make the identification $\Omega^2_G(P, \mathfrak{g}) \cong \Omega^2_G(M, \mathrm{Ad}(P))$. The map ω is non-degenerate, skew-symmetric, bilinear and constant in the sense that it does not depend on the base point A. Therefore, it has the right to be called a symplectic form on $\mathcal{A}$, so the pair $(\mathcal{A}, \omega)$ is an infinite-dimensional symplectic manifold.

For $\xi \in \mathrm{Lie}(\mathcal{G}) \cong \Omega^0_G(P, \mathfrak{g})$ the infinitesimal action of $\mathcal{G}$ on $\mathcal{A}$ is given by the vector field

$$\mathcal{A} \to T\mathcal{A} \cong \Omega^1_G(P, \mathfrak{g}), \quad A \mapsto -d\xi - [A, \xi].$$

One can easily check that for each fixed ξ this is a Hamiltonian vector field with Hamiltonian function

$$\mathcal{A} \to \mathbb{R}, \quad A \mapsto \int_M \langle F_A \wedge \xi \rangle.$$

One can identify $\Omega^2_G(P, \mathfrak{g})$ as the dual space of $\mathrm{Lie}(\mathcal{G}) \cong \Omega^0_G(P, \mathfrak{g})$ via the pairing

$$\Omega^0_G(P, \mathfrak{g}) \times \Omega^2_G(P, \mathfrak{g}) \to \mathbb{R}, \quad \langle \xi, F \rangle := \int_M \langle \xi \wedge F \rangle.$$

It follows that a moment map is given by the curvature

$$\mu\colon \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^* \cong \Omega^2_G(P, \mathfrak{g}), \quad A \to F_A = dA + \frac{1}{2}[A, A].$$

Then the reduction of $(\mathcal{A}, \omega)$ at level zero $\mathcal{M} := \mu^{-1}(0)/\mathcal{G}$ with respect to $\mathcal{G}, \mu$ is the space of flat connections modulo gauge equivalence, known as the **moduli space of flat connections**. It turns out that $\mathcal{M}$ is a finite-dimensional symplectic orbifold. Moreover, if G is simply connected the quotient $\mathcal{M}$ can be identified with the space of conjugacy classes of representations $\rho\colon \pi_1(\Sigma) \to G$, see [MS17].

There are other very interesting examples of symplectic reduction in infinite dimensional settings, such as the space of compatible almost complex structure with constant scalar curvature on a symplectic manifold, see [MS17, Example 5.3.19], and the space of Lagrangian embeddings in a symplectic manifold modulo volume preserving diffeomorphisms, see [MS17, Example 5.3.20].

References

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR 702806

[CdS01] Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001. MR 1853077

[DHJvdH19] Shubham Dwivedi, Jonathan Herman, Lisa C. Jeffrey, and Theo van den Hurk, *Hamiltonian group actions and equivariant cohomology*, SpringerBriefs in Mathematics, Springer, Cham, 2019. MR 3970272

[Mer20] Will Merry, *Lecture notes on differential geometry*, 2020.

[MMeO+07] Jerrold E. Marsden, Gerard Misio ek, Juan-Pablo Ortega, Matthew Perlmutter, and Tudor S. Ratiu, *Hamiltonian reduction by stages*, Lecture Notes in Mathematics, vol. 1913, Springer, Berlin, 2007. MR 2337886

[MS17] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, third ed., Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2017. MR 3674984

[Sat56] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 79769

[Weh04] Katrin Wehrheim, *Uhlenbeck compactness*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2004. MR 2030823