Hamiltonian Vector Fields: Definitions and Examples

Full Version

Daniel Rutschmann

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Abstract

Using the non-degeneracy of the symplectic form of a symplectic manifold, we will seen a mechanism that allows us to associate to every smooth function a vector field. We will study this mechanism more closely, discuss various properties of the resulting vector field and its flow, and give various examples.

1 A refresher on the Lie derivative

We will briefly recall the Lie derivative of a tensor field. Let M be a (smooth) manifold and let $X \in \Gamma(TM)$ be a vector field. Let ρ_t be the local flow of X at time t. In other words, for every $p \in M$, $t \mapsto \rho_t(p)$ is the integral curve of X starting at p, defined on a maximal interval[1].

Given a (r, s)-tensor field $T \in \Gamma(T_{r,s}M)$, we would like to study how T changes along these integral curves.

Definition 1. The *Lie derivative* of T w.r.t. X at a point $p \in P$ is given by

$$(\mathcal{L}_X T)_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ((\rho_t)^* T)_p$$

One can check that this depends smoothly on p and gives a well-defined linear map

$$\mathcal{L}_X : \Gamma(T_{r,s}M) \to \Gamma(T_{r,s}M)$$

[1] When reading this for the first time, it might be useful to assume that X is complete, i.e. that that the flow $\rho_t : M \to M$ is defined on all of M for all $t \in \mathbb{R}$. This is true for example if X has compact support. The general case will then follow from local considerations.

Proposition 2 (Properties)**.** *The Lie derivative has the following properties*

1. *If* $f \in C^\infty(M)$*, then* $\mathcal{L}_X f = X(f)$*.*

2. *If* $S \in \Gamma(T_{r,s}M)$ *and* $T \in \Gamma(T_{r',s'}M)$ *are two tensor fields, then*

$$\mathcal{L}_X(S \otimes T) = \mathcal{L}_X(S) \otimes T + S \otimes \mathcal{L}_X(T)$$

3. $\mathcal{L}_X$ *commutes with contractions. More precisely, if* $C_{i,j} : \Gamma(T_{r,s}M) \to \Gamma(T_{r-1,s-1}M)$ *with* $1 \leq i \leq r$ *and* $1 \leq j \leq s$ *is a contraction and* $T \in \Gamma(T_{r,s}M)$ *a* (r,s)*-tensor field, then*

$$\mathcal{L}_X(C_{i,j}(T)) = C_{i,j}(\mathcal{L}_X(T))$$

4. *Formula for vector fields: If* $X, Y \in \Gamma(TM)$ *and* $f \in C^\infty(M)$ *then*

$$(\mathcal{L}_X Y)(f) = [X, Y](f) = X(Y(f)) - Y(X(f))$$

The Lie derivative on tensor fields induces a Lie derivative on differential forms

$$\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$$

which can be computed via Cartan's magic formula:

Theorem 3 (Cartan's Formula)**.** *We have*

$$\mathcal{L}_X \omega = \iota_X(\mathrm{d}\omega) + \mathrm{d}(\iota_X \omega)$$

Lemma 4. *The Lie derivative on differential form has the following properties*

1. *Commutativity with the exterior derivative*

$$\mathcal{L}_X(\mathrm{d}\omega) = \mathrm{d}(\mathcal{L}_X \omega)$$

2. *Let* $f \in C^\infty(M)$*, then*

$$\mathcal{L}_{fX}(\omega) = f\mathcal{L}_X(\omega) + \mathrm{d}f \wedge \iota_X \omega$$

3. *Product rule*

$$\mathcal{L}_X(\varphi \wedge \psi) = \mathcal{L}_X(\varphi) \wedge \psi + \varphi \wedge \mathcal{L}_X(\psi)$$

1.1 Future references

For a quick introduction, see pages 4-8 of the lecture notes by Erik van den Ban [vdB06]. For a detailed derivation and proofs, see Chapters 22 and 23 of W. J. Merry's lecture notes [Mer21]

2 Hamiltonian Vector Fields

Let (M, ω) be a symplectic manifold. As ω is non-degenerate, the musical map at every $p \in M$,

$$\omega_p^\flat : TM_p \to T^*M_p$$

is a linear isomorphism (see first talk today). One can check that this depends smoothly on p, so we get a bundle isomorphism $\omega^\flat : TM \to T^*M$. Let $H \in C^\infty(M)$, then $\mathrm{d}H \in \Gamma(T^*M)$, so we can associate the following vector field to H:

$$X_H := -(\omega^\flat)^{-1} \circ \mathrm{d}H$$

This vector field then satisfies

$$\iota_{X_H}\omega = \underbrace{\omega(X_H, .)}_{\omega^\flat \circ X_H} = -\,\mathrm{d}H$$

This mechanism allows us to associate a vector field to every smooth function on a symplectic manifold. We will study these vector fields and their flows more closely.

Definition 5. Let X be a vector field on a symplectic manifold (M, ω). Then:

- X is *symplectic* if $\iota_X\omega$ is closed.
- X is *hamiltonian* if $\iota_X\omega$ is exact.

Clearly, every hamiltonian vector field is also symplectic. Conversely, if $H^1_{\text{deRahm}}(M) = 0$, then every symplectic vector field is hamiltonian. For example, this is true if we're working locally on a contractible open set (Pointcare's Lemma)[2].

Example. In general, not every symplectic vector field is hamiltonian. For example, on $M = \mathbb{T}^2 = S^1 \times S^1$ with angular coordinates (ϕ, θ) and $\omega = \mathrm{d}\phi \wedge \mathrm{d}\theta$, the vector fields $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\theta}$ are symplectic but not hamiltonian.

[2] In last week's introductory talk, it was mentioned that in symplectic geometry, the global "structure" is often richer than the local one; this is one small example of this.

The vector field X_H that we constructed above is clearly a hamiltonian vector field. We will now show that the corresponding flow preserves both ω and H.

Lemma 6. *If X is* symplectic *vector field on a symplectic manifold (M, ω), then the flow ρ_t preserves ω in the sense that*

$$\mathcal{L}_X \omega = 0$$

Proof. By Cartan's formula, we have

$$\mathcal{L}_X \omega = \iota_X(\underbrace{\mathrm{d}\omega}_{=0}) + \underbrace{\mathrm{d}(\iota_X \omega)}_{=0}$$

as X is symplectic and ω is closed. (This explains why we required ω to be closed in the definition of a symplectic manifold.) □

If X is complete, then we can re-phrase this preservation in terms of pullbacks and flows.

Definition 7. Let (M, ω) be a symplectic manifold. A *symplectomorphism* is a diffeomorphism $\phi : M \to M$ that preserves ω, meaning

$$\phi^* \omega = \omega$$

Corollary 8. *If X is a* complete *symplectic vector field, then for every $t \in \mathbb{R}$, the flow $\rho_t : M \to M$ is a symplectomorphism.*

Proof. The flow is well-defined as X is complete. We will show that $\rho_t^* \omega$ does not depend on t: By the definition of the Lie derivative and Lemma 6

$$\frac{\mathrm{d}}{\mathrm{d}t} (\rho_t)^* \omega \Big|_{t=t_0} = (\rho_{t_0})^* \mathcal{L}_X(\omega) = 0$$

Therefore

$$(\rho_t)^* \omega = (\rho_0)^* \omega = \omega \qquad \forall t \in \mathbb{R}$$

□

Definition 9. If X is a hamiltonian vector field, then a function $H \in C^\infty(M)$ with

$$\iota_X \omega = -\,\mathrm{d}H$$

is called a *hamiltonian function* for X.

In a *connected* manifold, hamiltonian functions are unique up to adding a constant (recall that $H^0_{\text{deRahm}}(M) \cong \mathbb{R}$ are the constant functions if M is connected and the locally constant functions in general). Conversely, we've seen that every smooth function H is a hamiltonian function for the (unique) vector field X_H.

Lemma 10. *Let X be a hamiltonian vector field with a hamiltonian function H. Then the flow ρ_t of X preserves H.*

Proof.

$$\mathcal{L}_X H = X(H) = \mathrm{d}H(X) = -\iota_X \iota_X \omega = 0$$

□

Corollary 11. *Let X be a complete hamiltonian vector field with a hamiltonian function H. Let ρ_t be the flow of X, then*

$$\rho_t^* H = H$$

Proof. Similar to the previous corollary. □

In particular, every integral curve $t \mapsto \rho_t(p)$ is contained in a level set of H as

$$H(\rho_t(p)) = ((\rho_t)^* H)(p) = H(p) \quad \forall t$$

(Even if X is not complete, this still works if we consider local flows.) Physically, if we think of H as "energy", then this is some form of "energy conservation".

2.1 Examples

Plane Consider $\mathbb{R}^2$ with $\omega = \mathrm{d}x \wedge \mathrm{d}y$. In polar coordinates (r, θ), we have

$$\omega = r \cdot \mathrm{d}r \wedge \mathrm{d}\theta$$

Then $H(x, y) = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}r^2$ is a hamiltonian function corresponding to $X = \frac{\partial}{\partial \theta}$ as

$$\iota_X(\omega) = r \cdot \iota_X(\mathrm{d}r \wedge \mathrm{d}\theta) = r \cdot (\underbrace{\iota_X(\mathrm{d}r)}_{=0} \cdot \mathrm{d}\theta - \underbrace{\iota_X(\mathrm{d}\theta)}_{=1} \cdot \mathrm{d}r) = -dH$$

The flow of X corresponds to rotation around the origin. (This example generalizes to $\mathbb{R}^{2n}$.)

Sphere Consider the sphere in cylindrical coordinates, i.e. $M = S^2$ with $\omega = \mathrm{d}\theta \wedge \mathrm{d}h$. Let $H(\theta, h) = -h$, then

$$\mathrm{d}h = -\,\mathrm{d}H = \iota_{X_H}(\mathrm{d}\theta \wedge dh) = \iota_{X_H}(\mathrm{d}\theta) \cdot dh - \iota_{X_H}(\mathrm{d}h) \cdot \mathrm{d}\theta$$

therefore

$$X_H = \frac{\partial}{\partial \theta}$$

The corresponding flow lines have constant latitude, i.e. they preserve h. Moreover, the flow corresponds to rotation around the h-axis. This preserves the surface area of the sphere, hence ω.

Classical Mechanics Consider $M = \mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. Let $\omega = \sum_j dq_j \wedge dp_j$. Let $H \in C^\infty(M)$ be given. Suppose that $\rho_t(q(0), p(0)) = (q(t), p(t))$ is an integral curve of $X = X_H$. Then we have

$$\begin{aligned}
\iota_{X_H}\omega &= \sum_{j=1}^{n} \iota_{X_H}(\mathrm{d}q_j \wedge \mathrm{d}p_j) \\
&= \sum_{j=1}^{n} \Big((\iota_{X_H}\,\mathrm{d}q_j) \cdot \mathrm{d}p_j - (\iota_{X_H}\,\mathrm{d}p_j) \cdot \mathrm{d}q_j \Big) \\
&= \sum_{j=1}^{n} \Big(\frac{\mathrm{d}q(t)_j}{\mathrm{d}t} \cdot \mathrm{d}p_j - \frac{\mathrm{d}q(t)_j}{\mathrm{d}t} \cdot \mathrm{d}q_j \Big)
\end{aligned}$$

On the other hand, by the definition of X_H, we have

$$\iota_{X_H}\omega = -\,\mathrm{d}H = -\sum_{j=1}^{n} \Big(\frac{\partial H}{\partial p_j} \cdot \mathrm{d}p_j + \frac{\partial H}{\partial q_j} \cdot \mathrm{d}q_j \Big)$$

Comparing coefficients gives us the *Hamilton equations* that you might have seen in Numerical Analysis II:

$$\begin{aligned}
\frac{\mathrm{d}q(t)_j}{\mathrm{d}t} &= -\frac{\partial H}{\partial p_j} \\
\frac{\mathrm{d}p(t)_j}{\mathrm{d}t} &= \frac{\partial H}{\partial q_j}
\end{aligned}$$

Newtonian Physics Consider $M = \mathbb{R}^6$ with points of the form $(x, p) = (x_1, x_2, x_3, p_1, p_2, p_3) \in M$. As in the previous example, let $\omega = \sum_j \mathrm{d}x_j \wedge \mathrm{d}p_j$ Let $V : \mathbb{R}^3 \to \mathbb{R}$ be any potential. Put

$$H(x,p) = -V(x) - \frac{1}{2m}|p|^2 = -V(x_1, x_2, x_3) - \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2)$$

Then the Hamilton equations are

$$\begin{aligned} \frac{\mathrm{d}x_j}{\mathrm{d}t}(t) &= -\frac{\partial H}{\partial p_j} = \frac{1}{m} p_j(t) \\ \frac{\mathrm{d}p_j}{\mathrm{d}t}(t) &= \frac{\partial H}{\partial x_j} = -\frac{\partial V}{\partial x_j}(x(t)) \end{aligned}$$

which are precisely Newton's equations for a particle of mass m:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{p}{m} \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\nabla V(x) \end{aligned}$$

As a cotangent bundle Consider $N = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3).[3] Let $M = T^*N$, then M has points (x, p) with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $p = p_1 \,\mathrm{d}x_1 + p_2 \,\mathrm{d}x_2 + p_3 \,\mathrm{d}x_3 \in T^*N_x$. We can define a 1-form on M by

$$\lambda_{x,p} = p_1 \,\mathrm{d}x_1 + p_2 \,\mathrm{d}x_2 + p_3 \,\mathrm{d}x_3$$

Note that if $\pi : (x, p) \mapsto x$ is the projection map, then

$$\lambda_{x,p} = \pi^* p = p \circ \mathrm{d}\pi_{x,p}$$

(we will use this formula later for general cotangent bundles). This induces the following symplectic form on M

$$-\,\mathrm{d}\lambda_{x,p} = \mathrm{d}x_1 \wedge \mathrm{d}p_1 + \mathrm{d}x_2 \wedge \mathrm{d}p_2 + \mathrm{d}x_3 \wedge \mathrm{d}p_3$$

which is precisely the one we had before.

[3] Some notes use (q_1, q_2, q_3), but I find it incredibly easy to mix up q with p, especially since (q, p) goes against the natural alphabetical order. This is why I'm using (x, p) instead, which I find very intuitive based on physics: x is a position and p a momentum.

Why cotangent bundle and not tangent bundle? In the "Newtonian Physics" example, we could either work with velocity (vectors) or momentum (covectors):

$$H(x, v) = -V(x) - \frac{1}{2}m|v|^2$$
$$H(x, p) = -V(x) - \frac{1}{2m}|p|^2$$

Let's see what happens if we try to generalize these two approaches to a manifold: Let M be a manifold, let $V \in C^\infty(M)$ be a potential and let $m \in \Gamma(T_{0,2}M)$ a Riemannian metric. Then, we can consider velocity on the tangent bundle:

$$H(x, v) = -V(x) - \frac{1}{2}m_x(v, v), \qquad x \in M, v \in TM_x$$

and momentum on the cotangent bundle

$$H(x, p) = -V(x) - \frac{1}{2}m_x^{-1}(p, p), \qquad x \in M, p \in T^*M_x$$

where $m_x^{-1} : T^*M_x \times T^*M_x \to \mathbb{R}$ is the "matrix inverse" of the metric and can be derived from $m^\flat$.

In this example, from a physical perspective, we could work either with tangent bundles or cotangent bundles, as we can defined an "energy" H on either of them. (Intuitively, this comes from the fact that the metric m induces an isomorphism $m^\flat : TM \to T^*M$.)

But what about the symplectic structure? In the next section, we will that cotangent bundles are naturally symplectic spaces, which makes the momentum-based formulation on the cotangent bundle preferable.

2.2 Exact symplectic manifolds

Definition 12. A symplectic manifold (M, ω) is *exact* if ω is exact.

Cotangent bundles are exact symplectic manifolds, as $\omega = -\,\mathrm{d}\lambda$ where λ is the tautological 1-form.

Remark. Note: exact manifolds are never compact. If $\omega = -\,\mathrm{d}\lambda$, then $\omega^n = -\,\mathrm{d}(\omega^{n-1} \wedge \lambda)$ is an exact volume form. Hence by Stokes' theorem, M cannot be compact.

Lemma 13. *Let (M, ω) be an exact symplectic manifold with $\omega = -\,\mathrm{d}\lambda$. If the flow of a vector field X preserves λ, then it is a hamiltonian vector field with hamiltonian function $H = -\iota_X(\lambda)$.*

Proof. As the flow of X preserves λ, we have

$$0 = \mathcal{L}_X(\lambda) = \iota_X(\mathrm{d}\lambda) + \mathrm{d}(\iota_X(\lambda)) = \iota_X(-\omega) + \mathrm{d}\iota_X(\lambda)$$

hence

$$\iota_X(\omega) = \mathrm{d}\iota_X(\lambda)$$

so X is hamiltonian with $H = -\iota_X(\lambda)$. □

2.3 Cotangent bundles

I don't expect to reach this section in my talk, but I think this section gives some nice intuition on why cotangent bundles are cool. The main result we'll show is that every vector field $X \in \Gamma(TN)$ naturally induces a hamiltonian vector field $X \in \Gamma(T(T^*N))$ on the cotangent bundle, via lifting of flows.

2.3.1 A quick note on maps induced by pullbacks.

When working with cotangent bundles, it can be useful to define certain maps via pullbacks. While this can be confusing at first, it does in my opinion greatly reduce the amount of notational bulk needed and some proofs will flow very naturally as a result of this.

The main setting to keep in mind is the following: Let $f : M \to N$ be a smooth map. This, for every $p \in M$, induces a smooth pullback map on a cotangent space

$$\begin{aligned} f^* : T^*N_{f(p)} &\to T^*M_p \\ \theta &\mapsto \theta \circ \mathrm{d}f_p \end{aligned}$$

These maps piece together a smooth pullback map on 1-forms

$$\begin{aligned} f^* : \Gamma(T^*N) &\to \Gamma(T^*M) \\ \theta &\mapsto \theta \circ \mathrm{d}f \end{aligned}$$

Note that for the first pullback map, the basepoint p is assumed to be clear from context.

2.3.2 Back to cotangent bundles

Let N be an arbitrary manifold and let $M = T^*N$ be its cotangent bundle. We will use $x \in N$ for points and $p \in T^*N_x$ for co-vectors. Let $\pi : T^*N \to N$ be the projection map. We can define the *tautological* 1-form (also called the *Liouville 1-form*), by

$$\lambda_{x,p} = \pi^* p = p \circ \mathrm{d}\pi_{x,p}$$

Remark. If $\theta \in \Gamma(T^*N)$ is a 1-form, then

$$\theta^* \lambda = \theta^* \, \pi^* \, \theta = (\underbrace{\pi \circ \theta}_{=\mathrm{Id}})^* \theta = \theta$$

One can show that λ is uniquely determined by this property.

The tautological 1-form λ induces an exact 2-form

$$\omega = -\,\mathrm{d}\lambda$$

Using charts, one can show that ω is non-degenerate and hence symplectic (see first talk today). In particular, cotangent bundles are exact symplectic manifolds.

As in any symplectic manifolds, every $H \in C^\infty(T^*N)$ induces a hamiltonian vector field X_H on T^*N for which H is a hamiltonian function. In what other ways could we construct a hamiltonian vector field on T^*N? Let $X \in \Gamma(TN)$ be a complete[4] vector field on N. We would like to lift X to a hamiltonian vector field on T^*N, preferably in a very natural way.

Lemma 14. *Let $f : N_1 \to N_2$ be a diffeomorphism. Then there is a canonical symplectomorphism $f_\sharp : T^*N_1 \to T^*N_2$ that commutes with π. Moreover, $f_\sharp$ also preserves λ.*

$$\begin{array}{ccc} T^*N_1 & \xrightarrow{f_\sharp} & T^*N_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ N_1 & \xrightarrow{f} & N_2 \end{array}$$

Proof. Define

$$f_\sharp = \Big(f, (f^{-1})^*\Big)$$

i.e. for $(x,p) \in T^*N_1$, we have

$$f_\sharp(x,p) = \Big(f(x), (f^{-1})^* p\Big) = \Big(f(x), p \circ (df_{x,p})^{-1}\Big)$$

To show that f preserves ω, it suffices to show that f preserves λ. We have $\pi_2 \circ f_\sharp = f \circ \pi_1$, hence

$$(f_\sharp^* \lambda)_{x,p} = \underbrace{f_\sharp^* \pi_2^*}_{(\pi_2 \circ f_\sharp)^*} (f^{-1})^* p = \pi_1^* f^* (f^{-1})^* p = \pi_1^* p = \lambda_{x,p}$$

[4]We will assume completeness for simplicity. The non-complete case can be done similarly by working locally.

As the exterior derivative commutes with pullback and $\omega = -\,\mathrm{d}\lambda$, this implies

$$f_\sharp^* \omega_2 = \omega_1$$

Moreover, $(f_\sharp)^{-1} = (f^{-1})_\sharp$, hence $f_\sharp$ is indeed a symplectomorphism. □

Let $\overline{\rho}_t$ be the flow of X. Using the lemma, we can lift this to a flow ρ_t on T^*N, given by

$$\rho_t(x,p) = \Big(\overline{\rho}_t(x), \underbrace{(\overline{\rho}_{-t})^*\, p}_{\in T^*N_{\overline{\rho_t(p)}}}\Big)$$

This flow preserves λ, hence the corresponding vector field is hamiltonian by our discussion of exact symplectic manifolds.

For a more in-depth discussion of cotangent bundles, see Section 2 of [dS06].

References

[dS06] Ana Cannas da Silva. Lectures on Symplectic Geometry. `https://people.math.ethz.ch/~acannas/Papers/lsg.pdf`, 2006. [Online; accessed 30-September-2021].

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