Seminar - Contact Geometry

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Definition 1 (Contact Manifolds)**.** Let Y be a manifold of odd dimension $2n+1$. A contact structure is a maximally non-integrable hyperplane field $\xi = \ker \alpha \subset TY$. That is, the 1-form is required to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0$$

Such a 1-form is called a contact form, and the pair (Y, ξ) is called a contact manifold.

Remark 1. ξ is a field of hyperplanes on Y - that is, a sub-bundle of the tangent bundle TY with co-dimension 1.

Remark 2. In dimension 3, we have $n = 1$, and so the above condition simplifies to

$$\alpha \wedge d\alpha \neq 0$$

Theorem 1 (Martinet)**.** *Every 3-manifold admits a contact structure.*

Proof. Makes use of many concepts we don't really have access too, but it is developed in section 5.23 of [3]. □

Definition 2 (Important example of a Contact Structure)**.** On the 3-manifold $\mathbb{R}^3$, equipped with cylindrical coordinates (r, φ, z), we define the 1-form:

$$\alpha_{vrille} = \cos(r)dz + r\sin(r)d\varphi$$

We have:

$$\alpha_{vrille} \wedge d\alpha_{vrille} = \left(1 + \frac{\sin(r)}{r}\cos(r)\right) r dr \wedge d\varphi \wedge dz \neq 0 \tag{1}$$

so its kernel defines a contact structure on $\mathbb{R}^3$.

We will come back to this example later on in the seminar.

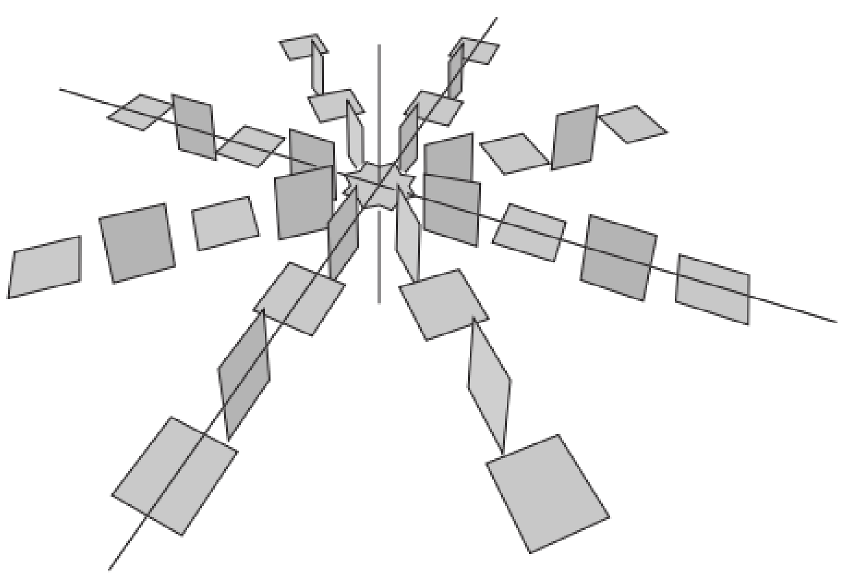

Figure 1: Standard Overtwisted Contact Structure [5]

Proof of (1). First, we compute $d\alpha_{vrille}$:

$$\begin{aligned} d\alpha_{vrille} &= \frac{\partial \cos(r)}{\partial r} dr \wedge dz + \frac{\partial r \sin(r)}{\partial r} dr \wedge d\varphi \\ &= sin(r) dr \wedge dz + (\sin(r) - r\cos(r)) dr \wedge d\varphi \end{aligned}$$

So we have:

$$\begin{aligned}
\alpha_{vrille} \wedge d\alpha_{vrille} &= [\cos(r)dz + r\sin(r)d\varphi] \wedge [sin(r)dr \wedge dz + (\sin(r) - r\cos(r))dr \wedge d\varphi] \\
&= \cos(r)(\sin(r) - r\cos(r))dz \wedge dr \wedge d\varphi + r\sin(r)^2 d\varphi \wedge dr \wedge dz \\
&= \left(1 + \frac{\sin(r)}{r}\cos(r)\right) r dr \wedge d\varphi \wedge dz
\end{aligned}$$

Which is the result we wanted. □

Remark 3. Note that $\forall p \in \mathbb{R}^3$, we have

$$\alpha_{vrille,p}\left(\frac{\partial}{\partial r}\Big|_p\right) = 0$$

since

$$dz_p\left(\frac{\partial}{\partial r}\Big|_p\right) = d\varphi_p\left(\frac{\partial}{\partial r}\Big|_p\right) = 0$$

So at every point in $\mathbb{R}^3$, the associated hyperfield - a plane since we are working in dimension 3 - contains the basis vector $\partial_{r,p}$. This corresponds to what we observe on the picture.

We can verify that this vector together with $\cos(r)\partial_{\varphi,p} - r\sin(r)\partial_{z,p}$ spans the entire hyperfield. This gives us contact planes that make infinitely many turns as one moves out in the radial direction.

Definition 3 (Overtwisted Disc)**.** An embedded disc Δ in a contact manifold (Y, ξ) is an overtwisted disc if its boundary $\partial\Delta$ is such that $\forall p \in \partial\Delta$, we have:

$$\xi_p = T_p D$$

In other words, the interior of the disc must be transversal to ξ everywhere, and its boundary must be tangent to ξ.

Remark 4. The fact that these definitions are indeed equivalent is not trivial... In particular the re-wording includes a condition on transversality inside the disc which does not appear anywhere above.
In the same manner, some definitions of overtwisted disc also include that 1 point in the interior of the disc need not fulfill the transversality condition. As it turns out, all of these definitions end up being equivalent, and we won't delve any deeper into why for the purpose of this talk.
If you are interested in further analysis of this concept, you can take a look at section 4.5 of [1].

This is what our overtwisted disc will standardly look like:

Figure 2: The curves represent the foliation obtained by intersection of the hyperplane field with the disc. The dot is the point where the disc fails to be transversal. Image Source [1]

Definition 4 (Tight, Overtwisted contact structure)**.** A contact structure ξ on a 3-manifold is called overtwisted if it contains an overtwisted disc. Otherwise, it is called tight.

Proposition 1. *The contact structure α_{vrille} on $\mathbb{R}^3$ is overtwisted. In fact, it is called the standard overtwisted contact structure on $\mathbb{R}^3$.*
An overtwisted disc for this structure is for instance $\Delta = \{z = 0, r \leq \pi\}$.

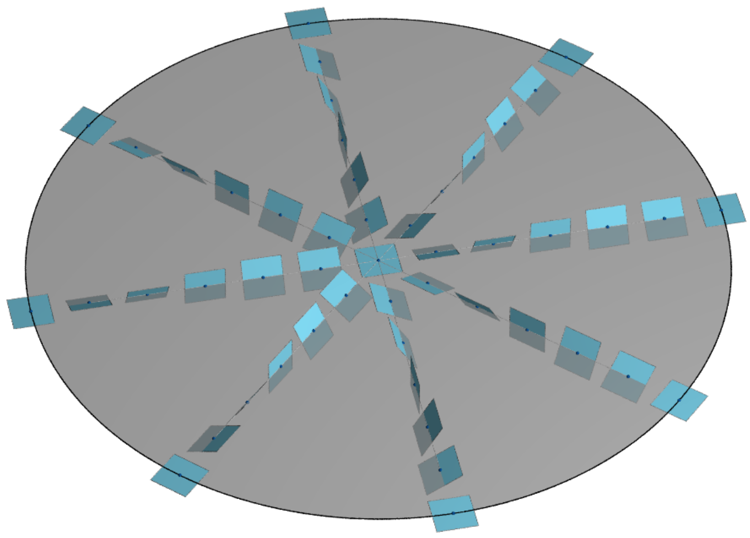

Figure 3: The embedded disk Δ in the standard overtwisted contact structure [6]

We observe, as per the definition, that at $r = \pi$ the hyperplane field is horizontal and thus corresponds with the tangent planes to the disc.

Let us now take a look at a standard example of a tight contact structure.

Definition 5 (Standard Contact Structure on $\mathbb{R}^3$)**.** The standard contact structure on $\mathbb{R}^3$, which we already met in some other seminars, is the kernel of the 1-form

$$\alpha_0 = dz - ydx$$

From the definition, it is clear that at any point $\frac{\partial}{\partial y}$ is in the kernel. Furthermore, we can easily check that at $p = (x, y, z)$, the vector $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ is also in the kernel. Thus, these 2 vectors form a base of the hyperplane at p.

The hyperplane field associated to this contact structure consists of planes twisting about the y-axis.

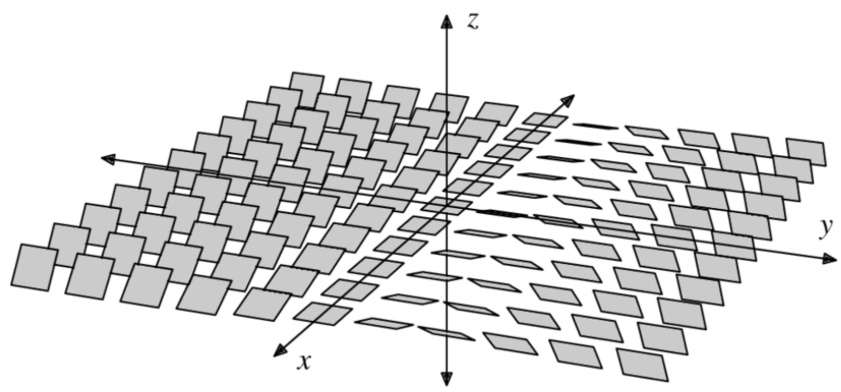

Figure 4: The slopes go to ∞ but the planes never become vertical. [7]

Proposition 2. *The standard contact structure on $\mathbb{R}^3$ is tight.*

We can intuitively understand why looking at Figure 4. For a detailed proof, see [4] (in French).

Definition 6 (A Final Contact Structure)**.** Consider the 3 dimensional torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and $n \in \mathbb{Z}_+$. Then

$$\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy$$

is such that

$$\alpha_n \wedge d\alpha_n = 2\pi n dx \wedge dy \wedge dz \neq 0$$

and so it is a well defined contact structure on $\mathbb{T}^3$.
As before, it is clear that ∂_z is in the kernel, and we can verify that $\cos(2\pi nz)\partial_x - sin(2\pi nz)\partial_y$ is another vector in the kernel, thus giving us a basis. The hyperplanes propel along the z direction, completing n full twists.

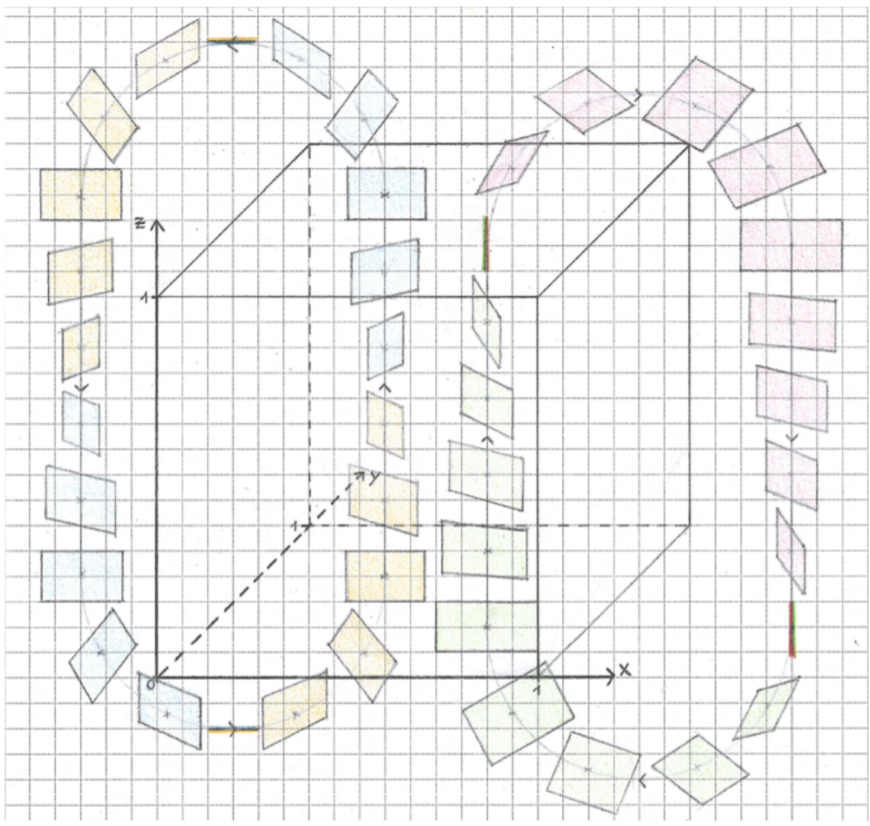

Figure 5: Image Source: Alessandro Imparato [8]. Both the cases $n = 1, 2$ are drawn.

Proposition 3. *This is an overtwisted contact structure for* $\Delta = \{p, p_y = 0, ||p|| \leq 1/n\}$.

Remark 5 (Why care about this classification?). Though tight vs. overtwisted is obviously a dichotomy, it is not clear that it is a useful one. It turns out, overtwisted contact structures are somewhat "easy" to deal with, whereas tight contact structures are quite a bit more difficult to understand. Moreover, a tight contact structure is capable of detecting subtle global properties of the manifold supporting it.

By "easy to deal with", we mean that on 3-manifolds, overtwisted contact structures actually have a classification theorem!

Definition 7 (Weak Homotopy Equivalence)**.** A weak homotopy equivalence $f : X \to Y$ is a continuous map for which

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all n.

Theorem 2 (Eliashberg's overtwisted classification)**.** *Define* $Cont_{OT}(Y) \subset Cont(Y)$ *the set of overtwisted contact structures. Let* (Δ, ζ) *be an overtwisted disk together with a contact structure whose characteristic foliation is the standard one.*
Then let $Cont_{OT}(Y)$ *be the set of overtwisted contact structures which agree with* ζ *on* Δ*, and let* $Dist(Y, \Delta)$ *be the set of 2-plane fields which agree with* ζ *at the center of* Δ*. Then we have a weak homotopy equivalence:*

$$Cont_{OT}(Y, \Delta) \hookrightarrow Dist(Y, \Delta)$$

We now present some applications of the concepts introduced in this seminar.

Theorem 3 (Etnyre-Honda)**.** *There exists a closed-compact 3-manifold that does not support any tight contact structure.*

This means that no matter what 1-form we pick to obtain a contact structure, we will *always* be able to find an overtwisted disc for it!

Definition 8 (Filling a Manifold)**.** A compact symplectic 4-manifold (X, ω) is said to fill a contact 3-manifold (Y, ξ) if $\partial X = Y$ and $\omega_{|_\xi}$ is an area form on ξ.

Theorem 4. *If a contact structure can be filled by a compact symplectic manifold then it is tight.*

Notably, Theorem 4 implies that the standard contact structure on S^3 is tight. Proofs of these theorems are discussed in [3], section 4, page 9.

References

[1] Hansjörg Geiges, 2009, An Introduction to Contact Topology.

[2] Ko Honda, Notes for Contact Geometry.

[3] John B. Etnyre, Introductory Lectures on Contact Geometry.

[4] Douady, Adrien, volume 1982/83, exposes 597-614, Asterisque, no. 105-106 (1983), Expose no. 604, 20 p.

[5] Image Credit: S. Schönenberger

[6] Image: Wikimedia Commons, $https//commons.wikimedia.orgwiki/FileOvertwisted_contact_structure.p$

[7] Image Credit: Wikipedia, URL: $https//en.wikipedia.org/wiki/Contact_geometry$

[8] URL: $https//people.math.ethz.ch/ bacubulut/assets/week4_alessandro_full.pdf$