Contact and Reeb Vector Fields

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Abstract

In this talk we will introduce contact vector fields. Just like we have seen in symplectic geometry that Hamiltonian vector fields have a correspondence to smooth functions, it's similar for contact vector fields. We will also see the flow of a contact vector field gives a smooth family of contactmorphisms. In the case that the contact structure is given by a contact form, we have a special contact vector field, called the Reeb vector field.

1 Contact Vector Fields

Recall: Let (Y, ξ) be a contact manifold, then we have:

$$\omega_\xi : \Gamma(\xi) \otimes \Gamma(\xi) \to \Gamma(TY/\xi),\ X \otimes Z \mapsto \pi([X, Z])$$

is non-degenerate on every $p \in Y$, which implies:

$$\forall X \neq 0 \in \Gamma(\xi), \text{there always exists } Z \in \Gamma(\xi) \text{ such that } [X, Z] \notin \Gamma(\xi).$$

If moreover there is a contact form $\alpha \in \Omega^1(Y)$ with $\xi = \ker \alpha$, then:
1. $\alpha \wedge (d\alpha)^{\wedge n}$ is a volume form;
2. $d\alpha_p$ is non-degenerate on ξ_p for any $p \in Y$;

Definition 1.1 (Contact Vector Fields)**.** Let (Y, ξ) be a contact manifold. We say $X \in \Gamma(TY)$ is a *contact vector field* if $\mathcal{L}_X Z \in \Gamma(\xi)$ for all vector fields $Z \in \Gamma(\xi)$.

Remark. A non zero contact vector field is never tangent to ξ.

We denote by $Tcont(Y, \xi)_{\text{id}}$ the space of all contact vector fields (read as the tangent fiber to the group of contactmorphisms). This is clearly a vector space, but it's not $C^\infty(Y)$-linear (note that $[fX, gZ] = fg[X, Z] + fX(g)Z - gZ(f)X$). Just like we have seen in symplectic geometry that there is a correspondence between smooth functions and Hamiltonian vector fields, we also have an analogue in contact geometry setting. In the following, we give an identification of $Tcont(Y, \xi)_{\text{id}}$:

Theorem 1. *Let (Y, ξ) be a contact manifold. Let*

$$\pi : Tcont(Y, \xi)_{\text{id}} \to \Gamma(TY/\xi)$$

be the restriction of the projection $\Gamma(TY) \to \Gamma(TY/\xi)$.
Then π is an isomorphism of vector spaces.

Proof. To prove the injectivity, we just need to show $\ker(\pi) = 0$.
If $\pi(X) = 0$, then X is tangent to ξ everywhere. Since X is contact, it must be 0.
To show the surjectivity, for every $\sigma \in \Gamma(TY/\xi)$ we construct a contact vector field X_σ.
Choose a random vector field Z (may not be contact) such that $\pi(Z) = \sigma$.
Define:

$$E : \Gamma(TY) \to \Gamma(TY/\xi),\ W \to \pi([Z, W]).$$

Note that Z is contact iff $E(W) = 0$ for all $W \in \Gamma(\xi)$.
Our goal is to find some $V \in \xi$ such that $E(V) = E(W)$.
Since E is $C^\infty(Y)$-linear, it suffices to find for each point $p \in Y$ a vector $V_p \in \xi_p$ such that $\pi \circ [V_p, W_p] = E(W_p)$.
This is however given by the non-degeneracy of ω_ξ from the definition of a contact structure.
Let $X_\sigma = Z - V$ then we are done. ∎

Since we have defined contact vector fields, it would be natural to think about their flows. The following definition and proposition will explain this:

Definition 1.2 (Contact Isotopies)**.** Let (Y, ξ) be a contact manifold. A contact isotopy is an isotopy through contact morphisms, i.e. a smooth 1-parameter family of contactmorphisms $\phi_t : Y \to Y$ such that:

$$\phi_0 = \mathrm{id};$$

$$\phi_{t*}(\xi) = \xi.$$

Proposition 1.1. Let ϕ_t be a contact isotopy on a contact manifold (Y, ξ). Then

$$X^t := \frac{\partial \phi_t}{\partial t} \circ \phi_t^{-1}$$

is a contact vector field for all t.
Conversely, let X^t be a smooth family of contact vector fields, then the flow of X^t is a contact isotopy.

Proof. Suppose ϕ_t is a contact isotopy, to show is $\mathcal{L}_{X^t} Z \in \xi$ for any $Z \in \xi$.
Note that

$$\mathcal{L}_{X^t} Z = \phi_{t*}^{-1} \lim_{s \to 0} \frac{\phi_{t+s*} Z - \phi_{t*} Z}{s}.$$

Since ϕ_t is a contact isotopy we have $\mathcal{L}_{X^t} Z \in \xi$.
Conversely, let ϕ_t be the flow corresponding to the vector field X^t such that

$$X^t := \frac{\partial \phi_t}{\partial t} \circ \phi_t^{-1}$$

$$\phi_0 = \mathrm{id}.$$

To show is $\phi_{t*} Z \in \xi$ for any $Z \in \xi$ and any t.
Let $c(t) = \phi_{t*} Z$. Since $c(0) \in \xi$ and $c'(t) = \mathcal{L}_{X_t} Z \in \xi$ we are done. □

From the Theorem and the Proposition above we see that contact isotopies are easy to construct. They can be locally constructed by the following corollary:

Corollary 1.1 (isotopy extension)**.** *Let (Y_1, ξ) and (Y_2, ζ) be two contact manifolds of equal dimension. Let*

$$f_t : Y_2 \to Y_1$$

be a smooth path of contact embeddings.
Then there is a contact isotopy ϕ_t on Y_1 such that

$$\phi_t \circ f_0 = f_t.$$

Furthermore, we can assume that ϕ_t is the identity outside any open set $V_t \subset Y_1$ containing the closure of $f_t(U)$.

Proof. Define $X_t : f_t(Y_2) \to TY_1$ as

$$X_t(f_t(p)) = \frac{\partial f_t(p)}{\partial t}.$$

By **Theorem** 1 X_t can be viewed as a section in $\Gamma(TY_1/\xi)|_{f_t(U)}$.
This section can be extended to all of Y_1 by cutting it off. The flow of this section defines the desired ϕ_t. □

2 From The Point of View of Contact Forms

In this section, we assume the existence of contact form. In this case we have a rather simple characterization of contact iosotopies and contact vector fields.

Proposition 2.1. Let $(Y, \ker \alpha)$ be a contact manifold with contact form α.
A diffeomorphism $\phi : Y \to Y$ is a contactomorphism if and only if

$$\phi^* \alpha = f \alpha$$

for some nowhere vanishing function $f \in C^\infty(Y)$.
A vector field X is a contact vector field if and only if

$$\mathcal{L}_X \alpha = g \alpha$$

for some (possibly zero) function $g \in C^\infty(Y)$.

Proof. Since $\alpha(\phi_* Z) = (\phi^*\alpha)Z$, ϕ is a contactmorphism iff ker $\alpha =$ ker $\phi^*\alpha$. Hence they are differed by a nowhere vanishing smooth function.
For the second statement, let Z be a vector field tangent to ξ. Then

$$0 = \mathcal{L}_X(\alpha(Z)) = (\mathcal{L}_X\alpha)(Z) + \alpha([X, Z]).$$

X being contact means that $\alpha([X, Z]) = 0$ for all $Z \in \ker\alpha$. This means that $\ker\alpha \subset \ker(\mathcal{L}_X\alpha)$, which implies the claim. □

Proposition 2.2 (Reeb Vector Field)**.** Let $(Y, \ker\alpha)$ be a contact manifold with contact form α. α induces a map

$$\alpha : Tcont(Y, \ker\alpha)_{\mathrm{id}} \to C^\infty(Y).$$

This map is an isomorphism of vector spaces.
We define the Reeb vector field R_α to be the contact vector field that is sent to the constant function $1 \in C^\infty(Y)$.

Proof. Since there exists a contact form, the line bundle TY/ξ is trivial. Therefore $C^\infty(Y) \cong \Gamma(TY/\xi)$. The statement follows immediately from **Theorem** 1. Since $\alpha : Tcont(Y, \ker\alpha)_{\mathrm{id}} \to C^\infty(Y)$ is an isomorphism, R_α is well defined. □

In the following we give an alternative definition of Reeb vector fields:

Proposition 2.3 (Alternative Definition of Reeb Vector Fields)**.** Let $(Y, \ker\alpha)$ be a contact manifold, a vector field R_α is called the Reeb vector field if it holds the following two conditions:

$$\iota_{R_\alpha}\alpha = 1;$$

$$\iota_{R_\alpha}d\alpha = 0.$$

Proof. We first prove **Proposition** 2.2 implies **Proposition** 2.3.
$\iota_{R_\alpha}\alpha = 1$ is clear from the construction. It's left to prove $\iota_{R_\alpha}d\alpha = 0$.
Indeed, by Cartan's formula we have $\iota_{R_\alpha}d\alpha = \mathcal{L}_{R_\alpha}\alpha - d\iota_{R_\alpha}\alpha = \mathcal{L}_{R_\alpha}\alpha - 0$. Since $\mathcal{L}_{R_\alpha}\alpha = g\alpha$ for some smooth function g, we have $\iota_{R_\alpha}d\alpha = 0$ on ξ. It's easy to see $\mathcal{L}_{R_\alpha}R_\alpha$ is also 0. Therefore we conclude that $\iota_{R_\alpha}d\alpha = 0$.
Conversely let's assume $\iota_X\alpha = 1$ and $\iota_X d\alpha = 0$. We just need to show X is a contact vector field.
Indeed, again by Cartan's formula $\mathcal{L}_X\alpha = d\iota_X\alpha + \iota_X d\alpha = 0$. By Proposition 2.1 and taking $g = 0$ we are done. □

Remark. From **Proposition** 2.2 we see that for each $(Y, \ker\alpha)$ there exists a unique Reeb vector field and it preserves the contact form α in the sense that $\mathcal{L}_{R_\alpha}\alpha = 0$. Note that for X to be a contact vector field we only require $\mathcal{L}_X\alpha = g\alpha$. All contact vector fields constructed as in **Corollary** 1.1 are not Reeb vector fields.

Now we give an example how Reeb vector fields are related to Hamiltonian vector fields. Recall from the previous talk:

Definition 2.1. A Liouville vector field X on a symplectic manifold (W, ω) is a vector field satisfying the equation $\mathcal{L}_X\omega = \omega$. In this case, the 1–form $\alpha := \iota_X\omega$ is a contact form on any hypersurface H transverse to X. Such hypersurfaces are said to be of contact type.

Lemma. If a codimension 1 submanifold $N \subset T^*W$ is both a hypersurface of contact type with contact form $\alpha = \iota_X\omega$ for some Liouville vector field X (ω is the symplectic form on T^*W) and the level set of a Hamiltonian function $H : T^*W \to \mathbb{R}$, then the Reeb vector field R_α and the Hamiltonian vector field agree up to scaling.

Proof. Note that $d\alpha = d\iota_X\omega = \mathcal{L}_X\omega = \omega$. On each fiber of TN it has a kernel of dimension 1. This kernel is defined both by the Reeb vector field R_α and the Hamiltonian vector field X_H as following:

$$\iota_{R_\alpha}d\alpha|_{TN} = 0;$$

and

$$\iota_{X_H}d\alpha|_{TN} = -dH|_{TN} = 0.$$

Hence R_α and X_H are differed only by scaling. □

3 Examples

1. Consider $(Y, \ker\alpha)$ given by $Y = \mathbb{R}^{2n+1}$ and $\alpha = \sum_i x_i dy_i + dz$.
   Then $R_\alpha = \frac{\partial}{\partial z}$ is the corresponding Reeb vector field.
   Recall **Proposition** 1.1 we have the corresponding contact isotopy
   $$\phi_t^R(x_1, y_1, .., x_n, y_n, z) = (x_1, y_1, .., x_n, y_n, z + t).$$
   We also give an example of contact vector field which is not a Reeb vector field:
   $$X = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}$$
   with $\mathcal{L}_X\alpha = \alpha$.

2. Consider the odd sphere $S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ as
   $$\{(x_1, y_1, ..., x_n, y_n) | \sum(x_i^2 + y_i^2) = 1\}.$$
   Consider the 1-form σ on $\mathbb{R}^{2n}$,
   $$\sigma = \frac{1}{2}\sum(x_i dy_i - y_i dx_i).$$
   Let $\alpha = i^*\sigma$.
   Claim: α is a contact form on S^{2n-1}.

   Proof. We just need to show
   $$\alpha \wedge (d\alpha)^{\wedge(n-1)} \neq 0.$$
   Consider the 1-form $\nu \in \Gamma(T^*\mathbb{R}^{2n})$ given by $d\sum(x_i^2 + y_i^2)$. It holds that $T_pS^{2n-1} = \ker\nu_p$ at $p \in S^{2n-1}$. Verify that $\nu \wedge \sigma \wedge (d\sigma)^{\wedge(n-1)}$. □

   The distribution $\xi = \ker\alpha$ is called the *standard contact structure* on S^{2n-1}. The Reeb vector field is given by $R_\alpha = 2\sum(x_i\frac{\partial}{\partial y_i} - y_i\frac{\partial}{\partial x_i})$ and is also known as the *Hopf vector filed* on S^{2n-1}.

3. (Cogeodesic Flow)[1]
   Let (M, g) be a Riemannian manifold. The Riemannian metric g on M allows us to define a bundle isomorphism Ψ from the tangent bundle TM to the cotangent bundle T^*M, which is fiberwise given by:
   $$\Psi_p : T_pM \to T_p^*M,\ V \mapsto g_p(V, -).$$
   This induces a bundle metric g^* on T^*B, defined by
   $$g_p^*(u_1, u_2) = g_p(\Psi_p^{-1}(u_1), \Psi_p^{-1}(u_2))$$
   for $u_1, u_2 \in T_p^*M$.
   The unit tangent bundle STM is defined fiberwise by
   $$ST_pM = \{X \in T_pM : g_p(X, X) = 1\};$$
   likewise the unit cotangent bundle ST^*M is defined in terms of g^*.

   Definition 3.1 (Geodesic Flows and Geodesic Fields)**.** Let (M, g) be a Riemmanian manifold. There is a unique vector field G on the tangent bundle TM whose trajectories are of the form
   $$t \mapsto \frac{\partial\gamma}{\partial t} \in T_{\gamma(t)}M \subset TM,$$
   where γ is a geodesic on M.
   This vector field G is called the *geodesic field*, and its flow the *geodesic flow*.

   Theorem 2. *Let (M, g) be a Riemannian manifold.*
   (a) The Liouville form λ on the cotangent bundle T^*M induces a contact form on the unit cotangent bundle ST^*M. The Reeb vector field R_λ of this contact form is dual to the geodesic vector field G in the sense that
   $$\Gamma\Psi(G) = R_\lambda.$$
   (b) Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian function defined by
   $$H(u) = \frac{1}{2}g^*(u, u).$$
   Then along $ST^*M = H^{-1}(2)$, the Reeb vector field R_λ equals the Hamiltonian vector field X_H (with respect to the symplectic form $\omega = d\lambda$ on T^*M).

   Remark. The flow of $\Gamma\Psi(G)$ on ST^*M is called the cogeodesic flow. The theorem says that the cogeodesic flow is equivalent both to the Reeb flow of λ and the Hamiltonian flow of H on ST^*M.

References

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