# Darboux's Theorem and Moser's Argument

Frederik Semmel

**Abstract**

The goal of this weeks student seminar is to present the famous Moser's Argument. It will be used to prove the Darboux-Weinstein Theorem which shows that a symplectic form is locally determined up to symplectomorphism by its value on a submanifold. These notes contain most of the prerequisites needed to explain Moser's Trick and do the proof.

## 1 Introduction

Let us first state the theorem which we are discussing in this weeks student seminar, formulated as in the Lectures on Symplectic Geometry [Can01].

**Theorem 1.1** (Darboux Weinstein Theorem or Relative Moser Theorem)**.** *Let $Q$ be a compact submanifold of $M$, $i : Q \to M$ the inclusion map, $\omega_0$ and $\omega_1$ symplectic forms on $M$ with $\omega_0|_p = \omega_1|_p$ for every $p \in Q$. Then there exist neighborhoods $\mathcal{U}_0$, $\mathcal{U}_1$ of $Q$ in $M$ and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\varphi|_Q = \mathrm{id}_Q \quad \textit{and} \quad \varphi^*\omega_1 = \omega_0$$

In particular, this theorem encompasses the case $Q = \{p\} \subset M$ and can be used to prove the original Darboux's Theorem.

**Theorem 1.2** (Darboux's Theorem)**.** *Let $(M, \omega)$ be a symplectic manifold, and let $p$ be any point in $M$. Then we can find a coordinate system $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at $p$ such that on $U$*

$$\omega = \sum dx_i \wedge dy_i.$$

One only needs to find a chart $(U, x'_1, \ldots, x'_n, y'_1, \ldots, y'_n) = (U, x', y')$ around $p$ such that

$$\sum dx'_i \wedge dy'_i \Big|_p = \omega|_p.$$

Then the Darboux Weinstein Theorem provides the diffeomorphism $\varphi$, which maps the standard symplectic form corresponding to the primed coordinates onto $\omega$:

$$\omega = \varphi^* \left( \sum dx'_i \wedge dy'_i \right) = \sum d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$$

On an appropriate open set $V$ the coordinate functions $x_i = x'_i \circ \varphi$ and $y_i = y'_i \circ \varphi$ form the desired chart from the theorem.

## 2 Standard Form in Vector Spaces

We discuss the standard form on vector spaces to understand why it is possible to find a the chart $(U, x', y')$ in the proof of the Darboux's Theorem. The idea is that $\omega_p$ is a non-degenerate skew-symmetric bilinear map, and hence it is possible to find a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of $T_pM$ such that $\omega_p$ is in normal form with respect to this basis. This basis of $T_pM$ can then be extended to a chart $(U, x', y')$ by only requiring that

$$\frac{\partial}{\partial x'_i}\Big|_p = e_i \quad \text{and} \quad \frac{\partial}{\partial y'_i}\Big|_p = f_i,$$

which is always possible.

**Theorem 2.1** (Standard Form for Skew-symmetric Bilinear Maps)**.** *Let $V$ be an $m$-dimensional vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. Assume that the map $\Omega$ is skew-symmetric, i.e., $\Omega(u, v) = -\Omega(v, u)$ for all $u, v \in V$.*

*Then there is a basis $(u_1, \dots, u_l, e_1, \dots, e_n, f_1, \dots, f_n)$ of $V$ such that*

$$\begin{aligned}
&\Omega(u_i, v) = 0, && \text{for all } i \text{ and all } v \in V \\
&\Omega(e_j, e_k) = 0 = \Omega(f_j, f_k), && \text{for all } j, k \\
&\Omega(e_j, f_k) = \delta_{jk}, && \text{for all } j, k
\end{aligned}$$

A very similar result was proven in the student seminar from week 3 by Reto Kaufmann, which covered the case of a symplectic vector space. The proof for this extended theorem is identical, except for the first step, which deals with the possible degeneracy of the bilinear map.

*Proof.* Let $U = \{u \in V \mid \Omega(u, v) = 0 \quad \forall v \in V\}$ and choose any basis $u_1, \dots, u_l$ of $U$.

This is a proof by induction on $n$ that constructs the basis elements $e_1, \dots, e_n, f_1, \dots, f_n$. The following are the induction assumptions:

1) There are subspaces $W_j = \operatorname{span}(e_j, f_j)$ for vectors $e_j, f_j$ in $V$ with $\Omega(e_j, f_j) = 1$ for $j = 1, \dots, k$.

2) There is a subspace $R_k \neq \{0\}$ such that $V = U \oplus W_1 \oplus \cdots \oplus W_k \oplus R_k$ and all the summands are pairwise orthogonal with respect to $\Omega$.

3) $\Omega$ is non-degenerate on $R_k$

For the initial step at $k = 0$, choose $R_0$ any subspace with $V = U \oplus R_0$. $\Omega$ cannot be degenerate on $R_0$ because all the degenerate directions are captured in $U$.

Now let the assumptions hold for some $k - 1 \in \mathbb{N}$. As $\Omega$ is non-degenerate on $R_{k-1}$ and $R_{k-1} \neq \{0\}$ we find vectors $e_k$ and $f_k$ in $R_{k-1}$ with $\Omega(e_k, f_k) = 1$.

Define:

$$\begin{aligned}
W_k &:= \operatorname{span}(e_k, f_k) \\
R_k &:= \{w \in R_{k-1} \mid \Omega(w, v) = 0 \quad \forall v \in W_k\}
\end{aligned}$$

If $R_k = \{0\}$ then the construction of the basis is finished, so we can assume $R_k \neq \{0\}$.

Assumption 1) is satisfied by definition of $W_k$ and the vectors $e_k$ and $f_k$. As $R_k$ is orthogonal to $W_k$, we only need $R_{k-1} = W_k \oplus R_k$ for assumption 2) to hold. Let $v \in W_k \cap R_k$. Then $v = ae_k + bf_k$. As $v \in R_k$ we have

$$\Omega(v, e_k) = 0 = -b$$
$$\Omega(v, f_k) = 0 = a$$

which forces $v = 0$, proving $W_k \cap R_k = \{0\}$.

Notice that $R_k \supseteq \ker(\Omega(\cdot, e_k)) \cap \ker(\Omega(\cdot, f_k))$. These kernels are hyperplanes in $R_{k-1}$ meaning that

$$\dim(\ker(\Omega(\cdot, e_k))) = \dim(R_{k-1}) - 1.$$

And therefore for the intersection these two hyperplanes we have

$$\dim(R_k) \geq \dim(R_{k-1}) - 2.$$

Because we know that $\dim(W_k) = 2$ the assumption 2) holds also for $k$.

Finally assumption 3) is satisfied via the following argument. Suppose towards contradiction that there is a $0 \neq w \in R_k$ with $\Omega(w, w') = 0$ for any $w' \in R_k$.

Then let $v \in V$ be arbitrary. By assumption 2) for $k$ we can write

$$v = au + b^1 w_1 + \cdots + b^k w_k + cw'$$

for some $u \in U$, $w_i \in W_i$ and $w' \in R_k$. As all these subspaces are orthogonal to each other we get $\Omega(w, v) = 0$. This is a contradiction as then $w$ should be in $U$.

The process eventually stops because at each induction step we reduce the dimension of $R_k$ by 2 and $V$ was finite dimensional. □

This theorem is the reason why we can only find non-degenerate skew-symmetric bilinear forms on even-dimensional vector spaces. Intuitively this makes sense, as we are trying to find a basis for $V$ such that the matrix representation of $\Omega$ is

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}.$$

This matrix can only be of full rank if $\dim V$ is even.

## 3 Darboux-Weinstein Theorem and Preliminaries

The question we are trying to answer is the following. If two symplectic forms on $M$ are equal on a subset of $M$, is there a symplectomorphism on neighborhoods of this subset? We will impose the additional constraint that the aforementioned subset is a compact submanifold, then the answer is yes. The compactness constraint can be somewhat relaxed, but it simplifies the proof and we will only need this version for the remainder of the seminar.

Figure 1 shows an example of what the goal is for a 2-dimensional manifold. Here the question translates to: Given any two volume forms of a 2-dimensional Manifold can one find an area preserving diffeomorphism around two neighborhoods of $Q$?

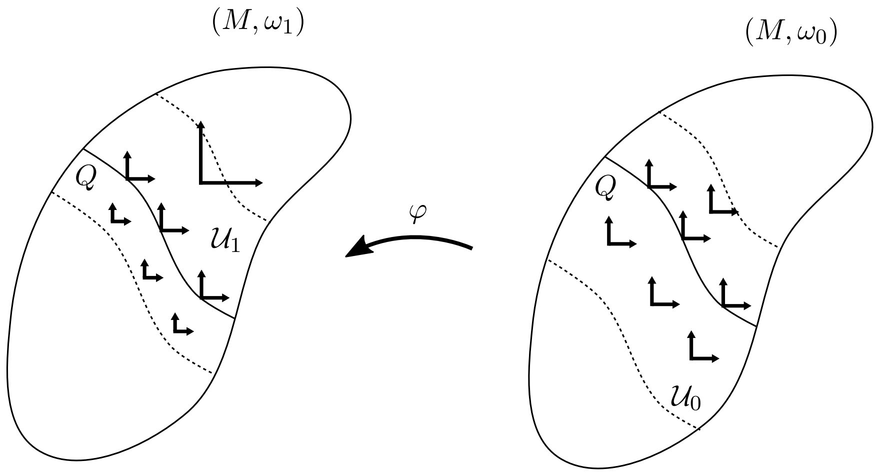


Figure 1: This picture shows an area preserving diffeomoprhism around the submanifold Q. The arrows represent two tangent vectors $v_1, v_2$ that span a paralellogram of unit area, i.e. $\omega(v_1, v_2) = 1$

The proof needs a few preliminaries which will be explained in this text. The proof itself, using Moser's Trick, will be presented in the next student seminar. We will build upon the following tools from Differential Geometry.

**Theorem 3.1** (Cartan's Magic Formula)**.** *Let $X \in \mathfrak{X}(M)$. Then*

$$\mathcal{L}_X = d \circ i_X + i_X \circ d$$

We will think of the Lie derivative as the infinitesimal flow via the following characterization.

**Proposition 3.2.** *Let $X \in \mathfrak{X}(M)$ with flow $\Phi_t$. Then for any $\omega \in \Omega(M)$ the Lie derivative is given by:*

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} \Phi_t^* w$$

The proof for 3.1 and 3.2 can be found in [Mer19]

A central tool for Moser's Trick are time dependent vector fields and isotopies.

**Definition 3.3.** *A time dependent vector field on $M$ is a smooth map $X : M \times \mathbb{R} \to TM$ such that*

$$X(t, p) = X_t(p) \in T_pM$$

There is a one-to-one relationship between time dependent vector fields defined on the whole of $M \times \mathbb{R}$ and isotopies, when $M$ is compact.

**Definition 3.4.** *An isotopy is a map $\rho : M \times \mathbb{R} \to M$ such that $\rho_t := \rho(\cdot, t)$ is a diffeomorphism for every $t \in \mathbb{R}$ and $\rho_o = \mathrm{id}_M$.*

As we will concentrate on compact spaces it is enough to force them to be defined on the whole of $M \times \mathbb{R}$.

**Example 3.5.** *A flow of a complete vector field is an isotopy with the added property that $\rho_{t+s} = \rho_t \circ \rho_s$.*

An isotopy $\rho$ always generates a time dependent vector field defined on $\Omega = M \times \mathbb{R}$ by considering the velocity vectors of the curves $t \mapsto \rho(p, t)$ for any $p \in M$.

$$\begin{aligned} X_t(p) &= \frac{d}{ds}\Big|_{s=t} \rho_s(\rho_t^{-1}(p)) \\ &= \frac{d}{ds}\Big|_{s=0} \rho_{t+s}(\rho_t^{-1}(p)) \end{aligned}$$

Similar to the way flows and vector fields have a one-to-one relationship, isotopies and time dependent vector fields behave similarly. If we restrict ourselves to compact manifolds, then any time dependent vector field can be integrated to an isotopy by solving a non-autonomous ODE. This doesn't require any new proof, as one can use the proof of the integral curves uniqueness and existence of local flow of time independent vector fields by defining a vector field $\tilde{X} \in \mathfrak{X}(M \times \mathbb{R})$ via

$$\tilde{X}(p, t) := X_t(p) + \frac{\partial}{\partial t}\Big|_{(p,t)}.$$

Another fact from Differential Geometry which we will use is the following formula for the Lie derivative of time dependent vector fields.

**Proposition 3.6.** *Let $X_t$ be a time dependent vector field on $M$, then*

$$\mathcal{L}_{X_t}\omega = \frac{d}{ds}\Big|_{s=0} (\rho_{t+s} \circ \rho_t^{-1})^* \omega.$$

With this we can proof the following formula, which is one of the key steps in Moser's Trick.

**Proposition 3.7.** *Let $\omega_t$ be a smooth family of d-forms and $\rho_t$ an isotopy with the corresponding time dependent vector field $X_t$. Then*

$$\frac{d}{dt}\rho_t^* \omega_t = \rho_t^* \left( \mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt} \right)$$

Note that here we are using the notation $\frac{d\omega_t}{dt} = \frac{d}{ds}\big|_{s=t}\omega_s = \lim_{s \to t} \frac{\omega_s - \omega_t}{s-t}$, where this limit is defined pointwise.

*Proof.* First, if we fix some $p \in M$ and $\xi_1, \dots, \xi_d \in T_pM$ then

$$f(x,y) = \rho_x^* \omega_y(p)(\xi_1, \dots, \xi_d)$$

is a real function of two variables, hence we can apply the chain rule:

$$\frac{d}{ds} f(s,s) = \frac{d}{dx}\Big|_{x=t} f(x,t) + \frac{d}{dy}\Big|_{y=t} f(t,y)$$

Dropping the function application at $p$ and $\xi_1, \dots, \xi_d$ to save space, we can calculate:

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=t} \rho_s^* \omega_s &= \frac{d}{dx}\Big|_{x=t} \rho_x^* \omega_t + \frac{d}{dy}\Big|_{y=t} \rho_t^* \omega_y \\
&= \frac{d}{dx}\Big|_{x=0} \left(\rho_{t+x} \circ \rho_t^{-1} \circ \rho_t\right)^* \omega_t + \rho_t^* \frac{d\omega_t}{dt} \\
&= \rho_t^* \left[ \frac{d}{dx}\Big|_{x=0} \left(\rho_{t+x} \circ \rho_t^{-1}\right)^* \omega_t \right] + \rho_t^* \frac{d\omega_t}{dt} \\
&= \rho_t^* \mathcal{L}_{X_t} \omega_t + \rho_t^* \frac{d\omega_t}{dt} \\
&= \rho_t^* \left( \mathcal{L}_{X_t} \omega_t + \frac{d\omega_t}{dt} \right)
\end{aligned}$$

□

To come back to the goal of this student seminar, let us restate the theorem.

**Theorem 1.1** (Darboux Weinstein Theorem or Relative Moser Theorem)**.** *Let $Q$ be a compact submanifold of $M$, $i : Q \to M$ the inclusion map, $\omega_0$ and $\omega_1$ symplectic forms on $M$ with $\omega_0|_p = \omega_1|_p$ for every $p \in Q$. Then there exist neighborhoods $\mathcal{U}_0$, $\mathcal{U}_1$ of $Q$ in $M$ and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\varphi|_Q = \mathrm{id}_Q \quad \textit{and} \quad \varphi^* \omega_1 = \omega_0$$

Note, that $\omega_0 = w_1$ on $Q$ implies $[\omega_0] = [\omega_1]$ on any neighborhood homotopic to $Q$ by the homotopy invariance of the de Rahm cohomology.

The idea, brought forth by Moser in 1964 [Mos65], is to find a smooth family of symplectic forms $w_t$ connecting $w_0$ and $w_1$ on a neighborhood of $Q$. Then one needs to construct an isotopy $\rho_t$ such that

$$\rho_t^* \omega_t = w_0 \quad \text{and} \quad \rho_t|_Q = \mathrm{id}_Q \quad \text{for all } t \in \mathbb{R}. \tag{1}$$

This task can then be translated into finding a time dependent vector field by differentiating the condition (1), i.e. setting

$$\frac{d}{dt} \rho_t^* \omega_t = 0.$$

## References


[Can01] Ana Cannas da Silva. *Lectures on symplectic geometry*. Vol. 1764. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001. ISBN: 3-540-42195-5. DOI: 10.1007/978-3-540-45330-7. URL: https://doi.org/10.1007/978-3-540-45330-7.

[Mer19] Will J. Merry. *Differential Geometry*. ETH Zürich, 2018-2019. URL: https://doi.org/10.1007/978-3-540-45330-7.

[Mos65] Jürgen Moser. "On the volume elements on a manifold". In: *Trans. Amer. Math. Soc.* 120 (1965). ISSN: 0002-9947. DOI: 10.2307/1994022. URL: https://doi.org/10.2307/1994022.