Symplectic Manifolds: Definition, Examples and Non-Examples

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Abstract

A symplectic manifold is a smooth manifold equipped with a closed, nondegenerate two-form. The goal of this talk is to understand this definition and to know some examples and non-examples of symplectic manifolds.

1 Symplectic Linear Algebra

Definition 1. Let V be a real vector space equipped with a bilinear form $\omega : V \times V \to \mathbb{R}$. This form ω is called **skew-symmetric** if for all $v, w \in V$ we have

$$\omega(v, w) = -\omega(w, v).$$

We will denote such pairs as (V, ω).

Recall. There is a natural identification of alternating multilinear maps $V \times \cdots \times V \to \mathbb{R}$ and elements of $\bigwedge^k V^*$. Hence we may regard ω as an element of $\bigwedge^2 V^*$ which will be useful when passing to manifolds and the wedge-product.

Definition 2. A pair (V, ω) is called **non-degenerate** or **symplectic** if for every non-zero $v \in V$ there exists $w \in V$ such that

$$\omega(v, w) \neq 0.$$

This is only one of several equivalent characterisations of nondegeneracy. Since the different viewpoints offer different insights, we will explore some of them before we move on to manifolds. The straightforward generalistion of these results to smooth manifolds will provide us with examples and non-examples of symplectic manifolds.

Definition 3. Given a real vector space V and a bilinear skew-symmetric form ω, we can define a so-called **musical map**

$$\begin{aligned} \omega^\flat &: V \to V^* \\ v &\mapsto \omega^\flat(v) = \omega(v, \cdot) = \iota_v \omega \end{aligned}$$

Proposition 4. (V, ω) *is symplectic if and only if* $\omega^\flat$ *is an isomorphism.*

Proof. Note first that $\omega^\flat$ is linear (since ω is bilinear) and that $\dim(V) = \dim(V^*)$ so that $\omega^\flat$ being an isomorphism is equivalent to $\ker(\omega^\flat) = \{0\}$. We observe that

$$\begin{aligned}\ker(\omega^\flat) &= \{v \in V : \omega(v, \cdot) = 0\} \\ &= \{v \in V : \omega(v, w) = 0 \; \forall w \in V\} \\ &= \{v \in V : \nexists w \in V \text{ such that } \omega(v, w) \neq 0\}\end{aligned}$$

and hence that ω is non-degenerate if and only if $\ker(\omega^\flat) = \{0\}$. ■

Recall. Recall the canonical identification $V \cong (V^*)^*$ defined by

$$\begin{aligned}V &\to (V^*)^* \\ v &\mapsto \eta_v\end{aligned}$$

where η_v is the element of $(V^*)^*$ acting on $\varphi \in V^*$ by $\langle \eta_v, \varphi \rangle = \langle \varphi, v \rangle \in \mathbb{R}$. In particular, any map $\epsilon : V^* \to V$ induces a map

$$\begin{aligned}\tilde{\epsilon} : V^* &\to (V^*)^* \\ \varphi &\mapsto \eta_{\epsilon(\varphi)}\end{aligned}$$

which then clearly satisfies

$$\langle \tilde{\epsilon}(\varphi), \psi \rangle = \left\langle \eta_{\epsilon(\varphi)}, \psi \right\rangle = \langle \psi, \epsilon(\varphi) \rangle .$$

Since the identification $V \cong (V^*)^*$ is canonical we will drop it from the notation after the next proposition.

This identification allows us to define a second musical map which goes the other way. This will then give us yet another equivalent criterium for a symplectic vector space.

Definition 5. Given a real vector space V and a bilinear form $\pi : V^* \times V^* \to \mathbb{R}$ i.e. $\pi \in \bigwedge^2 (V^*)^*$ on its dual space we can define another **musical map**

$$\begin{aligned}\pi^\sharp : V^* &\to (V^*)^* = V \\ \varphi &\mapsto \pi^\sharp(\varphi) = \pi(\varphi, \cdot) = \iota_\varphi \pi\end{aligned}$$

Proposition 6. (V, ω) *is symplectic if and only if there exists a unique skew-symmetric bilinear form* $\pi : V^* \times V^* \to \mathbb{R}$ *on the dual space such that as maps* $V^* \to (V^*)^* = V$ *we have*

$$\pi^\sharp = \widetilde{(\omega^\flat)^{-1}}$$

Proof. It is clear that if such a form π exists, $\omega^\flat$ is an isomorphism and hence (V, ω) is symplectic by the previous proposition.

Reversely, if (V, ω) is symplectic, the previous proposition asserts that there is a well-defined map $\pi^\sharp := (\omega^\flat)^{-1} : V^* \to (V^*)^*$. It follows that we can define (uniquely)

$$\begin{aligned} \pi : V^* \times V^* &\to \mathbb{R} \\ (\varphi, \psi) &\mapsto \left\langle \pi^\sharp(\varphi), \psi \right\rangle = \left\langle \psi, (\omega^\flat)^{-1}(\varphi) \right\rangle. \end{aligned}$$

We are left to check that π is skew-symmetric. Since $\omega^\flat$ is an isomorphism we may write $\varphi = \omega^\flat(v)$ and $\psi = \omega^\flat(w)$ for some unique $v, w \in V$. Hence

$$\begin{aligned} \pi(\varphi, \psi) &= \left\langle \psi, (\omega^\flat)^{-1}(\varphi) \right\rangle \\ &= \left\langle \omega^\flat(w), v \right\rangle \\ &= \omega(w, v) \\ &= -\omega(v, w) \\ &= -\left\langle \omega^\flat(v), w \right\rangle \\ &= -\left\langle \varphi, (\omega^\flat)^{-1}(w) \right\rangle \\ &= -\pi(\psi, \varphi) \end{aligned}$$

which proves the proposition. ■

Examples. 1. Take $V = \mathbb{R}^{2n}$ with basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ and define a symplectic form by

$$\begin{cases} \omega_0(e_i, e_j) = 0 = \omega_0(f_i, f_j) \\ \omega_0(e_i, f_j) = \delta_{ij} \end{cases} \qquad \forall i \leq j$$

We call the pair $(\mathbb{R}^{2n}, \omega_0)$ the standard symplectic vector space.

2. Take L a real vector space of dimension n and L^* its dual space. We can define a symplectic form on the direct sum $L \oplus L^*$ by setting

$$\omega(v + \varphi, w + \psi) := \langle \varphi, w \rangle - \langle \psi, v \rangle$$

3. Let H be a complex vector space together with a sesquilinear form

$$h : H \times H \to \mathbb{C}$$

such that $h(x, y) = \overline{h(y, x)}$, $h(x, x) \geq 0$ and $h(x, x) = 0$ if and only if $x = 0s$. Consider then H as a vector space over $\mathbb{R}$ and set

$$\omega(v, w) := \text{Im}(h(x, y)).$$

Note then that

- ω is skew symmetric since

$$\omega(y,x) = \operatorname{Im}(h(y,x)) = \operatorname{Im}(\overline{h(x,y)}) = -\operatorname{Im}(h(x,y)) = -\omega(x,y)$$

- and ω is nondegenerate as for $x \neq 0$ we have

$$\omega(x,ix) = \operatorname{Im}(h(x,ix)) = \operatorname{Im}(ih(x,x)) = h(x,x) \neq 0.$$

Hence $(H_{\mathbb{R}}, \omega)$ is a symplectic vector space.

Definition 7. Let $U \subset V$ be a subspace of a symplectic vector space (V, ω). The **symplectic complement** of U is

$$U^{\omega} = \{v \in V \mid \omega(v,u) = 0 \; \forall u \in U\}$$

Remark. The symplectic complement is related to the 'usual' orthogonal complement

$$U^{\perp} = \{\varphi \in V^* \mid \langle \varphi, u \rangle = 0 \; \forall u \in U\}$$

by $\omega^{\flat}(U^{\omega}) = U^{\perp}$ as $\langle \omega^{\flat}(u), v \rangle = \omega(u,v)$. It follows immediately that

$$\dim(U^{\omega}) = \dim(U^{\perp}) = \dim(V) - \dim(U).$$

Proposition 8. *$(U^{\omega})^{\omega} = U$.*

Proof. Note first that

$$\begin{aligned} \dim((U^{\omega})^{\omega}) &= \dim(V) - \dim(U^{\omega}) \\ &= \dim(V) - (\dim(V) - \dim(U)) \\ &= \dim(U) \end{aligned}$$

and then that

$$U \subset (U^{\omega})^{\omega} = \{v \in V \mid \omega(v,u) = 0 \; \forall u \in U^{\omega}\}$$

by definition of U^{ω}. ∎

This symplectic orthogonal is very important and we will use it later in the seminar to define special kinds of subspaces. For the moment however, we will just use it to prove that the first example above is essentially the only symplectic vector space there is.

Theorem 9. *Let (V, ω) be symplectic. Then there is a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that*

$$\begin{cases} \omega(e_i, e_j) = 0 = \omega(f_i, f_j) \\ \omega(e_i, f_j) = \delta_{ij} \end{cases} \qquad \forall i, j.$$

Proof. We prove this by induction and start by taking any nonzero $e_1 \in V$. Since ω is nondegenerate, there is a $f_1 \in V$ such that $\omega(e_1, f_1) \neq 0$. By rescaling we might assume (without loss of generality) that $\omega(e_1, f_1) = 1$ and set $U = \mathrm{Span}\{e_1, f_1\}$.

Suppose now that $v \in U \cap U^\omega$. Like any other element of U it can be written in the form $v = ae_1 + bf_1$. As an element of U^ω on the other hand we have

$$\begin{aligned} 0 &= \omega(v, e_1) = -b \\ 0 &= \omega(v, f_1) = a \end{aligned}$$

so that actually $v = 0$ giving $U \cap U^\omega = \{0\}$. Since $\dim(V) = \dim(U) + \dim(U^\omega)$ it follows that

$$V = U \oplus U^\omega.$$

The restriction of ω to U^ω is nondegenerate since $(U^\omega)^\omega = U$ and therefore we can repeat the argument for U^ω.

The process eventually stops as V is finite dimensional and the elements $e_i, f_i \in V$ form a basis with the desired property. ■

Corollary 10. *Any symplectic vector space has even dimension.*

Equipped with this standard form, we now look at a last equivalent way of seeing nondegeneracy.

Proposition 11. *(V, ω) is symplectic if and only if $\mathcal{L} = \frac{1}{n!}\omega^n \neq 0$ where $n = \frac{1}{2}\dim(V)$.*

Proof.

- Suppose ω is degenerate i.e. there is an element $v \in V$ such that $\iota_v\omega = 0$. Then

$$\begin{aligned} \iota_v\mathcal{L} &= \frac{1}{n!}\iota_v(\omega \wedge \cdots \wedge \omega) \\ &= \frac{n}{n!}(\iota_v\omega) \wedge \omega^{n-1} \\ &= 0 \end{aligned}$$

  and since v can be completed into a basis of V, this shows that $\mathcal{L} = 0$.

- Suppose now that (V, ω) is symplectic. We write ω in the basis which is dual to the basis constructed in the previous theorem

$$\omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*$$

  and observe that

$$\mathcal{L} = \frac{1}{n!}\omega^n = e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^* \neq 0.$$

■

2 Symplectic Manifolds

Let now M be a smooth manifold and $\omega \in \Omega^2(M)$ be a two-form on M, that is, for each $p \in M$, the map

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

is skew-symmetric bilinear and the dependence on $p \in M$ is smooth.

Definition 12. The two-form $\omega \in \Omega^2(M)$ is **symplectic** if

1. ω_p is nondegenerate for all $p \in M$ and
2. ω is closed i.e. $\mathrm{d}\omega = 0$ where d is the exterior differential.

Now we have all the ingredients:

Definition 13. A **symplectic manifold** is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Remark.

- We have seen that symplectic vector spaces must have even dimension. Since (T_pM, ω_p) is a symplectic vector space and $\dim(T_pM) = \dim(M)$, we conclude that **symplectic manifolds have even dimension.**
- For $n = \frac{1}{2}\dim(M)$, the form $\mathcal{L} = \frac{1}{n!}\omega^n$ is non-vanishing top-form and is thus a volume form. We conclude that **symplectic manifolds are orientable with canonical orientation $\mathcal{L} = \frac{1}{n!}\omega^n$ induced by ω.** $\mathcal{L}$ is called the Liouville form.
- For each $p \in M$ there is an isomorphism $\omega_p^\flat : T_pM \to T_p^*M$. These maps piece together to a smooth map

  $$\begin{aligned} \omega^\flat &: TM \to T^*M \\ X &\mapsto \omega^\beta(X) = \iota_X\omega = \omega(X, \cdot) \end{aligned}$$

  Analoguously to the discussion for vector spaces we get a skew-symmetric map

  $$\pi : T^*M \times T^*M \to C^\infty(M)$$

  Together with the exterior differential we can use this to produce a vector field out of a function via

  $$X_f := -\pi^\sharp(df)$$

  The vector fields of this type will be the topic of the second talk of today.

Examples. 1. Let $M = \mathbb{R}^{2n}$ and equip it with the standard linear coordinates $x_1, ..., x_n, y_1, ..., y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$$

is symplectic and hence $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold.

2. Let M be an orientable manifold of dimension 2 with volume form $\omega \in \Omega^2(M)$. Since ω is a topform is is clearly closed and it is also non-degenerate since it is a volume form. It follows that all orientable surfaces are symplectic manifolds e.g. the sphere S^2 admits a symplectic structure.

3. Let Q be an n-dimensional manifold and consider its cotangent bundle T^*Q with the footpoint map π. If $(U, q^1, \ldots, q^n)$ is a chart on Q, we can construct a canonical chart on $\pi^{-1}(U)$ by using that the differentials $\mathrm{d}q^i_x$ of the coordinate maps span the cotangent space T^*_xQ. Concretely, any $\xi_x \in T^*_xQ$ can be written as

$$\xi_x = \sum_{i=1}^{n} p_i(\xi_x)\mathrm{d}q^i_x \quad \text{with} \quad p_i(\xi_x) = \xi_x\left(\frac{\partial}{\partial q^i}\bigg|_x\right) \in \mathbb{R}$$

which induces a chart

$$\begin{aligned} \pi^{-1}(U) &\to \mathbb{R}^{2n} \\ \xi_x &\mapsto (q^1(x), \ldots, q^n(x), p_1(\xi_x), \ldots, p_n(\xi_x)). \end{aligned}$$

One can check that the transition maps are smooth and hence that T^*Q is a $2n$-dimensional manifold.

The **tautological form** or **Liouville 1- form** is the 1-form given by

$$\lambda = \sum_{i=1}^{n} p_i \mathrm{d}q^i.$$

We check that this is well-defined i.e. that it does not depend upon the choice of coordinates: If $(U', q'^1, \ldots, q'^n, p'_1, \ldots, p'_n)$ is another chart, then we have on $U \cap U'$

$$\lambda' = \sum_{i=1}^{n} p'_i \mathrm{d}q'^i = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial q^j}{\partial q'^i} p_j\right)\left(\sum_{k=1}^{n} \frac{\partial q'^i}{\partial q^k}\mathrm{d}q^k\right) = \sum_{i=1}^{n} p_i \mathrm{d}q^i = \lambda.$$

The **canonical symplectic form** on the cotangent bundle is then

$$\omega = -\mathrm{d}\lambda = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$$

where both closedness and non-degeneracy are straightforward by the previous results.

Non-examples.

1. We saw above that all symplectic manifolds must have even dimension. Hence no odd-dimensional manifold can be symplectic e.g. all the odd dimensional spheres S^{2n+1} do not admit symplectic structures.

2. We also saw above that symplectic manifolds are canonically oriented. Therefore no non-orientable manifold such as e.g. the Klein-bottle admits a symplectic structure.

3. Let M now be a compact manifold so that in particular we can integrate over it. Then the cohomology class $[\mathcal{L}]$ of the Liouville form $\mathcal{L} = \frac{1}{n!}\omega^n$ is non trivial. It is clearly closed as a top form so suppose it were also exact and there existed $\alpha \in \Omega^{2n-1}(M)$ such that $\mathcal{L} = \mathrm{d}\alpha$. But then we get a contradiction as $\mathcal{L}$ is a volume form and by Stokes Theorem

   $$0 \neq \int_M \mathcal{L} = \int_M \mathrm{d}\alpha = \int_{\partial M} \alpha = 0.$$

   But this implies in turn that also the cohomology class of ω itself is non-trivial. If it were exact and there existed $\beta \in \Omega^1(M)$ such that $\omega = \mathrm{d}\beta$ then also $\mathcal{L}$ would be exact as

   $$\omega^n = (\mathrm{d}\beta)^n = \mathrm{d}\big(\beta(\mathrm{d}\beta)^{n-1}\big)$$

   If follows that **for compact manifolds we need some room in the second cohomology group i.e. $H^2(M) \neq 0$ to allow for a symplectic structure.** If follows in particular that the even-dimensional spheres S^{2n} for $n > 1$ do not admit a symplectic structure.

   However, this is only a necessary condition and even under the additional assumption of an almost complex structure on the manifold, there is no guarantee for a symplectic structure to exist.

3 Why this definition?

I can really recommend readin the (short) text by Henry Cohn (which you can find here) explaining why from the point of view of classical mechanics, the definition of a symplectic manifold is exactly what we need it to be.

References

[dS04] A.C. da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2004.

[Lee03] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003.