Contact Reduction

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Abstract

This talk introduces a method to construct contact manifolds using the Quotient Manifold Theorem. A simple example for the reduction of the standard overtwisted structure on $\mathbb{R}^3$ is given.

1 Introduction

Given a contact manifold (Y, α) with a contact action preserving α one can reduce the dimension of the manifold by taking the 0-level set of the α-moment and identifying all points in an orbit. The main condition for this to be possible is that 0 is a regular value of the α-moment map. In this construction the dimension of Y gets reduced by $2 \dim G$.

This fits in with the result we have seen last week about toric G-manifolds. Recall that a toric G-manifold is a contact manifold (Y, ξ) with an effective contact action of a torus G such that $2 \dim G = \dim Y + 1$. In this setting contact reduction should not be possible, as the dimension would be reduced to -1. This is in fact the case, as the α-moment map does not vanish for any G-invariant contact form α on the toric G-manifold.

2 Contact Reduction

We will start at the linear algebra level by considering how one-forms descent to a quotient manifold. In order to canonically define the one-form on the quotient, it needs to be identical on all the representatives of an element in the quotient.

This lemma is the counterpart to the Lemma 2.1. from the previous talk by Elia, where two-forms are defined on quotients by coisotropic subspaces.

Lemma 2.1. *Let V be a vector space with a one-form $\alpha \in V^*$, and let $W \subset \ker(\alpha)$ be a subspace. Then α induces a canonical one-form on V/W.*

Proof. Let $[v] \in V/W$. Define $\tilde{\alpha}$ on V/W by $\tilde{\alpha}([v]) := \alpha(v)$. That $\tilde{\alpha}$ is a well defined one-form follows from $\tilde{\alpha}([v+w]) = \alpha(v+w) = \alpha(v)$, for an arbitrary $w \in W$. □

As in the symplectic case, we will make use of the Quotient Manifold Theorem which is restated here as formulated in [Mer19].

Theorem 2.2 (Quotient Manifold Theorem)**.** *Let σ be a smooth action of a Lie group G on M which is both proper and free. Then the quotient space M/G admits the structure of a topological manifold of dimension* $\dim M - \dim G$*. Moreover there exists a unique smooth structure on M/G such that the quotient map $\rho : M \to M/G$ is a smooth submersion.*

Recall the definition of the α-moment map for actions which preserve the contact form. This is not a restriction to the applicability of α-moment maps, as a contact form can always be rescaled such that it is preserved by the action.

Definition 2.3. *Let (Y, α) be a contact manifold equipped with an action $\sigma : G \times Y \to Y$ which preserves α, i.e. $\sigma_g^* \alpha = \alpha$. The α-moment map $\mu_\alpha : Y \to \mathfrak{g}^*$ is defined at $p \in Y$ by*

$$\mu_\alpha(p)(X) := \alpha_p(X_p^\sharp)$$

Also, recall that α-moment maps are G-equivariant.

Lemma 2.4. *The α-moment map is G-equivariant, i.e.*

$$\mu_\alpha(g \cdot p) = \mathrm{Ad}_g^* \, \mu_\alpha(p)$$

The theorem and proof are slightly adapted from [Gei97] to match our notation.

Theorem 2.5 (Contact Reduction)**.** *Let (Y, α) be a contact manifold equipped with a contact action of a compact Lie group G which preserves α.*

If 0 is a regular value of the α-moment map $\mu_\alpha : Y \to \mathfrak{g}^*$, then the action of G on $\mu_\alpha^{-1}(0)$ is locally free. If this action is free, the quotient manifold $\mu_\alpha^{-1}(0)/G$ has an induced contact form α^0.

Proof. If 0 is a regular value of μ_α, then $\mu_\alpha^{-1}(0)$ is a submanifold of dimension $\dim Y - \dim G$. We claim that G acts locally freely on $\mu_\alpha^{-1}(0)$. Indeed if $p \in \mu_\alpha^{-1}(0)$ then

$$\mu_\alpha(g \cdot p) = \mathrm{Ad}_g^* \, \mu_\alpha(p) = 0,$$

and hence $G \cdot p \subseteq \mu_\alpha^{-1}(0)$ and G acts on $\mu_\alpha^{-1}(0)$. This also implies that

$$X_p^\sharp \in T_p \mu_\alpha^{-1}(0) \cap \xi_p$$

for any $X \in \mathfrak{g}$ and $\xi = \ker \alpha$.

In order to prove that G acts locally freely on the level set we need to show that $X_p^\sharp \neq 0$ for any $p \in \mu_\alpha^{-1}(0)$. This follows from (Y, α) being contact and 0 being a regular value as follows. Denote by $\mu_\alpha^X : Y \to \mathbb{R}$ a component of the α-moment map, i.e. $\mu_\alpha^X(p)(X) := \alpha_p(X_p^\sharp)$, for $X \in \mathfrak{g}$. We can calculate

$$d\mu_\alpha^X = d(\alpha(X^\sharp)) = \mathcal{L}_{X^\sharp} \alpha - i_{X^\sharp} d\alpha = -i_{X^\sharp} d\alpha.$$

As $d\alpha$ is non degenerate on ξ we have $d\mu_\alpha^X(p) = 0 \Leftrightarrow X_p^\sharp = 0$. Therefore, p being a regular point of μ_α implies that G acts locally freely on $\mu_\alpha^{-1}(0)$. Let us assume, as stated in the theorem, that this action is free. This allows us to use the Quotient Manifold Theorem 2.2 to obtain the smooth manifold $\mu_\alpha^{-1}(0)/G$.

Now we claim α reduces to a canonical contact form on $\mu_\alpha^{-1}(0)/G$. For each point $p \in \mu_\alpha^{-1}(0)$ we obtain a canonical one-form $\alpha^0_{[p]}$ on

$$T_p\mu_\alpha^{-1}(0)/T_p\mathcal{O}_p \cong T_{[p]}(\mu_\alpha^{-1}(0)/G)$$

as $\alpha_p(X_p^\sharp) = 0$. This is well-defined because α is G-invariant. To prove that $d\alpha^0|_{\ker \alpha^0}$ is non-degenerate is equivalent to showing that the kernel of $d\alpha_p$ on $\ker \alpha_p \cap T_p\mu_\alpha^{-1}(0)$ is $T_p\mathcal{O}_p$. We know that $T_p\mathcal{O}_p$, which has dimension $\dim G$, is in this kernel by

$$i_{X^\sharp} d\alpha = -d\mu_\alpha^X \qquad \text{and} \qquad d\mu_\alpha^X(p)|_{T_p\mu_\alpha^{-1}(0)} = 0.$$

On the other hand the kernel of $d\alpha_p$ restricted to $\ker \alpha_p \cap T_p\mu_\alpha^{-1}(0)$ has dimension less than or equal to $\dim G$ because $T_p\mu_\alpha^{-1}(0)$ has codimension $\dim G$. Therefore the kernel is exactly $T_p\mathcal{O}_p$.

□

Note that contact reduction can only be made on the 0-level set $\mu_\alpha^{-1}(0)$, unlike symplectic reduction, which is possible for regular value. The reason is that for a different level set it is not possible to push forward the one-form α onto the quotient manifold.

3 Examples and application

First, a concrete example in $\mathbb{R}^3$, which is almost trivial, as any 1-dimensional contact manifold can only have a trivial contact structure.

Example 3.1. *Consider the standard overtwisted contact structure on $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and contact form*

$$\alpha = \alpha_{vri} = \cos(r)dz + r\sin(r)d\theta.$$

Let $G = \mathbb{S}^1$ act on $\mathbb{R}^3$ via rotation around the z-axis, hence $X^\sharp = \frac{\partial}{\partial\theta}$ for $X = \frac{\partial}{\partial\phi} \in \mathfrak{g}$. For $p = (r, \theta, z)$ the α-moment map is

$$\mu_\alpha(p)\left(\frac{\partial}{\partial\phi}\right) = \alpha_p\left(\frac{\partial}{\partial\theta}\Big|_p\right) = r\sin(r)$$

Notice, that 0 is not a regular value because for p_z in the z-axis $Z = \{p \in \mathbb{R}^3 \mid r(p) = 0\}$, we have $\mu_\alpha(p_z) = 0$ and $d\mu_\alpha(p_z) = 0$. Sidestepping the problem by cuttin out the z-axis, the contact manifold $\mathbb{R}^3 \setminus Z$ fulfills the requirements of the reduction theorem. Concretely, the reduced contact manifold $(\mu_\alpha^{-1}(0)/\mathbb{S}^1, \alpha^0)$ can be constructed as follows.

The 0-level set of μ_α are the cylinders

$$\mu_\alpha^{-1}(0) = \left\{(r, \theta, z) \in \mathbb{R}^3 \mid r = 2\pi k \text{ for } k \in \mathbb{N}\right\}$$

The orbit of $p = (r, \theta, z)$ *is*

$$\mathcal{O}_p = \big\{(r, \theta', z) \mid \theta' \in [0, 2\pi)\big\}$$

Therefore

$$\mu_\alpha^{-1}(0)/\mathbb{S}^1 = \{(2\pi k, z) \mid k \in \mathbb{N} \text{ and } z \in \mathbb{R}\}$$

This is a contact manifold with $\ker \alpha^0 = \{0\}$*, which is not particularly interesting, but at least the overtswisted structure is not completely lost.*

Another source of examples is the following proposition from [Gei97], which follows from the theorem. It can also be formulated for hamiltonian $\mathbb{T}^n$ actions, instead of $\mathbb{S}^1$ actions.

Proposition 3.2. *Let* (Y, α) *be a contact manifold such that* $\mathbb{S}^1$ *acts on it by the flow along the reeb vector field* ξ *of* α*. Let* $(M, d\beta, \mathbb{S}^1, \mu_1 := -\beta(X_1^\sharp))$ *be a hamiltonian* $\mathbb{S}^1$*-space where the action preserves* β*. Then* $\xi + KX_1^\sharp$ *generates a free* $\mathbb{S}^1$*-action on* $Y \times \mu_1^{-1}(1/K)$ *for any* $K \in \mathbb{Z} \setminus \{0\}$*, and the contact form* $\alpha + \beta$ *on* $Y \times M$ *descends to a contact form on*

$$\left(Y \times \mu_1^{-1}\left(\frac{1}{K}\right)\right) / \, \mathbb{S}^1.$$

To see why the theorem applies, consider the following calculation. Let $(p, q) \in Y \times \mu_1^{-1}(1/K)$, then:

$$\begin{aligned}(\alpha + \beta)_{(p,q)} \left(\xi(p) + KX_1^\sharp(q)\right) &= \alpha_p(\xi(p)) + K\beta_q(X_1^\sharp(q)) \\ &= 1 - K\frac{1}{K} \\ &= 0\end{aligned}$$

The converse also holds, hence the manifold from the proposition is exactly the 0-level set of the $(\alpha + \beta)$-moment map.

The Hirzenbruch surface W_k is an example for a hamiltonian $\mathbb{T}^2$-space for which the proposition applies.

Example 3.3. *Consider the hypersurface*

$$V \subset \left(\mathbb{C}^2 \setminus \{0\}\right) \times \left(\mathbb{C}^3 \setminus \{0\}\right)$$

using coordinates (a, b, x, y, z) *on* $\mathbb{C}^2 \times \mathbb{C}^3$ *defined by* $a^k y - b^k x = 0$ *for some* $k \in \mathbb{Z}$*. Let* $\mathbb{T}^2$ *act on* V *by*

$$(s, t) \cdot (a, b, x, y, z) = (sa, sb, tx, ty, tz).$$

We choose the moment map $\mu : V \to \mathbb{R}^2$ *by*

$$\mu(a, b, x, y, z) = \left(|a|^2 + |b|^2, |x|^2 + |y|^2 + |z|^2\right).$$

For $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$ *the symplectic quotient* $\mu^{-1}(u,v)/\mathbb{T}^2$ *is called the Hirzebruch surface* W_k.

Now let (Y,α) *be any contact manifold with reeb vector field* ξ*, choose* $K, L \in \mathbb{Z} \setminus \{0\}$ *and let* $\mathbb{T}^2$ *act on* $Y \times V$ *by the flow of the two vector fields* $\xi + X_1^\sharp$ *and* $\xi + X_2^\sharp$*. Then by the proposition we obtain the contact quotient*

$$\left(Y \times \mu^{-1}\left(\frac{1}{K}, \frac{1}{L}\right)\right) / \ \mathbb{T}^2.$$

References

[Gei97] Hansjörg Geiges. “Constructions of contact manifolds”. In: *Math. Proc. Cambridge Philos. Soc.* 121.3 (1997), pp. 455–464. ISSN: 0305-0041. DOI: 10.1017/S0305004196001260. URL: https://doi.org/10.1017/S0305004196001260.

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