Equivariant Darboux

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Abstract

In this talk we discuss a modified version of the Darboux theorem. If we have in addition a compact Lie group G acting on the manifold M, then a natural question would be whether the diffeomorphism that we have seen in the Darboux-Weinstein theorem can be chosen to be G-equivariant. Then we focus on hamiltonian torus actions and get a normal form near a $\mathbb{T}^k$-invariant point.

1 Equivariant Darboux

Let's first recall the Darboux-Weinstein theorem:

Theorem 1 (Darboux-Weinstein Theorem)**.** *Let M be a manifold, X a submanifold of M, $i : X \hookrightarrow M$ the inclusion map, ω_0, ω_1 symplectic forms on M. Suppose that the two symplectic forms coincide on X. Then there exist neighborhoods U_0 and U_1 of X in M and a diffeomorphism $\phi : U_0 \to U_1$ which is identity on X and $\phi^*\omega_1 = \omega_0$.*

sketch of proof.

1. Pick a tubular neighborhood U_0 of X. The 2-form $\omega_1 - \omega_0$ is closed on U_0, and $(\omega_1 - \omega_0)_p = 0$ at all $p \in X$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on U_0 such that $(\omega_1 - \omega_0) = d\mu$ and $\mu_p = 0$ at all $p \in X$.
2. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\mu$ of closed 2-forms on U_0. Shrinking U_0 if necessary, we can assume that U_0 is compact and ω_t is symplectic for $0 \le t \le 1$.
3. Solve the Moser equation: $\iota_{V_t}\omega_t + \mu = 0$ and then integrate V_t. This gives an isotopy $\rho : U_0 \times [0,1] \to U_0$ with $\rho_t^*\omega_t = \omega_0$, for all $t \in [0,1]$. Just let $U_1 = \rho_1(U_0)$ and $\phi = \rho_1$.

□

The first step in the proof follows from the follwoing result:

Proposition 1.1. Let X be a submanifold of M and let U be a tubular neighborhood of X. If a closed l-form ω on U, is 0 on X, then ω is exact, i.e. $\omega = d\mu$ for some $\mu \in \Omega^{l-1}(U)$. Moreover, we can choose μ to be 0 on X.

Proof. View U as a convex neighborhood of X in NX, the normal bundle of X. Define $\rho_t : U \to U$ for $t \in [0,1]$ to be

$$\rho_t(q, v) = (q, (1-t)v)$$

Then ρ_0 is the identity and $\rho_1 = i \circ \pi$ where $\pi : U \to X$ is the projection and $i : X \hookrightarrow U$ is the embedding.

We have that $(i \circ \pi)^* - \mathrm{id} = dQ + Qd$, where d is the de Rham differential and $Q : \Omega^l(U) \to \Omega^{l-1}(U)$ is defined as:

$$Q(\omega) = \int_0^1 \rho_t^*(\iota_{V_t}\omega)dt$$

where V_t is the vector field associated with ρ_t.

If ω is closed and vanishes on X, then we have $\omega = -dQ(\omega)$. Let $\mu = -Q(\omega)$ then we are done. □

Now suppose that a compact Lie group G acts on M and with submanifold X being G-invariant and that the two symplectic forms are also G-invariant. Then ϕ may be chosen to be G-equivariant:
Consider the proof of Theorem 1. ρ_t being G-equivariant is equivalent to V_t being G-invariant. To have V_t being G-invariant, it suffices to let both ω_t and μ to be G-invariant. Since $\omega_t = \omega_0 + td\mu$, we just need to choose μ to be G-invariant. Indeed, by choosing a G-invariant metric (for example averaging over G any riemannian metric) we can get a G-invariant tubular neighbourhood of X. And by equivariant tubular neighborhood theorem we are allowed to consider this as a convex subspace of the normal bundle. Since the homotopy operator Q is G-equivariant, we are done.

In this way we have proved the equivariant Darboux theorem:

Theorem 2 (Equivariant Darboux theorem)**.** *Let M be a manifold and G be a compact Lie group acting on M. Let X be a G-invariant submanifold of M, $i : X \hookrightarrow M$ the inclusion map. ω_0, ω_1 are two G-invariant symplectic forms on M. Suppose that the two symplectic forms coincide on X. Then there exist neighborhoods U_0 and U_1 of X in M and a G-equivariant diffeomorphism $\phi : U_0 \to U_1$ which is identity on X and $\phi^*\omega_1 = \omega_0$.*

2 Toric Darboux theorem

In the following we will focus on the case of a hamiltonian torus action. Let $(M, \omega, \mathbb{T}^k, \mu)$ be a hamiltonian torus space and $p \in M$ be a fixed point under the action of $\mathbb{T}^k$. Then there is an induced representation of $\mathbb{T}^k$ on the tangent space T_pM. This is called the *isotropy representation* at the point p.

Definition 2.1 (Isotropy representation)**.** Let (G, σ) be a Lie group action on M. Denote by G_p the stablilizer of a point $p \in M$. For each $g \in G_p$, $\sigma_g : M \to M$ fixes p. Therefore differentiating σ_g gives

$$D\sigma_g(p) : T_pM \to T_pM$$

Note that $D\sigma_{gh}(p) = D\sigma_g(p)D\sigma_h(p)$. Hence we get a representation $r : G_p \to GL(T_pM)$.

Now we want to make T_pM into a complex vector space so we can view the isotropy representation as a complex representation. To do this we need some preparation work.

2.1 Compatible almost complex structures

As usual, we start with linear spaces.

Definition 2.2 (Complex vector space)**.** Let V be a vector space. A complex structure on V is a linear map

$$J : V \to V$$

with

$$J^2 = -\text{Id}.$$

The pair (V, J) is called a complex vector space.

Definition 2.3 (Compatible complex structure)**.** Let (V, Ω) be a symplectic vector space. A complex structure J on V is said to be compatible if

$$G_j(u, v) := \Omega(u, Jv)$$

is a positive inner product on V, i.e.

$$\begin{cases} \Omega(Ju, Jv) = \Omega(u, v) \\ \Omega(u, Ju) > 0, \forall u \neq 0 \end{cases}$$

Remark. For a given symplectic vector space, compatible complex structure always exists. (proof see [1]12.2)

Definition 2.4 (Almost complex structure)**.** An almost complex structure on a manifold M is a smooth field of complex structures on the tangent spaces:

$$x \mapsto J_x : T_xM \to T_xM \text{ and } J_x^2 = -\text{Id}.$$

The pair (M, J) is then called an almost complex manifold.

Definition 2.5 (Compatible almost complex structure)**.** Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called compatible if

$$x \mapsto g_x : T_xM \times T_xM \to \mathbb{R}$$

is a riemannian metric on M, where $g_x(u, v) = \omega_x(u, J_xv)$.

Remark. Let M be a manifold with symplectic form ω, riemannian metric g and almost complex structure J. Then the triple (ω, g, J) is called a compatible triple if $g(., .) = \omega(., J.)$.

Proposition 2.1. Let M be a manifold with symplectic form ω and riemannian metric g. Then there exists a canonical almost complex structure J on M which is compatible.

Remark. Note that the construction of the compatible almost complex structure J depends on the riemannian metric g, but the riemannian metric defined by $\omega(., J.)$ is not necessarily equal to g, i.e. (ω, g, J) may not be a compatible triple ([1]12.3.)

Now, back to our story, starting with a $\mathbb{T}^k$-invariant riemannian metric $\hat{g}$ (for example, averaging over $\mathbb{T}^k$ any riemmanian metric), we get a $\mathbb{T}^k$-equivariant compatible almost complex structure J, which makes (T_pM, J_p) into a complex vector space by defining $iv = J_pv$ for any $v \in T_pM$. As being said, we now view the isotropy representation as a complex representation.

2.2 Some representation theory

Now we make a digression into representation theory of compact Lie groups. We will quickly go through some important ideas and results. Some proofs are omitted.

2.2.1 Character

Let G be a compact Lie group and $\sigma : G \to GL(V)$ be a complex representation of G.

Definition 2.6 (Character)**.** If V is a representation of G, then its character χ_V is a complex valued function

$$\chi_V : G \to \mathbb{C},\ g \mapsto \mathrm{Tr}(\sigma_g).$$

Remark. Note that if g and h are in the same conjugacy class, then $\chi_V(g) = \chi_V(h)$. This kind of functions are called *class functions.*

Proposition 2.2. Let V and W be representations of G. Then

- $\chi_{V\oplus W} = \chi_V + \chi_W$
- $\chi_{V\otimes W} = \chi_V \cdot \chi_W$
- $\chi_{V^*} = \overline{\chi}_V$

The importance of characters is that any representation is determined by its character up to isomorphism, which we will see in the following.

Denote by V^G the subspace of all fixed points in V. Since G is compact, there exists a left and right invariant (normalized) Haar measure on it. Hence we can define the map

$$\rho : V \to V,\ v \mapsto \int_G \sigma_g v dg$$

which is a projection onto V^G.

We can then calculate

$$\dim(V^G) = \mathrm{Tr}(\rho) = \int_G \chi_V(g)dg \qquad (*)$$

Let W be another representation of G. Let G act on $\mathrm{Hom}(V, W)$ by $(g \cdot f)v = \sigma_g f(\sigma_g^{-1}v)$. With this action $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$, where $\mathrm{Hom}_G(V, W)$ is the space of G-module homomorphisms from V to W. Therefore we have

$$\rho : \mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W),\ f \mapsto \int_G g \cdot f dg.$$

If V and W are irreducible, then by Schur's lemma

$$\begin{cases} \mathrm{Hom}_G(V, W) \cong \mathbb{C} & \text{if } V \cong W \\ \mathrm{Hom}_G(V, W) = 0 & \text{if } V \ncong W \end{cases}$$

Since $\mathrm{Hom}(V, W) \cong V^* \otimes W$, we have $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi}_V \cdot \chi_W$. Replace V by $\mathrm{Hom}(V, W)$ in (*) we get

$$\begin{cases} \int_G \overline{\chi}_V(g) \cdot \chi_W(g)dg = 1 & \text{if } V \cong W \\ \int_G \overline{\chi}_V(g) \cdot \chi_W(g)dg = 0 & \text{if } V \ncong W \end{cases}$$

Consider the space of all square integrable class functions on G, denoted by $\mathbb{C}^2_{\text{class}}(G)$. Let's define an Hermitian inner product on it:

$$(\alpha, \beta) = \int_G \overline{\alpha}(g) \cdot \beta(g)dg.$$

Hence we have proved the following result:

Theorem 3. *In terms of this inner product, the characters of the irreducible representations of G are orthonormal.*

Corollary 2.1. *Any representation of G is determined by its character.*

Proof. If $V = V_1^{\oplus a_1} \oplus ... \oplus V_r^{\oplus a_r}$ where V_i's are distinct irreducible representations (representations of compact Lie groups are semisimple), then $\chi_V = a_1\chi_{V_1} + ... + a_r\chi_{V_r}$. Since χ_{V_i}'s are linear independent, we are done. □

Corollary 2.2. *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

Let G and H be two compact Lie groups. Assume that V and W are irreducible representations of G and H respectively. Then using Corollary 2.2 we can check that $V \otimes W$ is an irreducible representation of $G \times H$ with the action $(g, h)(v \otimes w) = (g \cdot v) \otimes (h \cdot w)$. Moreover, we can show that any irreducible representation of $G \times H$ is a tensor product of this form.

2.3 Representations of tori

A torus $\mathbb{T}^k$ is equal to $S^1 \times ... \times S^1$. From the last subsection we know that, if we want to study the irreducible representations of $\mathbb{T}^k$, it suffices to study irreducible representations of S^1. Our aim is to find out all the irreducible representations of S^1. Let's view S^1 as $\{\theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ with addition as group operation.

First note that any irreducible representation of S^1 is one-dimensional. Actually, any irreducible representation V of an abelian Lie group G is one-dimensional. Indeed, since G is abelian, $\sigma_g : V \to V$ is a G-module homomorphism for all $g \in G$. By Schur's lemma this is multiplication by some constant c, which implies that any subspace of V is G-invariant. This forces V to be one-dimensional. In this case, the character is just the representation itself.

First note that $\{(\mathbb{C}, \rho_k)\}$ are irreducible representations of S^1, where $\rho_k(\theta)$ acts vis multiplication by $e^{ik\theta}$. We claim that all irreducible representations of S^1 are of this kind. Indeed, $\mathbb{C}^2_{\text{class}}(S^1)$ is actually $L^2(-\pi, \pi; \mathbb{C})$. Since $L^2(-\pi, \pi; \mathbb{C})$ is separable, it admits a Hilbertian basis given by $\{e^{ik\theta}; k \in \mathbb{Z}\}$, hence there can not exist other characters orthogonal to $\{e^{ik\theta}; k \in \mathbb{Z}\}$, i.e. no other irreducible representations. Moreover, note that these irreducible representations are also unitary.

Now it's easy to conclude all irreducible representations of $\mathbb{T}^k$. Denote $(\theta_1, ..., \theta_k) \in \mathbb{T}^k$ by $[\theta]$. Any irreducible representation must be of the form $(\mathbb{C}, \hat{\rho}_k)$, where

$$\hat{\rho}_k([\theta]) = e^{i\sum_{j=1}^{k} a_j \theta_j}$$

for some integers $a_1, ...a_k$. This is again unitary.

Therefore, for arbitrary representation V of $\mathbb{T}^k$ we have a *weight space* decomposition

$$V = \bigoplus_j W_j^{\oplus \alpha_j}$$

where W_j are distinct irreducible representations of $\mathbb{T}^k$ on which $[\theta]$ acts via multiplication by $e^{i\cdot\langle\lambda^j,\theta\rangle}$ for weights $\lambda^j = (a_1^j, ..., a_k^j)$. $W_j^{\oplus\alpha_j}$ is called a weight space.

2.4 Toric Darboux theorem and its application

Theorem 4 (Toric Darboux)**.** *Let $(M, \omega, \mathbb{T}^k, \mu)$ be a $2n$-dimensional hamiltonian torus space and p be a fixed point. Let $\lambda^1, ..., \lambda^n$ be the weights (may not be distinct) of the isotropy representation of $\mathbb{T}^k$ on T_pM. Then there is a $\mathbb{T}^k$-invariant neighborhood U of p in M and coordinate functions $(x_1, ..., x_n, y_1, ..., y_n)$ centered at p with respect to which we have*

1. *$\omega_{|U} = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j$, where $z_j = x_j + iy_j$ and $\overline{z}_j = x_j - iy_j$;*
2. *The action of $\mathbb{T}^k$ on U with this chart becomes*

$$[\theta] \cdot (z_1, ..., z_n) = (e^{i\cdot\langle\lambda^1,\theta\rangle} z_1, ..., e^{i\cdot\langle\lambda^n,\theta\rangle} z_n)$$

$$[\theta] \cdot (\overline{z}_1, ..., \overline{z}_n) = (e^{-i\cdot\langle\lambda^1,\theta\rangle} \overline{z}_1, ..., e^{-i\cdot\langle\lambda^n,\theta\rangle} \overline{z}_n)$$

3. *The moment map on U with this chart becomes*

$$\mu(z) = \mu(p) + \frac{1}{2}\sum_{j=1}^n \lambda^j |z_j|^2$$

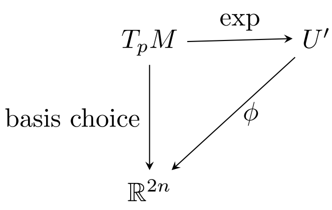
Proof. The idea is the same as proving the Darboux theorem by applying the Darboux-Weinstein theorem.

Let J be the $\mathbb{T}^k$-equivariant compatible almost complex structure. By the weight decomposition we can give a basis $\{v_1, ..., v_n, Jv_1, ..., Jv_n\}$ to T_pM, where v_j and Jv_j, viewed as complex vectors, belong to W_j. Since ω is $\mathbb{T}^k$ invariant, we have that $\omega_p(v_i, v_j) = 0$ for v_i, v_j from different weight spaces. By the Gram-Schmidt method, even if v_i and v_j are in the same weight space, we can choose them to be orthogonal. By normalizing this basis we get a symplectic basis for ω, denoted by $(v'_1, ..., v'_n, u'_1, ..., u'_n)$.

Consider the metric given by $\omega(\cdot, J\cdot)$. Using the exponential map with respect to this metric, we construct coordinates $(x'_1, ..., x'_n, y'_1, ..., y'_n)$ centered at p and valid on some neighborhood U', with $\frac{\partial}{\partial x'_j}|_p = v'_j$ and $\frac{\partial}{\partial y'_j}|_p = u'_j$. Then we have

$$\omega_p = \sum_{j=1}^n dx'_j \wedge dy'_j|_p.$$

It remains to check that on $(U', (x'_1, ..., x'_n, y'_1, ..., y'_n))$ the symplectic form $\sum_{j=1}^n dx'_j \wedge dy'_j$ is $\mathbb{T}^k$-invariant. Indeed, consider the following triangle:



Clearly $\sum_{j=1}^n dx'_j \wedge dy'_j$ is $\mathbb{T}^k$-invariant on $\mathbb{R}^{2n}$. Note that both the exponential map and basis choice are $\mathbb{T}^k$-equivariant, we can choose ϕ to be $\mathbb{T}^k$-equivariant as well (just compose the inverse of exponential map and basis choice).

By applying equivariant Darboux theorem and by the same argument as in the proof of Darboux theorem we prove the first statement.

The last two statements are trivial given the representation of $\mathbb{T}^k$ on T_pM.

□

Remark. Note that the first statement can be generalized to general G-actions. In our case we form a symplectic basis explicitly by weight space decomposition, but actually a symplectic basis always exists. Another point that we have to pay attention to is that unlike in the Darboux theorem, we can only find a normal form near a fixed point. This is because we need to use the fact that the exponential map is G-equivariant.

In the last talk we have seen the Delzant's correspondence, which says there is a one to one correspondence between symplectic toric manifolds (up to equivalence) and unimodular polytopes (up to translation). The idea is to consider the moment polytope of a symplectic toric manifold. By applying toric Darboux theorem we can prove that the moment polytope of a symplectic toric manifold is indeed unimodular.

Proposition 2.3. Let $(M, \omega, \mathbb{T}^n, \mu)$ be a symplectic toric manifold. Then the image Δ of μ is a unimodular polytope.

Proof. By the Atiyah-Guillemin-Sternberg convexity theorem the image Δ is the convex hull of the images of the fixed points of the action (discussed in Alessandro's talk).

Let τ be a vertex of Δ. Then there is $p \in M$ fixed by $\mathbb{T}^n$ with $\mu(p) = \tau$.

By toric Darboux we have a Darboux chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that:

$$\mu(z_1, ..., z_n) = \tau + \frac{1}{2}\sum_{j=1}^n \lambda^j |z_j|^2.$$

Since the $\mathbb{T}^n$ action is effective, $\{\lambda^j\}$ forms a basis of $\mathbb{Z}^n$. The points in image of U by μ then have the form

$$\tau + \sum_{j=1}^n t_k \lambda^k$$

with $t_k \geq 0$. Hence the simplicity, rationality and smoothness are satisfied locally at vertex τ.

Moreover, by the Atiyah-Guillemin-Sternberg theorem $\mu^{-1}(\tau)$ is connected. The arguments above shows that $\mu^{-1}(\tau) = \{p\}$. Indeed, if not, then there exists $(t_1, ..., t_n) \neq 0$ such that $\sum_{j=1}^n t_k \lambda^k = 0$, which contradicts the fact that $\{\lambda^k\}$'s are linear independent. We apply the arguments above to all the vertices in the moment polytope then the simplicity, rationality and smoothness are satisfied globally.

□

References

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