Pfaff's Theorem and Gray's Stability Theorem

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Abstract

In this talk, we will discuss about the Delzant's Construction which, for any Delzant's Polytope Δ, gives us a Symplectic Toric Manifold with moment polytope Δ. This serves as proving of the existence part in Delzant's Theorem. The content is based on [1].

Theorem 1. *Symplectic toric manifolds are classified up to equivalence by Delzant's polytopes up to translation. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\begin{array}{ccc} \{\textit{symplectic toric manifolds}\} & \xrightarrow{1-1} & \{\textit{unimodular polytopes}\} \\ \textit{(mod equivalence)} & & \textit{(mod translation)} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longmapsto & \mu(M)\ . \end{array}$$

We shall now construct a Symplectic Toric Manifold with moment polytope any Delzant's polytope.

Let Δ be a Delzant's polytope with d facets. Let $v_k \in \mathbb{Z}^n, k = 1, ..., d$ (where $d > n$), be the primitive inward-pointing normal vectors to the facets. For some $c_k \in \mathbb{R}$, we can write

$$\Delta = \{x \in (\mathbb{R}^n)^* | \langle x, v_k \rangle \geq c_k, k = 1, ..., d\}.$$

Let $e_1 = (1, 0, ..., 0), ..., e_d = (0, ..., 0, 1)$ be the standard basis of $\mathbb{R}^d$. Consider

$$\begin{aligned} \Pi : \mathbb{R}^d &\longrightarrow \mathbb{R}^n \\ e_k &\longmapsto v_k. \end{aligned}$$

Then the map Π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$ since, for each vertex, the v_k's corresponding to the facets meeting at that vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Therefore, Π induces a surjective map, still called Π, between tori:

$$\mathbb{Z}^d/(2\pi\mathbb{Z}^d) \xrightarrow{\Pi} \mathbb{R}^n/(2\pi\mathbb{Z}^n)$$
$$\mathbb{T}^d \xrightarrow{\Pi} \mathbb{T}^n.$$

Let N be the kernel of Π, a $(d-,)$-dimensional Lie subgroup of $\mathbb{T}^d$, with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of Lie groups

$$\mathbb{1} \to N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\Pi} \mathbb{T}^n \to \mathbb{1}$$

induces an exact sequence of Lie algebras

$$0 \to \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\Pi} \mathbb{R}^n \to 0$$

with dual exact sequence

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\Pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \to 0.$$

Now consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k$, and standard hamiltonian action of $\mathbb{T}^d$ given by

$$(e^{i\theta_1}, ..., e^{i\theta_d}) \cdot (z_1, ..., z_d) = (e^{i\theta_1}z_1, ..., e^{i\theta_d}z_d).$$

The moment map is $\phi : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, ..., z_d) = \tfrac{1}{2}(|z_1|^2, ..., |z_d|^2) + Constant,$$

where we will choose the constant to be $(c_1, ..., c_d)$, where the c_k are the constants above. Moreover, the subgroup N acts on $\mathbb{C}^d$ in a hamiltonian way with moment map

$$i^* \circ \phi : \mathbb{C}^d \longrightarrow \mathfrak{n}^*.$$

Let $Z = (i^* \circ \phi)^{-1}(0)$ be the zero-level set. Note that Z is connected, because $(i^*)^{-1}(0)$ is a linear subspace of $\mathbb{R}^d$ and the fibers $\phi^{-1}(x)$ are path-connected.

The submanifold Z is compact and N acts freely on Z.

We postpone the proof of this claim until further down.

Now Z is the $zero-level$ of a moment map for the action of the torus N on $\mathbb{C}^d$. Knowing that N acts freely on Z ensures that this is a regular level. Hence, Z is a submanifold of $\mathbb{C}^d$ of dimension $2d - (d-n) = d + n$. We now use the following theorem from Lie theory:

Lemma 2. *If a compact Lie group N acts freely on a manifold Z, then the orbit space Z/N is a manifold and the point-orbit $p : Z \to Z/N$ is a principal N-bundle.*

In our case, Z is a compact $(d+n)$-dimensional manifold, so the orbit space $M_\Delta := Z/N$ is a compact manifold of dimension

$$\dim\, M_\Delta = \dim\, Z - \dim\, N = (d+n) - (d-n) = 2n.$$

The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the inclusion $j : Z \hookrightarrow \mathbb{C}^d$. The Marsden-Weinstein-Meyer theorem guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^*\omega_\Delta = j^*\omega_0.$$

Since Z is connected, the compact symplectic $2n$-dimensional manifold $(M_\Delta, \omega_\Delta)$ is also connected. Let introduce a last small lemma before handling the proof of Lemma ???. We here consider Δ' to be the image of Δ by Π^*.

Lemma 3. *Let $y \in (R^d)^*$. Then*

$$y \in \Delta' \Longleftrightarrow y \in \phi(Z).$$

Proof. The value y is in the image of Z by ϕ if and only if both of the following conditions hold:

1. y is in the image of ϕ;
2. $i^*y = 0$.

Using the expression for *phi* and the dual exact sequence, we see that these conditions are equivalent to:

1. $\langle y, e_k \rangle \le c_k$ for $k = 1, ..., d$;
2. $y = \Pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that the second condition holds, so that $y = \Pi^*(x)$. Then

$$\begin{aligned} \langle y, e_k \rangle \le c_k, \forall k &\Longleftrightarrow \langle \Pi^*(x), e_k \rangle \le c_k, \forall k \\ &\Longleftrightarrow \langle x, \Pi(e_k) \rangle \le c_k, \forall k \\ &\Longleftrightarrow \langle x, v_k \rangle \le c_k, \forall k \\ &\Longleftrightarrow x \in \Delta. \end{aligned}$$

Thus,

$$y \in \phi(Z) \Longleftrightarrow y \in \Pi^*(\Delta) = \Delta'.$$

This concludes the proof of this intermediate lemma. □

We now come back to the proof of Lemma 2.

Proof. The set Z is clearly closed, hence in order to show that it is compact it suffices, thanks to the Heine-Borel theorem, to show that Z is bounded. We will show that $\phi(Z) = \Delta'$.
Since we have that Δ' is compact, that ϕ is a proper map and that $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact. It remains to show that N acts freely on Z.
Pick a vertex τ of Δ, and let $I = \{k_1, ..., k_n\}$ be the set of indices for the n facets meeting at τ. Pick $z \in Z$ such that $\phi(z) = \Pi^*(\tau)$. Then τ is characterised by n equations $\langle \tau, v_k \rangle = c_k$ where k ranges in I:

$$\begin{aligned}
\tau, v_k\rangle = c_k &\Longleftrightarrow \langle \tau, \Pi(e_k)\rangle = c_k \\
&\Longleftrightarrow \langle \Pi^*(\tau), e_k\rangle = c_k \\
&\Longleftrightarrow \langle \phi(z), e_k\rangle = c_k \\
&\Longleftrightarrow i\text{-th coordinate of } \phi(z) \text{ is equal to } c_k \\
&\Longleftrightarrow \frac{1}{2}|z_k|^2 + c_k = c_k \\
&\Longleftrightarrow z_k = 0.
\end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, ..., n\}$. The stabilizer of z is

$$(\mathbb{T}^d)_z = \{(e^{i\theta_1}, ..., e^{i\theta_n}, 1, ..., 1) \in \mathbb{T}^d\}.$$

As the restriction $\Pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, ..., e_n$ to a $\mathbb{Z}$-basis $v_1, ..., v_n$ of $\mathbb{Z}^n$ respectively, at the level of groups the map $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = ker(\Pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{\mathbb{1}\}$, i.e., $N_z = \{\mathbb{1}\}$. Hence, all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'} (z' \in Z)$ are contained in stabilizers for points z which map to vertices. This concludes the proof of Lemma 2. □

Given a Delzant's polytope Δ, we have constructed a symplectic manifold $(M_\Delta, \omega_\Delta)$ where $M_\Delta := Z/N$ is a compact $2n$-dimensional manifold manifold and ω_Δ is the reduced symplectic form.

Lemma 4. *The manifold $(M_\Delta, \omega_\Delta)$ inherits a hamiltonian $\mathbb{T}^n$-action with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$.*

Proof. Let z be such that $\phi(z) = \Pi^*(\tau)$ where τ is a vertex of Δ, as in the proof above. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\Pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. Since we have found a *section* in the exact sequence

$$\mathbb{1} \longrightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\Pi} \mathbb{T}^n \longrightarrow \mathbb{1},$$
$$\xleftarrow{\sigma}$$

the exact sequence *splits* and hence provides us with an isomorphism

$$(i, \sigma) : N \times \mathbb{T}^n \xrightarrow{\simeq} \mathbb{T}^d.$$

The action of the $\mathbb{T}^n$ factor descends to the quotient $M_\Delta = Z/N$.

It remains to show that the $\mathbb{T}^n$-action on M_Δ is hamiltonian with appropriate moment map.

Consider the maps $p : Z \longrightarrow M_\Delta$, $j : Z \hookrightarrow \mathbb{C}^d$ along with the sequence

$$Z \hookrightarrow \mathbb{C}^d \xrightarrow{\phi} (\mathbb{R}^d)^* \simeq \mathfrak{n}^* \bigoplus (\mathbb{R}^n)^* \xrightarrow{\sigma^*} (\mathbb{R}^n)^*$$

where the last horizontal map is simply projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies

$$\mu_\Delta \circ p = \sigma^* \circ \phi \circ j.$$

Thanks to reduction for product groups, this is a moment map for the action $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. Finally, the image of μ_Δ is:

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (^* \circ \Pi^*)(\Delta) = \Delta,$$

because $\phi(Z) = \Pi^*(\Delta)$ and $\sigma^* \circ \Pi^* = (\Pi \circ \sigma)^* = id$. □

The above $\mathbb{T}^n$-action is effective because $\mathbb{T}^d$, and hence $\mathbb{T}^n$, act freely on the open dense subset

$$\phi^{-1}(\Pi^*(\Delta^0)) \subset Z,$$

where Δ^0 denotes the interior of Δ.
We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required symplectic toric manifold corresponding to Δ.

References

[1] Ana Cannas. *Seminar on Symplectic Toric Manifolds*. 2021. URL: https://people.math.ethz.ch/~acannas/Papers/stm_seminar.pdf.