Pfaff's Theorem and Gray's Stability Theorem

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Abstract

In this talk, we will discuss about two main theorems of contact geometry: the Gray's Stability Theorem and the Pfaff's Theorem, both using the Moser's Trick in their proofs. We then finish by some quick applications of the latter. The content is mainly based on [1] and [2].

1 Gray's Stability Theorem

The Gray stability theorem that we are going to prove in this section says that there are no non-trivial deformations of contact structures on closed manifolds. First a preparatory lemma

Lemma 1. *Let α_t, $t \in [0,1]$, be a smooth family of differential k–forms on a manifold Y and $(\phi_t)_{t\in[0,1]}$ an isotopy of Y. Define a time-dependant vector field on X_t on Y by $X_t \circ \phi_t = \dot{\phi}_t$, so that ϕ_t is the flow of X_t. Then*

$$\frac{d}{dt}(\phi_t^* \alpha_t) = \phi_t^*(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha_t).$$

Proof. If α is a time-independant k-form, we directly have

$$\frac{d}{dt}(\phi_t^* \alpha) = \phi_t^*(\mathcal{L}_{X_t}\alpha).$$

since

1. the formula holds for functions,
2. if it holds for differential forms α and α', then also for $\alpha \wedge \alpha'$,
3. if it holds for α, then also for $d\alpha$,
4. locally functions and differentials of functions generate the algebra of differential forms.

We now compute the general case

$$
\begin{aligned}
\frac{d}{dt}(\phi_t^*\alpha_t) &= \lim_{h\to 0}\frac{\phi_{t+h}^*\alpha_{t+h}-\phi_t^*\alpha_t}{h}\\
&= \lim_{h\to 0}\frac{\phi_{t+h}^*\alpha_{t+h}-\phi_{t+h}^*\alpha_t+\phi_{t+h}^*\alpha_t-\phi_t^*\alpha_t}{}\\
&= \lim_{h\to 0}\phi_{t+h}^*(\frac{\alpha_{t+h}-\alpha_t}{h})+\lim_{h\to 0}\frac{\phi_{t+h}^*\alpha_t-\phi_t^*\alpha_t}{h}\\
&= \phi_t^*(\dot{\alpha}_t+\mathcal{L}_{X_t}\alpha_t).
\end{aligned}
$$

□

We now state and prove the Gray's Stability Theorem thanks to the previous lemma and to the Moser's trick.

Theorem 2 (Gray Stability)**.** *Let ξ_t, $t\in[0,1]$, be a smooth family of contact structures on a closed manifold Y. Then there is an isotopy $(\phi_t)_{t\in[0,1]}$ of Y such that*

$$\phi_t^*(\xi_t)=\xi_0,\ \forall t\in[0,1].$$

Proof. Moser's trick consists in finding conditions and work on time-dependent vector fields whose flow is our isotopy. We then find sufficient conditions on the latter and then reintegrate them to find our desired isotopy.

So thanks to the Moser's trick, we assume that ϕ_t is the flow of a time-dependant vector field X_t. We can hence translate the equation for ϕ_t into an equation for X_t. If that equation can be solved, the isotopy ϕ_t is found by integrating X_t which works well as the flow of X_t will be globally defined on such a closed manifold.

Let α_t be a smooth family of 1-forms with $ker\alpha_t=\xi_t$. The equation in theorem then translates into

$$\phi_t^*\alpha_t=f_t\alpha_0$$

where $f_t:Y\to\mathbb{R}^+$ is a suitable family of smooth functions. Differentiation of this equation with respect to t yields, with the help of the preceding lemma,

$$\phi_t^*(\dot{\alpha}_t+\mathcal{L}_{X_t}\alpha_t)=\dot{f}_t\alpha_0=\frac{\dot{f}_t}{f_t}\phi_t^*\alpha_t,$$

and, with the help of the Cartan's Magic Formula and setting $\mu_t=\frac{d}{dt}(logf_t)\circ\phi_t^{-1}$

$$\phi_t^*(\dot{\alpha}_t+d(\alpha_t(X_t))+i_{X_t}d\alpha_t)=\phi_t^*(\mu_t\alpha_t).$$

If we choose $X^t\in\xi_t$, this equation will be satisfied if

$$\dot{\alpha}_t+i_{X_t}d\alpha_t=\mu_t\alpha_t.$$

Plugging in the Reeb vector field R_{α_t} gives

$$R_{\alpha_t} = \mu_t.$$

So we can use the last equation to define μ_t, and then the non degeneracy of $d\alpha_t|_{\xi_t}$ and the fact that $R_{\alpha_t} \in ker(\mu_t \alpha_t - \dot{\alpha_t})$ allow us to find a unique solution $X_t \in \xi_t$ of the penultimate equation. □

2 Pfaff's Theorem

Theorem 3 (Pfaff's Theorem)**.** *Let α be a contact form on the $(2n+1)$-dimensional manifold Y and p a point on Y. Then there are coordinates $x_1, ..., x_n, y_1, ..., y_n, z$ on a neighbourhood $U \subset Y$ of p such that*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j dy_j.$$

Equivalently, if ξ is a contact structure of Y, then Y locally looks like $(\mathbb{R}^{2n+1}, \xi_0)$ with ξ_0 the standard contact structure associated to the standard contact form $\alpha_0 = dz - ydx$. This means that, for all $p \in Y$ there exists open sets $U \ni p$ of Y and $V \ni 0$ of $\mathbb{R}^{2n+1}$ such that $\phi : (U, \xi) \to (V, \xi_0), \phi(p) = 0$, is a contactomorphism.

Proof. We shall consider the second equivalence and will only prove the 3-dimensional case ($n = 1$). The full proof is available here [1].
So suppose $n = 1$. Without loss of generality we may assume that there is a local coordinate chart which maps p to $0 \in \mathbb{R}^3$ and the induced contact structure ξ is the xy-plane at 0, i.e., $\xi(0) = kerdz$. In a neighborhood of 0 we can write the contact 1-form as $\alpha = dz + fdx + gdy$

Now consider $y = 0$ (the xz-plane). On it, ξ restricts to the vector field $X = \partial_x - f\partial_z$. Observe that X is transverse to the z-axis $x = y = 0$. Using the fundamental theorem of ODE's, we can integrate along this vector field, starting along the z-axis. We let $\Phi(x, z)$ be the time x flow, starting at $(0, 0, z)$. Hence there are new coordinates (also called (x, z)) such that the vector field is ∂_x, i.e., $f = 0$ along $y = 0$.

In other words, we have normalized the contact structure on the $xz - plane$. Finally, consider the restrictions of ξ to $x = const$, which give the vector field $\partial_y - g\partial_z$. We then have a vector field on all of a neighborhood of 0 in $\mathbb{R}^3$, which is transverse to $y = 0$. If we integrate along the vector field, we obtain new coordinates so that we can write $\alpha = dz + f(x, y, z)dx$ with $f(x, y, 0) = 0$ and $\partial_y f < 0$ (from the contact condition). Now simply change coordinates $(x, y, z) \to (x, -f(x, y, z), z)$. □

3 Moser's Trick's applications

Example 4. Consider the Lie group Diff(Y, ξ) of contactomorphisms from Y to itself. We have $T_{id}\text{Diff}(Y, \xi) = \{C^\infty\text{-functions on } Y\}$

Proof. If $X \in T_{id}\text{Diff}(Y, \xi)$, then $\mathcal{L}_X \alpha = g\alpha$ for some smooth function g. Let f be a smooth function aswell, we hence have $i_Y d\alpha + df = g\alpha$. Thus, if f is chosen first, there is a unique Y as above. ☐

Definition 5. A *transverse curve* is a curve γ such that $\dot{\gamma}$ is codimensional with ξ with ξ a contact structure.

Example 6. Suppose γ is a closed transverse curve in (Y, ξ). Then there is a neighborhood $N(\gamma)$ of γ such that $\xi|_{N(\gamma)}$ is given by

$$\alpha_0 = dz + \frac{1}{2}(-ydx + xdy) = dz + \beta_0.$$

Proof. We may choose coordinates (z, x, y) on $N(\gamma) = S^1 \times D^2$ so that ξ is given by $\alpha = dz + \beta$, where $\beta = fdx + gdy$ and $\beta(z, 0, 0) = 0$. We interpolate between α and α_0 by setting $\alpha_t = (1-t)\alpha_0 + t\alpha = dz + (1-t)\beta_0 + t\beta_1$.

As $\alpha_t \wedge d\alpha_t > 0$ in a neighborhood of γ (we consider all contact structures co-oriented), we use the Moser's trick to solve for X_t in:

$$\mathcal{L}_{X_t}\alpha_t = \frac{d\alpha_t}{dt} + f_t\alpha = (\beta_1 - \beta_0) + f_t\alpha.$$

Observing that we may take $f_t = 0$, $X_t = 0$. ☐

References

[1] Hansjörg Geiges. "Contact Geometry". In: *Handbook of Differential Geometry*. Vol. 2. F.J.E. Dillen and L.C.A. Verstraelen, 2004. ISBN: 9780444822406. URL: https://arxiv.org/abs/math/0307242v2.

[2] Ko Honda. *Notes For Math 599: Contact Geometry*. UCLA. 2019. URL: https://www.math.ucla.edu/~honda/math599/notes.pdf.