Contact Toric Manifolds

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Abstract

I In this talk, we will start by recalling some concepts and definitions about group actions on contact manifolds. Then, we will give the definition of contact toric manifolds and present some examples.

In the second half, we will state the classification theorem for contact toric manifolds and some of its applications, due to Lerman [3].

We will start by recalling a few concepts and definitions from the previous talks. Let $(Y, \xi = \ker \alpha)$ be a contact manifold. We will always assume that the contact structure is induced as the kernel of a contact 1-form α. In particular the contact structure ξ is co-orientable. We will also asssume that ξ is co-oriented, that is, a component ξ^0_+ of $\xi^0 \backslash 0 \subseteq T^*Y$ is chosen.

In general, we will denote a Lie group by G, and its Lie algebra by $\mathfrak{g}$.

An action of a Lie group G on a contact manifold $(Y, \xi = \ker \alpha)$ is called a ***contact action*** if for every element $g \in G$, the corresponding diffeomorphism $\sigma_g : Y \to Y$ preserves the contact structure and its co-orientation. That is, if $\sigma_g^* \alpha = f\alpha$ for some positive function f.

We have seen that if a compact Lie group G acts on $(Y, \xi = \ker \alpha)$, we may assume α is G-invariant. The proof for this fact can be found in [1].

Let us recall the definitions of α-moment map and the contact moment map for a co-oriented contact manifold $(Y, \xi = \ker \alpha)$ with co-orientation ξ^0_+.

Definition 1. Suppose a Lie group G acts on a manifold Y preserving a 1-form α. The corresponding ***α-moment map*** $\mu_\alpha : Y \to \mathfrak{g}^*$ is defined by

$$\mu_\alpha(x)(X) = \alpha_x(X^\#(x))$$

for all $x \in Y$ and $X \in \mathfrak{g}$, where $X^\#(x) = \frac{d}{dt}|_{t=0}(\exp tX) \cdot x$ is the vector field defined by $X \in \mathfrak{g}$

Definition 2. Suppose a Lie group G acts on a contact manifold $(Y, \xi = \ker \alpha)$ through a contact action. Define $\phi : T^*Y \to \mathfrak{g}^*$ by

$$\phi(p, q)(X) = q(X^{\#}(p))$$

Then, ***the contact moment map*** $\mu : \xi_+^\circ \to \mathfrak{g}^*$ is defined as the restriction $\mu = \phi|_{\xi_+^\circ}$

The ***moment cone*** $C(\mu)$ for the map $\mu : \xi_+^\circ \to \mathfrak{g}^*$ is the set $\mu(\xi_+^\circ) \cup \{0\}$. If α is a G-invariant contact form, then $C(\mu) = \{t\phi \in \mathfrak{g}^* : \phi \in \mu_\alpha(Y), t \geq 0\}$.

1 Contact Toric Manifolds

Recall that an action of the group G on a manifold M is called ***effective*** if the only element that fixes all the points of M is the identity.

Definition 3. A ***contact toric*** G***-manifold*** is a co-oriented contact manifold $(Y, \xi = \ker \alpha)$ with an effective action of a torus G preserving the contact structure and its co-orientation (i.e. an effective contact action of a torus G), such that $2 \dim G = \dim Y + 1$.

Remark. When we consider a contact toric G-manifold, we will consider it as a triple $(Y, \xi = \ker \alpha, \mu : \xi_+^\circ \to \mathfrak{g}^*)$ or $(Y, \xi = \ker \alpha, \mu_\alpha : Y \to \mathfrak{g}^*)$, analogous to the symplectic case.

Now we will consider a few examples of contact toric G-manifolds. The following examples of spheres is adapted from [2].

Example. Let $S^3 = \{(x_0, y_0, x_1, y_1) \in \mathbb{R}^4 : (x_0^2 + y_0^2) + (x_1^2 + y_1^2) = 1\}$ be the standard 3-sphere with the contact form

$$\alpha = (x_0 dy_0 - y_0 dx_0) + (x_1 dy_1 - y_1 dx_1)$$

The 2-torus $T^2 = S^1 \times S^1 = \{(\theta_0, \theta_1)\}$ acts on S^3 by rotation of (x_0, y_0)- and (x_1, y_1)-planes by θ_0 and θ_1 respectively. That is, T^2 action on S^3 is generated by the vector fields $H_i = (x_i \partial_{y_i} - y_i \partial_{x_i})$ for $i = 0, 1$ on S^3. From this, we can see that this action preserves the contact structure and is effective. Therefore, S^3 is a contact toric T^2-manifold.

Now consider the α-moment map $\mu_\alpha : S^3 \to \mathfrak{g}^*$, where we identify the lie algebra of the torus as $\mathfrak{g}^* \cong (\mathbb{R}^2)^* = span\{e_0^*, e_1^*\}$ with $e_i \in \mathfrak{g}$ generating H_i. We have:

$$\begin{aligned}
\mu_\alpha(x_0, y_0, x_1, y_1)(e_i) &= \alpha_{(x_0, y_0, x_1, y_1)}((e_i)^{\#}((x_0, y_0, x_1, y_1)) \\
&= \alpha_{(x_0, y_0, x_1, y_1)}((x_i \partial_{y_i} - y_i \partial_{x_i})) \\
&= (x_i^2 + y_i^2)
\end{aligned}$$

From this, we have $\mu_\alpha(x_0, y_0, x_1, y_1) = (x_0^2 + y_0^2)e_0^* + (x_1^2 + y_1^2)e_1^*$. As

$$(x_0^2 + y_0^2) + (x_1^2 + y_1^2) = 1$$

this shows that the image of the α-moment map is

$$\mu_\alpha(S^3) = \{t_0 e_0^* + t_1 e_1^* : t_0 + t_1 = 1, t_i \geq 0\}$$

That is, $\mu_\alpha(S^3)$ is the standard 1-simplex in $\mathfrak{g}^*$. Then, the moment cone is

$$C(\mu) = \{s_0 e_0^* + s_1 e_1^* : s_i \geq 0\}$$

That is, the first quadrant in $\mathfrak{g}^* \cong (\mathbb{R}^2)^*$.

Example. We may extend the above example to higher dimensions immediately. The torus T^{n+1} acts on the sphere S^{2n+1}, through the rotations generated by the vector fields $H_i = (x_i \partial_{y_i} - y_i \partial_{x_i})$ for $i = 0, 1, \ldots, n$ on S^{2n+1}. With this action, S^{2n+1} is a contact toric T^{n+1}-manifold.

By a similar argument as above, the image of the α-moment map is

$$\mu_\alpha(S^{2n+1}) = \{\sum_{i=0}^{n} t_i e_i^* : \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

That is, $\mu_\alpha(S^{2n+1})$ is the standard n-simplex in $\mathfrak{g}^* \cong (\mathbb{R}^n)^*$. Then, the moment cone is $C(\mu) = (\mathbb{R}^{n+1}_{\geq 0})^*$

Example. Consider the manifold $Y = S^1 \times T^2 = \{(x, \theta_0, \theta_1)\}$ with the contact form

$$\alpha = \cos x d\theta_0 + \sin x d\theta_1$$

The 2-torus $T^2 = \{(\phi_0, \phi_1)\}$ acts on $Y = S^1 \times T^2 = \{(x, \theta_0, \theta_1)\}$ by termwise addition on the second factor. This action is free and preserves the contact structure. Therefore, $S^1 \times T^2$ is a contact toric T^2-manifold.

Now consider the α-moment map $\mu_\alpha : S^1 \times T^2 \to \mathfrak{g}^* \cong span\{d\phi_0|_0, d\phi_1|_0\}$. We have:

$$\begin{aligned}
\mu_\alpha(x, \theta_0, \theta_1)(\partial_{\phi_0}|_0) &= \alpha_{(x,\theta_0,\theta_1)}((\partial_{\phi_0}|_0)^\#(x, \theta_0, \theta_1)) \\
&= \alpha_{(x,\theta_0,\theta_1)}(\frac{d}{dt}|_{t=0} \exp t(\partial_{\phi_0}|_0) \cdot (x, \theta_0, \theta_1)) \\
&= \alpha_{(x,\theta_0,\theta_1)}(\frac{d}{dt}|_{t=0}(x, \theta_0 + t, \theta_1)) \\
&= \alpha_{(x,\theta_0,\theta_1)}(\partial_{\theta_0}|_{(x,\theta_0,\theta_1)}) \\
&= \cos x
\end{aligned}$$

By a similar calculation, $\mu_\alpha(x,\theta_0,\theta_1)(\partial_{\phi_1}) = \sin x$. Therefore, we have $\mu_\alpha(x,\theta_0,\theta_1) = \cos x d\phi_0|_0 + \sin x d\phi_1|_0$. This shows that the image of the α-moment map is $\mu_\alpha(S^1 \times T^2) = \{\cos x d\phi_0|_0 + \sin x d\phi_1|_0 : x \in S^1\}$. From this, we see that the moment cone is

$$\begin{aligned} C(\mu) &= \{s\cos x d\phi_0|_0 + s\sin x d\phi_1|_0 : x \in S^1, s \geq 0\} \\ &= \{s_0 d\phi_0|_0 + s_1 d\phi_1|_0 : s_0, s_1 \in \mathbb{R}\} \\ &= \mathfrak{g}^* \cong (\mathbb{R}^2)^* \cong \mathbb{R}^2 \end{aligned}$$

We can generalize this argument to the same manifold $S^1 \times T^2 = \{(x,\theta_0,\theta_1)\}$ with the contact form

$$\alpha_n = \cos nx d\theta_0 + \sin nx d\theta_1$$

with n a positive integer. The moment cone is again $C(\mu) = \mathfrak{g}^* \cong \mathbb{R}^2$.

Definition 4. Two contact toric G-manifolds $(Y, \xi = \ker\alpha, \mu_\alpha : Y \to \mathfrak{g}^*)$ and $(Y', \xi' = \ker\alpha', \mu'_\alpha : Y' \to \mathfrak{g}^*)$ are ***isomorphic*** if there exists a G-equivariant co-orientation preserving contactomorphism $\varphi : Y \to Y'$. We will refer to such map as an ***isomorphism*** of contact toric manifolds.

We denote the group of isomorphisms of $(Y, \xi = \ker\alpha, \mu_\alpha : Y \to \mathfrak{g}^*)$ by $\mathrm{Iso}(Y, \xi = \ker\alpha, \mu_\alpha : Y \to \mathfrak{g}^*) = \mathrm{Iso}(Y)$

We will now state a lemma concerning the image of the moment map of a contact toric manifold.

Lemma 1. *Let $(Y, \xi, \mu : \xi_+^\circ \to \mathfrak{g}^*)$ be a contact toric G-manifold. Then the image of the moment map $\mu : \xi_+^\circ \to \mathfrak{g}^*$ does not contain 0.*

We may fix an inner product on $\mathfrak{g}$ and hence on $\mathfrak{g}^*$. Then there exists a unique G-invariant contact form preserving ξ and its co-orientation such that $\|\mu_\alpha(x)\| = 1$ for all $x \in Y$. This can be done by taking any contact form α' defining ξ and setting $\alpha_x = \frac{1}{\|\mu_{\alpha'}(x)\|}\alpha'_x$, which is possible by the previous lemma.

From now on, we will assume that an inner product on $\mathfrak{g}$ is fixed and for a contact toric G-manifold $(Y, \xi = \ker\alpha, \mu_\alpha : Y \to \mathfrak{g}^*)$, the α-moment map is normalized as above such that the image lies on the unit sphere $S(\mathfrak{g}^*)$ of the vector space $\mathfrak{g}^*$.

2 Classification of Compact Connected Contact Toric Manifolds

The classification of compact connected contact toric manifolds was completed by Lerman in [3]. The following formulation of the classification theorem (Theorem 2.18 from [3]) is from [4] by Marinković and Pabiniak:

Theorem 1 (Classification of Compact Connected Contact Toric Manifolds). *Compact connected contact toric manifolds* $(Y, \xi = \ker\alpha, \mu : \xi_+^\circ \to \mathfrak{g}^*)$ *are classified as follows:*

1. *Suppose* $\dim Y = 3$ *and the action of* T^2 *is free. Then,* $Y = T^3 = S^1 \times T^2 = \{(x, \theta_0, \theta_1)\}$ *and* $\alpha = \cos nxd\theta_0 + \sin nxd\theta_1$ *for some positive integer* n*. The moment cone is* $\mathfrak{g}^* \cong \mathbb{R}^2$*.*

2. *Suppose* $\dim Y = 3$ *and the action of* T^2 *is not free. Then* Y *is a lens space (this includes* S^3 *and* $S^1 \times S^2$*) equipped with one of the various contact structures. As a compact connected contact toric manifold,* $(Y, \xi = \ker\alpha, \mu : \xi_+^\circ \to \mathfrak{g}^*)$ *is classified by two rational numbers* r, q *with* $0 \leq r < 1$ *and* $r < q$*.*

3. *Suppose* $\dim Y = 2d - 1 > 3$ *and the action of* T^d *is free. Then* Y *is a principal* T^d*-bundle over* S^{d-1}*. Moreover, each principal* T^d*-bundle over* S^{d-1} *has a unique* T^d*-invariant contact structure making it a compact connected contact toric manifold. The moment cone is whole* $\mathfrak{g}^* \cong \mathbb{R}^d$*.*

   Principal T^d*-bundles over* S^{d-1} *are in one-to-one correspondence with the second cohomology classes of* S^{d-1} *with* $\mathbb{Z}^d$ *coefficients. Since the cohomology groups are* $H^2(S^{d-1}, \mathbb{Z}^d) = 0$ *for* $d - 1 \neq 2$*, it follows that for the case that* $\dim Y > 5$*, this bundle is the trivial bundle* $T^d \times S^{d-1}$*.*

4. *Suppose* $\dim Y = 2d-1 > 3$ *and the action of* T^d *is not free. Then such manifolds are classified by their (convex) moment cones in* $\mathfrak{g}^* \cong \mathbb{R}^d$*, up to* $GL(d, \mathbb{Z})$*-transformations corresponding to changing the splitting of a torus into a product of circles.*

We will not give the full proofs for the classification theorem. The full proofs can be found in [3].

For the discussion about the proofs, let Y/G be the orbit space of the action of the torus, with $\pi : Y \to Y/G$ as the orbit map. Then ***the orbital moment map*** $\overline{\mu_\alpha} : Y/G \to S(\mathfrak{g}^*)$ is the map induced by the α-moment map. Two contact toric G-manifolds $(Y, \xi = \ker\alpha, \mu_\alpha)$ and $(Y', \xi' = \ker\alpha', \mu_{\alpha'})$ are called ***locally isomorphic*** if

- There is a homeomorphism $\varphi : Y/G \to Y'/G$
- For any point $x \in Y/G$, there is a neighbourhood $U \subseteq Y/G$ of x and an isomorphism of contact toric manifolds $\varphi_U : \pi^{-1}(U) \to (\pi')^{-1}(\varphi(U))$ such that $\pi' \circ \varphi_U = \varphi \circ \pi$.

Also, define $\mathcal{S}$ as the sheaf of groups $S(U) = \mathrm{Iso}(\pi^{-1}(U))$. By Lemma 5.2 from [3], the elements of the sheaf cohomology group $H^1(Y/G, \mathcal{S})$ are in one-to-one correspondence with the isomorphism classes of the contact toric G-manifold that are locally isomorphic to a given contact toric G-manifold $(Y, \xi = \ker \alpha, \mu_\alpha : Y \to \mathfrak{g}^*)$.

For the proof of part (1), observe that since the action of T^2 is free, Y/T^2 is a compact connected 1-manifold, hence S^1. Moreover, since any T^2 bundle on S^1 is trivial, we have a diffeomorphism $Y \cong S^1 \times T^2$. Using the properties of the orbital moment map (see [3], 4.6 and 4.9), we can prove that $\overline{\mu_\alpha} : Y/G = S^1 \to S(\mathfrak{g}^*) = S^1$ is an n-sheeted covering map and we can obtain a local isomorphism between $(Y, \ker \alpha, \mu_\alpha)$ and $(S^1 \times T^2, \alpha_n = \cos nx d\theta_0 + \sin nx d\theta_1, \mu_{\alpha_n})$. By Corollary 5.4 [3], we have

$$H^1(Y/G, \mathcal{S}) \cong H^2(S^1, \mathbb{Z}^2) = 0$$

proving the isomorphism for part (1).

Similarly, for the proofs of parts (2) and (3), the orbital moment maps are used to construct a local isomorphism with compact connected contact toric manifolds given in the statements, and then the classification is completed by investigation of the sheaf cohomology groups $H^1(Y/G, \mathcal{S})$.

Lastly, for the proof of part (4), a similar construction to Delzant's construction is used to construct the manifold with the given "good" convex moment cone (see [3] for the definition of a good cone). And conversely, it is proved that every compact connected contact toric manifold has a good convex moment cone.

3 Applications of the Classification Theorem

Consider the cosphere bundle $S^*T^n := (T^*T^n \backslash 0)/\mathbb{R}$ of the n-torus where action of $t \in \mathbb{R}$ is given by the dilation $(p, q) \mapsto (p, e^t q)$. Then S^*T^n has a natural contact structure induced by the symplectic structure on T^*T^n. Similarly, consider the cosphere bundle $S^*S^2 := (T^*S^2 \backslash 0)/\mathbb{R}$ of the sphere. Then S^*S^2 has a natural contact structure induced by the symplectic structure on T^*S^2. Proofs of these facts can be found in [1].

Using the classification theorem for compact connected contact toric manifolds, we can prove the following two theorems (Theorems 1.3 and 1.5, [3]) regarding the contact actions on S^*T^n and S^*S^2:

Theorem 2. *Up to isomorphism, there is only one effective T^n-action on S^*T^n making it a contact toric manifold.*

Theorem 3. *Up to isomorphism, there is only one effective T^2-action on S^*S^2 making it a contact toric manifold.*

As a consequence of Theorem 2 and the classification theorem, one sees that any effective contact T^n-action on S^*T^n must be free. Using this result and the relation of the symplectic structure on $T^*T^n \backslash 0$ and the contact structure it induces on S^*T^n, one can prove:

Proposition 1. *Any effective T^n-action on $T^*T^n \backslash 0$ which preserves the symplectic form and commutes with dilations, is free.*

Through the work of Toth and Zeldich [5] this proposition implies that certain classes (called "toric integrable", see [3]) of metrics on tori are flat.

References

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