Delzant's Classification Theorem and Unimodular polytopes

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Abstract

In this talk we will introduce unimodular polytopes and give some examples and nonexamples to illustrate the concept. We will then move on to characterizing these polytopes on the plane and also give a dual characterization. Finally we will restate the unimodularity condition in the abstract setting and state delzant's classification theorem.

1 Setting and Reduction

We will begin by briefly giving an overview of the relevant facts on hamiltonian torus actions and symplectic toric manifolds.

1.1 Hamiltonian Torus Action

Recall that

Definition 1.1 (Hamiltonian Action)**.** Let (M, ω) be a symplectic manifold and G a lie group with lie algebra $\mathfrak{g}$. The action $\Psi : G \to \mathrm{Diff}(M)$ is called *hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{g}^*$ such that

- For any vector $X \in \mathfrak{g}$ with infinitesimal action $X^\sharp$ and $\mu^X := \mu(X) \in C(M)$ we have
$$i_{X^\sharp}\omega = -d\mu^X$$
- For any $g \in G$ we have
$$\mu \circ \Psi_g = \mathrm{Ad}_g^* \circ \mu$$

Definition 1.2 (Toric Lie Group)**.** A *toric lie group* is a lie group $\mathcal{T}^n$ which is abelian, compact and connected. We will denote its lie algebra by $\mathfrak{t} := T_e\mathcal{T}^n$. For $\exp_{\mathcal{T}^n} : \mathfrak{t} \to \mathcal{T}^n$ we define the *integral lattice* as the $\mathbb{Z}$-module $\mathfrak{t}_{\mathbb{Z}} := \ker(\exp_{\mathcal{T}^n}) \subset \mathfrak{t}$. The dual notion on $\mathfrak{t}^*$ is called the *weight lattice* and is defined as $\mathfrak{t}^*_{\mathbb{Z}} := \mathrm{Hom}_Z(\mathfrak{t}_Z, 2\pi\mathbb{Z})$.

After choosing a $\mathbb{Z}$-basis of $\mathfrak{t}_{\mathbb{Z}}$ we gain an isomorphism $\mathcal{J}$ such that $\mathcal{T}^n \simeq \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} \xrightarrow{\mathcal{J}} \mathbb{R}^n/(2\pi\mathbb{Z})^n \simeq \mathbb{T}^n$, where $\mathbb{T}^n = (S^1)^n$ is the *standard toric lie group*. The isomorphism $\mathcal{J}$ is called a *splitting* of $\mathcal{T}^n$. This allows us to identify any abstract toric lie group $\mathcal{T}^n$ with the standard toric lie group $\mathbb{T}^n$ and its lie algebra $\mathfrak{t}$ with $\mathbb{R}^n$. A more detailed account of this process can be found in Alessandro's talk [Imp].

Together with the canonical identification $(R^n)^* \simeq \mathbb{R}^n$, this will allow us to transform the abstract case of a hamiltonian torus action $\mathcal{T}^n$ with moment map $\mu : M \to \mathfrak{t}^*$ to the concrete case of the standard torus action $\mathbb{T}^n$ with moment map $\mu : M \to \mathbb{R}^n$. We can thus view the image of the moment map $\mu(M)$ as a subset of $\mathbb{R}^n$.

A remarkable fact about hamiltonian torus actions is that the image of the moment map is a convex polytope:

Theorem 1.3 (Atiyah, Guillemin-Sternberg)**.** *Let (M, ω) be a compact, connected symplectic manifold with a hamiltonian toric action $\mathbb{T}^n$ and moment map $\mu : M \to \mathbb{R}^n$. Then the image $\mu(M) \subset \mathbb{R}^n$ is a convex polytope.*

1.2 Symplectic Toric Manifolds

Definition 1.4 (Effective Action)**.** An action G on a manifold M is called *effective* or *faithful* if the action map $\Psi : G \to \mathrm{Diff}(M)$ is injective.

If we require our hamiltonian torus actions to be effective (which can always be achieved after taking the quotient by $\ker \Psi$) we incur the constraint $\dim \mathbb{T}^n \leq \frac{1}{2} \dim M$. Actions of maximal dimension are of particular interest:

Definition 1.5 (Symplectic Toric Manifold)**.** A symplectic manifold (M, ω) with a hamiltonian torus action $(\mathbb{T}^n, \mu)$ is called a *symplectic toric manifold* if $n \leq \frac{1}{2} \dim M$ and the action is effective.

As discussed, the image of the moment map $\mu(M) \subset \mathbb{R}^n$ is a convex polytope. In the case of symplectic toric manifolds, the image is even an *unimodular polytope.* ((Proven where?))

Definition 1.6 (Isomorphism)**.** Two symplectic toric manifolds $(\mathbb{T}^n, \mu_k, M_k, \omega_k), k \in \{1, 2\}$ are *isomorphic* if there exist a symplectomorphism $\Phi : M_1 \to M_2$ with $\Phi([\theta] \cdot p) = [\theta] \cdot \Phi(p)$ for all $p \in M_1, [\theta] \in \mathbb{T}^n$.

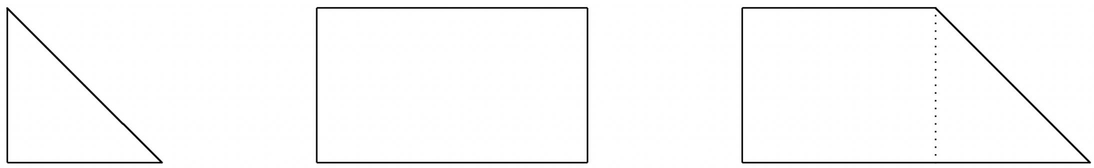


Figure 1: **Examples of unimodular polytopes in** $\mathbb{R}^2$ [Can]

2 Polytopes

In this section we will first introduce the basic terminology of (convex) polytopes. We will then define unimodular polytopes and illustrate the concept with some examples and nonexamples. Thereafter, we will give some basic characterizations of unimodular polytopes on the plane and finally look at a dual characterization of the unimodularity condition.

2.1 Convex Polytopes

Definition 2.1 (Convex Polytope)**.** A *convex polytope* $\Delta \subset \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$.

Convex polytopes can alternatively be characterized as intersections of affine half-spaces:

Theorem 2.2 (Weyl-Minkowsky)**.** *A* convex polyhedron *is a subset of* $\mathbb{R}^n$ *that is the intersection of a finite number of affine half-spaces. A subset* $\Delta \subset \mathbb{R}^n$ *is a convex polytope if and only if it is a compact convex polyhedron.*

The familiar notions of vertices, edges and facets are defined as follows:

Definition 2.3 (Faces)**.** A *face* of a convex polytope Δ is a nonempty intersection of Δ with a closed half-space H such that the intersection of the boundary of the half-space with the interior of Δ is empty. There exists only one face of dimension n, namely the polytope Δ itself.

A *vertex* is a face of dimension 0 and is also always one of the finitely many extremal points of Δ around which the convex hull is formed. An *edge* is a face of dimension 1 and a *facet* is a face of codimension 1 with respect to the dimension of the polytope.

2.2 Unimodularity

Equipped with these concepts we can now define unimodular polytopes.

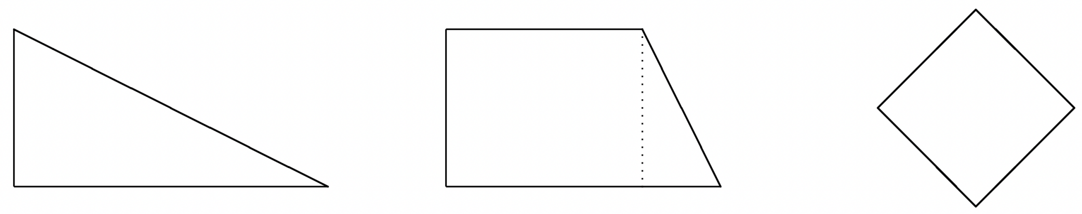


Figure 2: **Nonexamples of unimodular polytopes in $\mathbb{R}^2$**. The polytope on the left violates the smoothness condition at the upper- and rightmost vertex. The polytope in the middle violates the same condition on the two rightmost vertices. Finally, the one on the right violates the condition at every vertex.[Can]

Definition 2.4 (Unimodular Polytope)**.** A convex polytope $\Delta \subset \mathbb{R}^n$ is called *unimodular* if

- (Simplicity) there are n edges meeting at each vertex,
- (Rationality) the edges meeting at the vertex τ are rational in the sense that every edge E_k is of the form $\tau + tu_k$ where $t \in [0, T]$ and $u_k \in \mathbb{Z}^n$,
- (Smoothness) for each vertex with edges $E_1, \ldots, E_n$ the corresponding vectors $u_1, \ldots, u_n$ spanning the edges can be chosen to form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The following lemma will prove very useful for proving that a given set of vectors $u_1, \ldots, u_n$ is indeed a $\mathbb{Z}$-basis:

Lemma 2.5. *The vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if*

$$\det \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} = \pm 1$$

Proof. Since $u_1, \ldots, u_n \subset \mathbb{Z}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ iff the matrix is invertible, and the matrix is invertible iff its determinant is a unit, which in $\mathbb{Z}$ are exactly ± 1, the result follows. □

2.3 Unimodular Polytopes on the Plane

Proposition 2.6 (Triangle Polytopes on the Plane)**.** *Up to translation and linear transformation in* $\mathrm{GL}(2; \mathbb{Z})$ *the unimodular polytopes in* $\mathbb{R}^2$ *with three vertices are isosceles right triangles.*

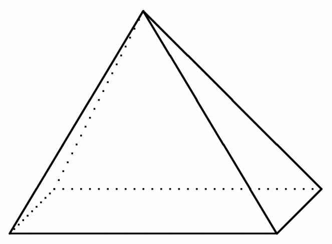


Figure 3: **Nonexample in** $\mathbb{R}^3$. The pyramid violates the simplicity condition at its upper vertex.[Can]

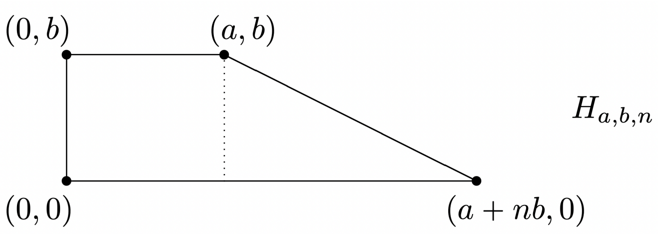


Figure 4: **Hirzebruch Trapezoid** [Can]

Proof. Let $\Delta \subset \mathbb{R}^2$ be a unimodular polytope with three vertices. We begin moving one vertex to the origin $(0,0)$, by the symplicity condition there are two outgoing edges. After a suitable linear transformation these edges lie on the coordinate axes and are spanned by vectors $u_1 = (1,0)$ and $u_2 = (0,1)$. The vertices are then given by $(0,0), (0,a)$ and $(b,0)$. Let $u_3 = (x,y) \in \mathbb{Z}^2$ be the vector spanning the edge from $(b,0)$ to $(0,a)$, we then have that $(-u_1, u_3)$ and $(-u_2, -u_3)$ are the vectors corresponding to the two outer vertices. After imposing the smoothness condition we get that $x, y = \pm 1$ and since Δ is a triangle we get that $u_3 = (-1,1)$. Hence, $a = b$ and Δ is (after linear transformation) an isosceles right triangle. □

We can also characterize the unimodular polytopes with four vertices:

Definition 2.7 (Hirzebruch Trapezoid)**.** The *Hirzebruch Trapezoid* with parameters $a, b \in \mathbb{R}, n \in \mathbb{N}$ denoted $H_{a,b,n}$ is the unimodular polytope with vertices

$$(0,0), (0,b), (a,b), (a+nb, 0)$$

Proposition 2.8 (Four-Vertex Unimodular Polytopes on the Plane)**.** *Let* $\Delta \in \mathbb{R}^2$ *be a four-vertex unimodular polytope. Then, after translation and linear transformation in* $\mathrm{GL}(2;\mathbb{Z})$*,* Δ *is a Hirzebruch Trapezoid.*

Proof. This proof works analogous to the proof of 2.6. □

2.4 Dual Characterization

The unimodularity condition can be translated into a condition on the normal vectors of the facets:

Proposition 2.9. *Let $\Delta \in \mathbb{R}^n$ be a unimodular polytope, $v \in \Delta$ a vertex and $u_1, \ldots, u_n \in \mathbb{Z}^n$ be the corresponding edge vectors forming a $\mathbb{Z}$-basis.*
Then there are n facets meeting at v (each one containing all but one of the u_k vectors) and the primitive inward-pointing normal vectors to these facets also form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Proof. After a suitable translation and linear transformation $A \in \mathrm{GL}(n; \mathbb{Z})$ v lies in the origin an the vectors u_k form the standard basis of $\mathbb{R}^n$, i.e. $u_k = e_k$ for all k. The facets meeting at v are then exactly given by

$$\begin{aligned} E_k &= \{(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \,|\, x_i \in \mathbb{R}\} \cap \Delta \\ &= \mathrm{span}\{u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n\} \cap \Delta \end{aligned}$$

The primitive inward-pointing normal vector v_k to E_k is thus exactly u_k and $(v_1, \ldots, v_n)$ is again a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ since $(u_1, \ldots, u_n)$ was also one. If we now reverse the linear transformation A, $((A^T)^{-1}v_1, \ldots, (A^T)^{-1}v_n)$ is still a $\mathbb{Z}$-basis and the proposition follows. □

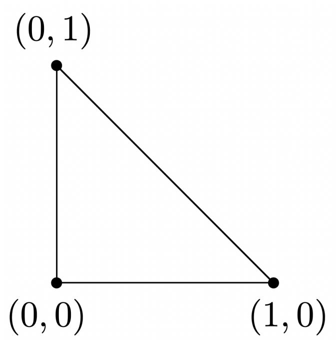
This allows us characterize a unimodular polytope $\Delta \in R^n$ with facets $E_k, k = 1, \ldots, d$ and corresponding primitive inward-pointing normal vectors v_k as follows:

$$\Delta = \{x \in \mathbb{R}^n \,|\, \langle x, v_k \rangle \geq c_k, c_k \in \mathbb{R}, k = 1, \ldots, d\}$$

Note that Δ is the intersection of the closed half-spaces $H_k = \{x \in \mathbb{R}^n \,|\, \langle x, v_k \rangle \geq c_k\}$.1§

Example 2.10. Let $\Delta = \{x \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$ be the standard isosceles right triangle. It can also be written as

$$\Delta = \{x \in \mathbb{R}^2 \,|\, \langle x, (1, 0)\rangle \geq 0, \langle x, (0, 1)\rangle \geq 0, \langle x, (-1, -1)\rangle \geq -1\}$$



3 Delzant's Classification Theorem

For $(\mathbb{T}^n, \mu, M, \omega)$ a hamiltonian torus action, the unimodularity condition for the abstract convex polytope $\mu(M) = \Delta \subset \mathfrak{t}^*$ can be restated as follows:

Definition 3.1 (Abstract Unimodularity)**.** $\Delta \subset \mathfrak{t}^*$ is *unimodular* if and only if for each vertex v there exist a neighborhood U of v and a $\mathbb{Z}$-basis $\eta_1, \ldots, \eta_n$ of the *weight lattice* $\mathfrak{t}^*_{\mathbb{Z}}$ such that

$$\Delta \cap U\{v + t_1\eta_1 + \cdots + t_n\eta_n \,|\, t_i \in [0, \varepsilon)\}$$

This definition encompasses simplicity, rationality and smoothness.

We can now state delzant's theorem which allows us to classify symplectic toric manifolds based on their moment map image:

Theorem 3.2 (Delzant)**.** *Symplectic toric manifolds are classified up to equivalence by unimodular polytopes up to translation. Specifically, the bijective correspondence between these two sets is give by the moment map:*

$$\frac{\{\textit{symplectic toric manifolds}\}}{(\textit{equivalence})} \xrightarrow{1-1} \frac{\{\textit{unimodular polytopes}\}}{(\textit{translation})}$$
$$(\mathbb{T}^n, \mu, M, \omega) \longmapsto \mu(M)$$

References

[Can] Ana Cannas. "Seminar on Symplectic Toric Manifolds". In: *Notes available at https://people.math.ethz.ch/~acannas/Papers/stm_seminar.pdf* ().

[Imp] Alessandro Imparato. "Hamiltonian torus actions". In: *Notes available at https://people.math.ethz.ch/~bacubulut/assets/week4_bangxin_full.pdf* ().