Seminar - Symplectic Geometry

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1 Preliminaries

Definition 1 (Action of a Lie Group on a Manifold)**.** Given a lie group G and a manifold M, an action of G on M is a group homomorphism

$$\begin{aligned}\psi : G &\to \text{Diff}(M)\\ g &\to \psi_g\end{aligned}$$

with Diff(M) the group of diffeomorphisms of M.
ψ has an associated evaluation map, which is the map

$$\begin{aligned}ev_\psi : M \times G &\to M\\ (p, g) &\to \psi_g(p)\end{aligned}$$

The action ψ is called smooth if the evaluation map is smooth.

Definition 2 (Symplectic Actions)**.** An action ψ of a Lie group G onto (M, ω) is said to be symplectic if the image of ψ lies in the group of symplectomorphisms of (M, ω):

$$\psi : G \to \text{Sympl}\,(M, \omega) \subset \text{Diff}(M)$$

Definition 3 (Moment Map and Hamiltonian Actions)**.** Suppose (M, ω) is a symplectic manifold, G a Lie group with associated action $\psi : G \to \text{Diff}(M)$, and $\mathfrak{g}$ the Lie algebra of G. The action ψ is called a hamiltonian action if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying the following conditions:

1. For each $X \in \mathfrak{g}$ let $\mu^X : M \to \mathbb{R}$, $\mu^X(p) = (\mu(p)/X)$ be the component of μ along X, and let $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{exp(tX), t \in \mathbb{R}\}$. Then

   $$d\mu^X = -\iota_{X^\#}\omega$$

   that is, μ^X is a hamiltonian function for the vector field $X^\#$.

2. μ is equivariant w.r.t ψ and Ad:

   $$\forall g \in G,\ \mu \circ \psi_g = \text{Ad}^*_g \circ \mu$$

   μ is called the moment map associated with the action.
   We recall from the talk 2 weeks ago that the Ad* map is trivial when the group is abelian; this will be the case we will find ourselves in.

Definition 4 (Torus Actions)**.** Consider the standard torus of rank $n \geq 1$, defined to be the product of n copies of the circle in $\mathbb{R}^2$:

$$\mathbb{T}^n = S^1 \times \ldots \times S^1$$

Elements of $\mathbb{T}^n$ are n-tuples of complex numbers lying on the unit circle:

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)$$

Note that since the multiplication on the torus is abelian, the Adjoint operator is trivial. In this context, additionally identifying the Lie Algebra (and thus its dual) to $\mathbb{R}^n$, the second hypothesis from the definition of the moment map simplifies to the following:

- Letting $\mu_1 \dots \mu_n$ be the coordinate functions of μ:

$$\mu_k\left(\left(e^{i\theta_1},\dots,e^{i\theta_n}\right)\cdot p\right) = \mu_k(p)\ \ \forall\left(e^{i\theta_1},\dots,e^{i\theta_n}\right)\in\mathbb{T}^n, p = 1,\dots,n$$

Example. *1. On $(\mathbb{C}, \omega_0 = \frac{i}{2}dz \wedge d\bar{z})$, we can consider the action of the standard circle S^1 by rotation:*

$$\psi_t(z) = t^l z$$

where $t \in S^1$ and $l \in \mathbb{Z}$ is fixed. The action $\psi : S^1 \to \mathrm{Diff}(C)$ is hamiltonian with moment map

$$\mu(z) = \frac{1}{2} l z\bar{z}$$

It is clear that μ is invariant by rotation. The other condition is less obvious but works out as well.

2. $\mathbb{T}^n = \{(t_1,\dots,t_n)\in\mathbb{C}^n,\, t_i\bar{(t_i)} = 1\,\forall i\}$ acting on $\mathbb{C}^n$ by

$$(t_1,\dots,t_n)\cdot(z_1,\dots,z_n) = (t_1z_1,\dots,t_nz_n)$$

is also a Hamiltonian action with a possible moment map μ given by:

$$\mu(z_1,\dots,z_n) = \frac{1}{2}(z_1\bar{z_1},\dots,z_n\bar{z_n})$$

2 Symplectic Toric Manifolds

Definition 5 (Effective Actions)**.** An action of a group G on a manifold M is said to be effective (sometimes the terminology faithful is also encountered) if $\psi : G \to \mathrm{Diff}(M)$ is injective; in other words, no non-identity element in G is mapped to the identity on M.

Definition 6. A symplectic toric manifold is a compact, connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a standard torus $\mathbb{T}^n$ where the following relation holds:

$$\dim(\mathbb{T}) = \frac{1}{2}\dim(M)$$

with a corresponding moment map

$$\mu : M \to (\mathbb{R}^n)^*$$

Remark 1. The compactness assumption is not strictly necessary to define symplectic toric manifolds, and there exist extensions of the notion that remove compactness from the list of required properties of the manifold.

We are interested specifically in *compact* manifolds because, as we will hear about in the next talk, there exists a theorem classifying symplectic toric manifolds by the image of the associated moment maps (Delzant). Compactness plays a crucial role in the proof of this theorem.

This is beyond the scope of today's talk, and indeed beyond the scope of the seminar itself, but I included a bunch of papers that deal with this in the references.

Remark 2. Another aspect of the definition that we can question is the condition that is imposed on the dimension of the torus $\mathbb{T}^n$. It is actually not restrictive, because of the following theorem.

Theorem 1. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a compact, connected symplectic manifold, along with an effective Hamiltonian action of the torus $\mathbb{T}^m$ onto (M, ω) with moment map μ.*
Then we have $\dim M \geq 2m$.

Proof. The moment map is constant on an orbit $\mathcal{O}$ of the action by hypothesis. So for $p \in \mathcal{O}$ the exterior derivative

$$d\mu_p : T_pM \to \mathfrak{g}^*$$

maps $T_p\mathcal{O}$ to 0.
Thus, $T_p\mathcal{O}$ in contained within the kernel of $d\mu_p$, the symplectic orthocomplement of $T_p\mathcal{O}$. This shows that the orbits are isotropic submanifolds of M; that is, the restriction of ω to such an orbit is trivial. Symplectic linear algebra yields the desired result on the dimensions. □

Definition 7 (Isomorphism of Symplectic Toric Manifolds)**.** Two symplectic toric manifolds $(M_1, \omega_1, \mathbb{T}_1, \mu_1)$, $(M_2, \omega_2, \mathbb{T}_2, \mu_2)$ are isomoprhic if there exists an equivariant symplectomorphism $\varphi : M_1 \to M_2$.

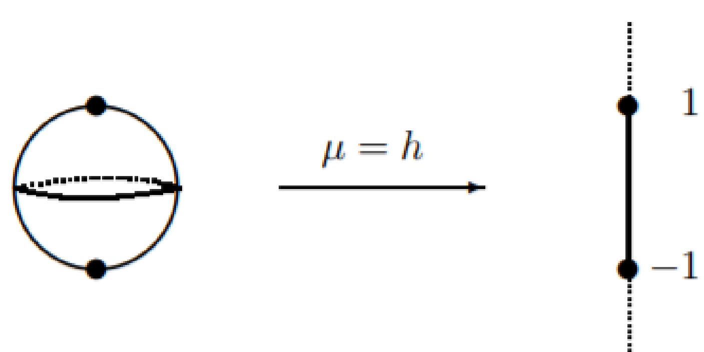
3 Examples of Symplectic Toric Manifolds

Example (Example 1: A non-example)**.** $(\mathbb{R}^{2n}, \omega_0)$ *does not have the structure of a symplectic toric manifold. Indeed, the compactness hypothesis is not respected.*

Example (Example 2: An actual example, the sphere)**.** *The circle S^1 acts on the sphere $(S^2, \omega_{std} = d\theta \wedge dh)$ by rotation:*

$$e^{it} \cdot (\theta, h) = (\theta + t, h)$$

with associated moment map $\mu = h$ the height function.



Example (Example 3: Complex projective spaces)**.** *Consider the complex projective space $\mathbb{CP}^n$, a $2n$ dimensional real manifold. It can be endowed with a diagonal action from $\mathbb{T}^n$:*

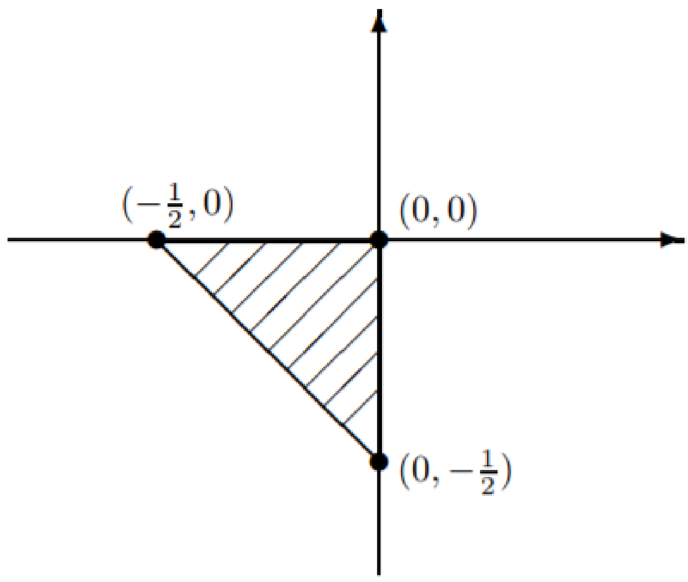
$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : e^{i\theta_1} z_1 : \ldots : e^{i\theta_n} z_n]$$

The Fubini-Study symplectic form on $\mathbb{CP}^n$ is given by

$$\omega_{FS} = d\theta_1 \wedge dH_1 + \ldots + d\theta_n \wedge dH_n$$

with H_k is the k-th component of the moment map $H : \mathbb{CP}^n \to \mathbb{R}^n$, and is given by:

$$H_k = \frac{|z_k|^2}{2(|z_1|^2 + \ldots + |z_n|^2)}$$



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