Contact and Reeb Vector Fields

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Abstract

In this talk we will introduce contact vector fields. Just like we have seen in symplectic geometry that Hamiltonian vector fields have a correspondence to smooth functions, it’s similar for contact vector fields. We will also see the flow of a contact vector field gives a smooth family of contactmorphisms. In the case that the contact structure is given by a contact form, we have a special contact vector field, called the Reeb vector field.

1 Contact Vector Fields

Recall: Let \((Y, \xi)\) be a contact manifold, then we have:

\[\omega_{\xi} : \Gamma(\xi) \otimes \Gamma(\xi) \to \Gamma(TY/\xi), \; X \otimes Z \mapsto \pi([X, Z])\]

is non-degenerate on every \(p \in Y\), which implies:

\[\forall X \neq 0 \in \Gamma(\xi), \; \text{there always exists } Z \in \Gamma(\xi) \text{ such that } [X, Z] \notin \Gamma(\xi).\]

If moreover there is a contact form \(\alpha \in \Omega^1(Y)\) with \(\xi = \ker \alpha\), then:

1. \(\alpha \wedge (d\alpha)^\wedge n\) is a volume form;
2. \(d\alpha_p\) is non-degenerate on \(\xi_p\) for any \(p \in Y\);

Definition 1.1 (Contact Vector Fields). Let \((Y, \xi)\) be a contact manifold. We say \(X \in \Gamma(TY)\) is a contact vector field if \(L_XZ \in \Gamma(\xi)\) for all vector fields \(Z \in \Gamma(\xi)\).

Remark. A non zero contact vector field is never tangent to \(\xi\).

We denote by \(T_{\text{cont}}(Y, \xi)_{\text{id}}\) the space of all contact vector fields (read as the tangent fiber to the group of contactmorphisms). This is clearly a vector space, but it’s not \(C^\infty(Y)\)-linear (note that \([fX, gZ] = fg[X, Z] + fX(g)Z - gZ(f)X\)). Just like we have seen in symplectic geometry that there is a correspondence between smooth functions and Hamiltonian vector fields, we also have an analogue in contact geometry setting. In the following, we give an identification of \(T_{\text{cont}}(Y, \xi)_{\text{id}}\):

Theorem 1. Let \((Y, \xi)\) be a contact manifold. Let

\[\pi : T_{\text{cont}}(Y, \xi)_{\text{id}} \to \Gamma(TY/\xi)\]

be the restriction of the projection \(\Gamma(TY) \to \Gamma(TY/\xi)\).

Then \(\pi\) is an isomorphism of vector spaces.

Proof. To prove the injectivity, we just need to show \(\ker(\pi) = 0\).

If \(\pi(X) = 0\), then \(X\) is tangent to \(\xi\) everywhere. Since \(X\) is contact, it must be 0.

To show the surjectivity, for every \(\sigma \in \Gamma(TY/\xi)\) we construct a contact vector field \(X_\sigma\).

Choose a random vector field \(Z\) (may not be contact) such that \(\pi(Z) = \sigma\).

Define:

\[E : \Gamma(TY) \to \Gamma(TY/\xi), \; W \mapsto \pi([Z, W]).\]

Note that \(Z\) is contact if \(E(W) = 0\) for all \(W \in \Gamma(\xi)\).

Our goal is to find some \(V \in \xi\) such that \(E(V) = E(W)\).

Since \(E\) is \(C^\infty(Y)\)-linear, it suffices to find for each point \(p \in Y\) a vector \(V_p \in \xi_p\) such that \(\pi \circ [V_p, W_p] = E(W_p)\).

This is however given by the non-degeneracy of \(\omega_{\xi}\) from the definition of a contact structure.

Let \(X_\sigma = Z - V\) then we are done.
Since we have defined contact vector fields, it would be natural to think about their flows. The following definition and proposition will explain this:

**Definition 1.2 (Contact Isotopies).** Let \((Y, \xi)\) be a contact manifold. A contact isotopy is an isotopy through contact morphisms, i.e. a smooth 1-parameter family of contact morphisms \(\phi_t : Y \to Y\) such that:

\[
\phi_0 = \text{id}; \quad \phi_{t*}(\xi) = \xi.
\]

**Proposition 1.1.** Let \(\phi_t\) be a contact isotopy on a contact manifold \((Y, \xi)\). Then

\[
X^t := \frac{\partial \phi_t}{\partial t} \circ \phi_t^{-1}
\]

is a contact vector field for all \(t\).

Conversely, let \(X_t\) be a smooth family of contact vector fields, then the flow of \(X_t\) is a contact isotopy.

**Proof.** Suppose \(\phi_t\) is a contact isotopy, to show is \(\mathcal{L}_{X^t} Z \in \xi\) for any \(Z \in \xi\).

Note that

\[
\mathcal{L}_{X^t} Z = \phi_{t*}^{-1} \lim_{s \to 0} \phi_{t+s*} Z - \phi_{t*} Z.
\]

Since \(\phi_s\) is a contact isotopy we have \(\mathcal{L}_{X^t} Z \in \xi\).

Conversely, let \(\phi_t\) be the flow corresponding to the vector field \(X_t\) such that

\[
\phi_0 = \text{id}.
\]

To show is \(\phi_{t*} Z \in \xi\) for any \(Z \in \xi\) and any \(t\).

Let \(c(t) = \phi_{t*} Z\). Since \(c(0) \in \xi\) and \(c'(t) = \mathcal{L}_{X_t} Z \in \xi\) we are done. \(\square\)

From the Theorem and the Proposition above we see that contact isotopies are easy to construct. They can be locally constructed by the following corollary:

**Corollary 1.1 (isotopy extension).** Let \((Y_1, \xi)\) and \((Y_2, \zeta)\) be two contact manifolds of equal dimension. Let \(f : Y_2 \to Y_1\)

be a smooth path of contact embeddings.

Then there is a contact isotopy \(\phi_t\) on \(Y_1\) such that

\[
\phi_t \circ f_0 = f_t.
\]

Furthermore, we can assume that \(\phi_t\) is the identity outside any open set \(V_t \subset Y_1\) containing the closure of \(f_t(U)\).

**Proof.** Define \(X_t : f_t(Y_2) \to TY_1\) as

\[
X_t(f_t(p)) = \frac{\partial f_t(p)}{\partial t}.
\]

By Theorem, \(X_t\) can be viewed as a section in \(\Gamma(TY_1/\xi)|_{f_t(U)}\).

This section can be extended to all of \(Y_1\) by cutting it off. The flow of this section defines the desired \(\phi_t\). \(\square\)

## 2 From The Point of View of Contact Forms

In this section, we assume the existence of contact form. In this case we have a rather simple characterization of contact isotopies and contact vector fields.

**Proposition 2.1.** Let \((Y, \ker \alpha)\) be a contact manifold with contact form \(\alpha\).

A diffeomorphism \(\phi : Y \to Y\) is a contactomorphism if and only if

\[
\phi^* \alpha = f \alpha
\]

for some nowhere vanishing function \(f \in C^\infty(Y)\).

A vector field \(X\) is a contact vector field if and only if

\[
\mathcal{L}_X \alpha = g \alpha
\]

for some (possibly zero) function \(g \in C^\infty(Y)\).
Proof. Since \( \alpha(\phi, Z) = (\phi^*\alpha)Z \), \( \phi \) is a contactomorphism iff \( \ker \alpha = \ker \phi^*\alpha \). Hence they are differed by a nowhere vanishing smooth function.

For the second statement, let \( Z \) be a vector field tangent to \( \xi \). Then
\[
0 = \mathcal{L}_X(\alpha(Z)) = (\mathcal{L}_X \alpha)(Z) + \alpha([X, Z]).
\]

\( X \) being contact means that \( \alpha([X, Z]) = 0 \) for all \( Z \in \ker \alpha \). This means that \( \ker \alpha \subset \ker(\mathcal{L}_X \alpha) \), which implies the claim. \( \square \)

**Proposition 2.2** (Reeb Vector Field). Let \((Y, \ker \alpha)\) be a contact manifold with contact form \( \alpha \). \( \alpha \) induces a map
\[
\alpha : T\text{cont}(Y, \ker \alpha)_{id} \rightarrow C^\infty(Y).
\]
This map is an isomorphism of vector spaces.

We define the Reeb vector field \( R_\alpha \) to be the contact vector field that is sent to the constant function \( 1 \in C^\infty(Y) \).

**Proof.** Since there exists a contact form, the line bundle \( TY/\xi \) is trivial. Therefore \( C^\infty(Y) \cong \Gamma(TY/\xi) \). The statement follows immediately from **Theorem 1**. Since \( \alpha : T\text{cont}(Y, \ker \alpha)_{id} \rightarrow C^\infty(Y) \) is an isomorphism, \( R_\alpha \) is well defined. \( \square \)

In the following we give an alternative definition of Reeb vector fields:

**Proposition 2.3** (Alternative Definition of Reeb Vector Fields). Let \((Y, \ker \alpha)\) be a contact manifold, a vector field \( R_\alpha \) is called the Reeb vector field if it holds the following two conditions:
\[
\iota_{R_\alpha} \alpha = 1;
\]
\[
\iota_{R_\alpha} d\alpha = 0.
\]

**Proof.** We first prove **Proposition 2.2** implies **Proposition 2.3**. 
\( \iota_{R_\alpha} \alpha = 1 \) is clear from the construction. It’s left to prove \( \iota_{R_\alpha} d\alpha = 0 \).

Indeed, by Cartan’s formula we have \( \iota_{R_\alpha} d\alpha = \mathcal{L}_{R_\alpha} \alpha - dR_\alpha \alpha = \mathcal{L}_{R_\alpha} \alpha - 0 \). Since \( \mathcal{L}_{R_\alpha} \alpha = g \alpha \) for some smooth function \( g \), we have \( \iota_{R_\alpha} d\alpha = 0 \) on \( \xi \). It’s easy to see \( \mathcal{L}_{R_\alpha} R_\alpha \) is also 0. Therefore we conclude that \( \iota_{R_\alpha} d\alpha = 0 \).

Conversely let’s assume \( \iota_X \alpha = 1 \) and \( \iota_X d\alpha = 0 \). We just need to show \( X \) is a contact vector field.

Indeed, again by Cartan’s formula \( \mathcal{L}_X \alpha = d\iota_X \alpha + \iota_X d\alpha = 0 \). By **Proposition 2.1** and taking \( g = 0 \) we are done. \( \square \)

**Remark.** From **Proposition 2.2** we see that for each \((Y, \ker \alpha)\) there exists a unique Reeb vector field and it preserves the contact form \( \alpha \) in the sense that \( \mathcal{L}_{R_\alpha} \alpha = 0 \). Note that for \( X \) to be a contact vector field we only require \( \mathcal{L}_X \alpha = g \alpha \). All contact vector fields constructed as in **Corollary 1.1** are not Reeb vector fields.

Now we give an example how Reeb vector fields are related to Hamiltonian vector fields. Recall from the previous talk:

**Definition 2.1.** A Liouville vector field \( X \) on a symplectic manifold \((W, \omega)\) is a vector field satisfying the equation \( \mathcal{L}_X \omega = \omega \). In this case, the 1-form \( \alpha := i_X \omega \) is a contact form on any hypersurface \( H \) transverse to \( X \). Such hypersurfaces are said to be of contact type.

**Lemma.** If a codimension 1 submanifold \( N \subset T^*W \) is both a hypersurface of contact type with contact form \( \alpha = i_X \omega \) for some Liouville vector field \( X \) (\( \omega \) is the symplectic form on \( T^*W \)) and the level set of a Hamiltonian function \( H : T^*W \rightarrow \mathbb{R} \), then the Reeb vector field \( R_\alpha \) and the Hamiltonian vector field \( X_H \) agree up to scaling.

**Proof.** Note that \( d\alpha = d(i_X \omega) = \mathcal{L}_X \omega = \omega \). On each fiber of \( TN \) it has a kernel of dimension 1. This kernel is defined both by the Reeb vector field \( R_\alpha \) and the Hamiltonian vector field \( X_H \) as following:
\[
\iota_{R_\alpha} d\alpha|_{TN} = 0;
\]
and
\[
\iota_{X_H} d\alpha|_{TN} = -dH|_{TN} = 0.
\]
Hence \( R_\alpha \) and \( X_H \) are differed only by scaling. \( \square \)


3 Examples

1. Consider \((Y, \ker \alpha)\) given by \(Y = \mathbb{R}^{2n+1}\) and \(\alpha = \sum x_i \, dy_i + dz\).
   Then \(R_\alpha = \sum \frac{x_i}{y_i} \) is the corresponding Reeb vector field.

   Recall Proposition \([1]\) we have the corresponding contact isotopy
   \[
   \phi^p_t(x_1, y_1, \ldots, x_n, y_n, z) = (x_1, y_1, \ldots, x_n, y_n, z + t).
   \]
   We also give an example of contact vector field which is not a Reeb vector field:
   \[
   X = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}
   \]
   with \(L_X \alpha = \alpha\).

2. Consider the odd sphere \(S^{2n-1} \hookrightarrow \mathbb{R}^{2n}\) as
   \[
   \{(x_1, y_1, \ldots, x_n, y_n) | \sum (x_i^2 + y_i^2) = 1\}.
   \]
   Consider the 1-form \(\sigma\) on \(\mathbb{R}^{2n}\),
   \[
   \sigma = \frac{1}{2} \sum (x_i \, dy_i - y_i \, dx_i).
   \]
   Let \(\alpha = \iota^* \sigma\).
   **Claim:** \(\alpha\) is a contact form on \(S^{2n-1}\).
   **Proof.** We just need to show
   \[
   \alpha \wedge (d\alpha)^{(n-1)} \neq 0.
   \]

   Consider the 1-form \(\nu \in \Gamma(T^* \mathbb{R}^{2n})\) given by \(d \sum (x_i^2 + y_i^2)\). It holds that \(T_p \mathbb{R}^{2n-1} = \ker \nu_p\) at \(p \in S^{2n-1}\).

   Verify that \(\nu \wedge \sigma \wedge (d\sigma)^{(n-1)}\).

   The distribution \(\xi = \ker \alpha\) is called the standard contact structure on \(S^{2n-1}\). The Reeb vector field is given by \(R_\alpha = \sum (x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i})\) and is also known as the Hopf vector field on \(S^{2n-1}\).

3. (Cogeodesic Flow)\([2]\)
   Let \((M, g)\) be a Riemannian manifold. The Riemannian metric \(g\) on \(M\) allows us to define a bundle isomorphism \(\Psi\) from the tangent bundle \(TM\) to the cotangent bundle \(T^* M\), which is fiberwise given by:
   \[
   \Psi_p : T_p M \rightarrow T_p^* M, \; V \mapsto g_p(V, -).
   \]
   This induces a bundle metric \(g^*\) on \(T^* B\), defined by
   \[
   g_p^*(u_1, u_2) = g_p(\Psi_p^{-1}(u_1), \Psi_p^{-1}(u_2))
   \]
   for \(u_1, u_2 \in T_p^* M\).

   The unit tangent bundle \(STM\) is defined fiberwise by
   \[
   ST_p M = \{X \in T_p M : g_p(X, X) = 1\};
   \]
   likewise the unit cotangent bundle \(ST^* M\) is defined in terms of \(g^*\).

   **Definition 3.1** (Geodesic Flows and Geodesic Fields). Let \((M, g)\) be a Riemannian manifold. There is a unique vector field \(G\) on the tangent bundle \(TM\) whose trajectories are of the form
   \[
   t \mapsto \frac{\partial \gamma}{\partial t} \in T_{\gamma(t)} M \subset TM,
   \]
   where \(\gamma\) is a geodesic on \(M\).

   This vector field \(G\) is called the geodesic field, and its flow the geodesic flow.

   **Theorem 2.** Let \((M, g)\) be a Riemannian manifold.
   (a) The Liouville form \(\lambda\) on the cotangent bundle \(T^* M\) induces a contact form on the unit cotangent bundle \(ST^* M\). The Reeb vector field \(R_\lambda\) of this contact form is dual to the geodesic vector field \(G\) in the sense that
   \[
   \Gamma \Psi(G) = R_\lambda.
   \]

   (b) Let \(H : T^* M \rightarrow \mathbb{R}\) be the Hamiltonian function defined by
   \[
   H(u) = \frac{1}{2} g^*(u, u).
   \]

   Then along \(ST^* M = H^{-1}(2)\), the Reeb vector field \(R_\lambda\) equals the Hamiltonian vector field \(X_H\) (with respect to the symplectic form \(\omega = d\lambda\) on \(T^* M\)).

   **Remark.** The flow of \(\Gamma \Psi(G)\) on \(ST^* M\) is called the cogeodesic flow. The theorem says that the cogeodesic flow is equivalent both to the Reeb flow of \(\lambda\) and the Hamiltonian flow of \(H\) on \(ST^* M\).
References

