A Version of Darboux's Theorem

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Note to the reader: This short note gives an overview of the proof of a strengthened version of Darboux's theorem, which is referred to by the name "Moser isotopy" in [MS17]. For a concise statement of Moser's trick, we refer to [MS17][p.108]. The aim of this exposition is not to express the main ideas concisely, but to fill in some of the details which may have been left to the reader in the consulted textbooks. Specfically, I am following the proofs given in [Lee12][Theorem 22.13] and [MS17][Lemma 3.2.1].

Theorem 1 (Moser isotopy)**.** *Let Q be a compact submanifold of the manifold M. Let ω_0, ω_1 be two symplectic forms on M that satisfy $\omega_0(q) = \omega_1(q)$ for all $q \in Q$. Then there exist two open neighborhoods U_1, U_2 of Q and a diffeomorphism $\Phi : U_1 \to U_2$ with the properties $\Phi^*(\omega_1) = \omega_0$ and $\Phi|_Q = \mathrm{id}_Q$.*

1 The Moser Trick

Suppose that we are in the situation described in the statement of Darboux's theorem. We claim that the following holds in this situation:

Claim 1. *There exists an open neighborhood U of Q in M such that, for a fixed $\epsilon > 0$, it holds that*

(i) $\tau \coloneqq (\omega_0 - \omega_1)|_U$ is exact,

(ii) there exists a 1-form β on U with $d\beta = \tau$ and $\beta(q) = 0$ for all $q \in Q$, and

(iii) $\omega_t \coloneqq \omega_0|_U + t(\omega_1|_U - \omega_0|_U)$ is symplectic for each $t \in (-2\epsilon, 1 + 2\epsilon)$.

Proof. The existence of an open neighborhood of Q for which the third bullet point holds is a consequence of the fact that non-degeneracy is an open condition and that the time interval is bounded (whence its closure is compact). The compactness argument is very similar to the one which is explained in detail in claim 2.

The claim made in the first bullet point, follows directly from choosing U to be a tubular neighborhood. The stronger statement made in bullet point two, however, is slightly more difficult to prove. While we could use any tubular neighborhood, for the sake of concreteness, we instead choose a Riemannian metric g on Y and use the associated exponential map. More precisely, we consider

$$\exp : TQ^{\perp} \mapsto M,$$

i.e. the exponential map on the tangent bundle TM of M restricted to the normal bundle $TQ^{\perp}$ of Q.

What we need is an open neighborhood of the zero section with the following two properties: Firstly, exp restricted to the open subset should be an embedding. Second, the intersection of the open neighborhood with the fiber $T_qQ^{\perp}$ should be star-shaped around 0_q for each $q \in Q$. In the proof of lemma 3.2.1 in [MS17], it is stated that there exists an $\epsilon > 0$ such that exp restricted to the ϵ-ball bundle

$$B_\epsilon \coloneqq \{v \in TQ^{\perp} \quad | \quad g(v,v) < \epsilon\}$$

is an embedding.[1]The argument is based on the fact that the zero section of a vector bundle is a deformation retract, the deformation retraction being given by

$$\begin{aligned} \theta : TQ^{\perp} \times [0,1] &\to TQ^{\perp} \\ ((q,v),t) &\mapsto (q, tv). \end{aligned}$$

As a side note, we remark that this is a generalisation of the map used to show that vector spaces are contractible. It is a straight forward matter to check that Q is a deformation retract of image$(\exp|_{B_\epsilon})$, the deformation retraction being given by

$$\begin{aligned}\Psi : \text{image}(\exp|_{B_\epsilon}) \times [0,1] &\to \text{image}(\exp|_{B_\epsilon})\\ (p,t) &\mapsto \exp|_{B_\epsilon} \circ \theta(\exp|_{B_\epsilon}^{-1}(p), t).\end{aligned}$$

The inclusion $\iota : Q \to \text{image}(\exp|_{B_\epsilon})$ being a homotopy equivalence implies the injectivity of the induced cohomology map. Since $\iota^*\tau$ is equal zero, it is exact. Hence, injectivity of the cohomology map induced by ι implies that $\tau|_{\text{image}(\exp|_{B_\epsilon})}$ is exact. Using only the existence of a primitive, I did not manage to show that the primitive of τ can be chosen to vanish on T_qY for all $q \in Q$. Therefore, we instead adopt the argument in [MS17][Lemma 3.2.1] which consists of finding an explicit formula for the primitive.

Define a time dependent 1-form σ on image$(\exp|_{B_\epsilon})) \times [0,1]$ by specifying the linear functional at the point (p,t) via the formula

$$\sigma_t|_p(v) \coloneqq \sigma|_{(t,p)} \coloneqq \tau_{\Psi_t(p)}(\frac{d}{ds}|_{s=t}\Psi_s(p), d_p\Psi_t(v)) \quad \text{for} \quad v \in T_pM. \tag{1}$$

The smoothness of Ψ and τ should directly imply the smoothness of σ. For any fixed point p, $t \mapsto \sigma_t|_p$ is a smooth curve in a finite dimensional vector space defined on a compact interval. Therefore, the integral over the parameter t exists. We claim that a desired primitive is given by

$$\sigma \coloneqq \int_{[0,1]} \sigma_t dt.$$

Before we check the validity of the claim, we define vector fields X_t for $t \in (0,1]$ via the usual formula

$$X_t \coloneqq (\frac{d}{ds}|_{s=t}\Psi_s) \circ \Psi_t^{-1}.$$

Note that the domain of X_t is given by image$(\exp|_{B_{t\epsilon}})$, which is an open neighborhood of Q that in some sense approaches Q as t goes to zero. Additionally, observe that we cannot define X_0, as Ψ_0 is a retraction and not an invertible map.

Having introduced these objects, we proceed to verify that σ is a primitive of τ:

$$\begin{aligned} d\sigma =& d\int_{[0,1]} \sigma_t dt\\ =& \int_{[0,1]} d\sigma_t dt\\ =& \int_{(0,1]} d\Psi_t^* \iota_{X_t}\tau dt\\ =& \int_{(0,1]} \Psi_t^* d\iota_{X_t}\tau dt\\ =& \int_{(0,1]} \Psi_t^* \mathcal{L}_{X_t}\tau dt\\ =& \int_{(0,1]} \frac{d}{dt}\Psi_t^*\tau dt\\ =& \int_0^1 \frac{d}{dt}\Psi_t^*\tau dt\\ =& \Psi_1^*\tau - \Psi_0^*\tau = \tau. \end{aligned}$$

Finally, we turn our attention to the behaviour of σ at points in Q. For this purpose, fix a point $q \in Q$. Observe that the equality $\sigma_t(q) = 0$ follows from equation 1 combined with the facts that $\Psi_t(q) = q$ and $\tau(q) = 0$. This directly implies the equality $\sigma(q) = 0$.

□

[1] For a proof of the existence of a neighborhood of the zero-section such that the restriction of exp to this neighborhood is a diffeomorphism onto its image, see [Wal16][Theorem 2.3.2]. The claim that the neighborhood can be chosen to be an epsilon-ball-bundle B_ϵ still requires justification.

From this claim we can deduce the following

Claim 2 (Moser's Trick). *There exists an open neighborhood $\tilde{U}$ of Q in M and a smooth map $\Phi : \tilde{U} \times (-\epsilon, 1+\epsilon) \to M$ that satisfies, for $t \in (-\epsilon, 1+\epsilon)$,*

(i) $\Phi_t := \Phi(\cdot, t)$ *is a diffeomorphism onto its image,*

(ii) $\Phi_0 = \mathrm{id}|_{\tilde{U}}$*,*

(iii) $\Phi_t|_Q = \mathrm{id}|_Q$*, and*

(iv) $\Phi_t^*(\omega_t) = \omega_0$*.*

Proof of theorem 1. The content of claim 2 directly implies theorem 1, since the map Φ_1 satisfies the properties stated in the theorem. □

Proof of claim 2. We want to construct Φ by finding an appropriate time dependent vector field and taking Φ to be the its flow. In this context, the flow of the time dependent vector field X_t refers to the map Φ that satisfies

$$\frac{d}{dt}|_t \Phi_t = X_t \circ \Phi_t \quad \text{and} \quad \Phi_0 = \mathrm{id}.$$

The properties stated in bullet points one and two are general properties of flows of time dependent vector fields. Thus, our choice of a time dependent vector field only has to be made in a manner that guarantees that the domain of the flow includes the set $\tilde{U} \times (-\epsilon, 1+\epsilon)$ and that points three and four hold.

Let U be an open neighborhood of Q and β be a 1-form on U with the properties stated in claim 1. We define a time dependent vector field X on $U \times (-2\epsilon, 1+2\epsilon)$ by

$$\iota_{X_t}\omega_t = \beta.$$

The above equation defines a vector field for fixed t, as ω_t is non-degenerate by claim 1.

First, we turn our attention to the domain of the flow Φ. Note that it suffices to show, for any $q \in Q$, the existence of an open neighborhood of q in M on which all integral curves exist for $t \in [-\epsilon, 1+\epsilon]$. The union of such neighborhoods over all points in Q then has the desired property. Let us fix a point $q \in Q$. Note that the 1-form β vanishes at the point q, wherefore we must have $X_t(q) = 0$ for all t. This means that the integral curve of X_t starting at q is given by the constant curve in q, whence $\{q\} \times [-\epsilon, 1+\epsilon]$ is contained in the domain of Φ. We now make use of the fact that the domain of Φ is an open subset of $U \times (-2\epsilon, 1+2\epsilon)$ the domain of X.[2] This implies that, for each $t \in (-2\epsilon, 1+2\epsilon)$, there exists a product open neighborhood $U_t \times I_t$ of (q, t) on which the flow of X is defined. By compactness of the interval $[-\epsilon, 1+\epsilon]$, there exists a finite family of such neighborhoods $U_{t_1} \times I_{t_1}, \ldots, U_{t_n} \times I_{t_n}$ which has the property that $\cup_{i=1}^n I_{t_i}$ contains $[-\epsilon, 1+\epsilon]$. Therefore, we conclude that $\cap_{i=1}^n U_{t_i}$ is a neighborhood of q in U on which the integral curves of X exist for $t \in (-\epsilon, 1+\epsilon)$.

In the previous paragraph, we showed that $\Phi_t(q)$ is the constant curve in q for every $q \in Q$, which is the same as saying that $\Phi_t|_Q = \mathrm{id}|_Q$. Thus, it only remains to tackle bullet point four.

The fourth bullet point follows directly from the choice of vector field. To see this, we note the following implications:

$$\begin{aligned}
& \iota_{X_t}\omega_t = \beta && \text{on } \tilde{U} \times (-\epsilon, 1+\epsilon) \\
\implies\ & 0 = d\iota_{X_t}\omega_t - d\beta && \text{on } \tilde{U} \times (-\epsilon, 1+\epsilon) \\
\implies\ & 0 = \Phi_t^*(d\iota_{X_t}\omega_t - d\beta) && \\
& \quad = \Phi_t^*(d\iota_{X_t}\omega_t + \iota_{X_t}d\omega_t + \omega_1 - \omega_0) && \\
& \quad = \Phi_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t) && \\
& \quad = \frac{d}{ds}|_{s=t}\Phi_s^*\omega_s && \text{on } \tilde{U} \times (-\epsilon, 1+\epsilon) \\
\implies\ & \omega_0 = \Phi_t^*\omega_t && \text{on } \tilde{U} \times (-\epsilon, 1+\epsilon).
\end{aligned}$$

□

[2] For this fact and other information on time dependent vector fields, see [Lee12][Theorem 9.49].

References

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