Lagrangian Submanifolds: Definitions, Examples

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1 Lagrangian Subspaces

Let (V, Ω) be a symplectic real vector space of dimension $2n$, and let $Y \subseteq V$ be a subspace of V. Then we define ***symplectic orthogonal*** Y^Ω as

$$Y^\Omega = \{v \in V : \Omega(v, u) = 0 \text{ for all } u \in Y\}$$

By linearity of the symplectic form, Y^Ω is a linear subspace of V.

Lemma 1. *For any subspace $Y \subseteq V$, we have*

$$\dim Y + \dim Y^\Omega = \dim V \text{ and } (Y^\Omega)^\Omega = Y$$

Proof. Consider the linear map $T : V \to Y^*$ defined as $v \mapsto \Omega(v, \bullet)|_Y$.

Let ϕ be in Y^* and let $\Phi \in V^*$ be any extension of ϕ, that is $\phi = \Phi|_Y$. Then by nondegeneracy of Ω, there exists a $v \in V$ such that $\Omega(v, \bullet) = \Phi$, and thus $\Omega(v, \bullet)|_Y = \phi$. Therefore, the map T is surjective.

On the other hand, if $v \in Y^\Omega$, then $T(v) = 0 \in Y^*$. Conversely, if $T(v)$ is the zero function, then $\Omega(v, u) = 0$ for all $u \in Y$, that is $v \in Y^\Omega$. Therefore, $\ker T = Y^\Omega$.

Then from the rank-nullity theorem, we have

$$\begin{aligned} \dim V &= \dim \operatorname{im} T + \dim \ker T \\ &= \dim Y^* + \dim Y^\Omega \\ &= \dim Y + \dim Y^\Omega \end{aligned}$$

as desired.

By skew symmetry $Y \subseteq (Y^\Omega)^\Omega$. Indeed, if $u \in Y$, then for all $w \in Y^\Omega$ we have $\Omega(u, w) = -\Omega(w, u) = 0$ by the definition of Y^Ω. Then by the dimension relation, the second assertion follows. □

From the definition and the above lemma, it is immediate that $Y \subseteq W$ as subspaces if and only if $W^\Omega \subseteq Y^\Omega$ as subspaces.

In contrast to the orthogonal complement with respect to an inner product, the symplectic orthogonal Y^Ω does not have to be a complementary subspace of the original subspace Y.

Definition 1. A subspace $Y \subseteq V$ is called:

- ***Symplectic***, if $Y \cap Y^\Omega = \{0\}$
- ***Isotropic***, if $Y \subseteq Y^\Omega$
- ***Coisotropic***, if $Y \supseteq Y^\Omega$
- ***Lagrangian***, if $Y^\Omega = Y$

This definition, combined with Lemma 1 gives us the following characterizations of Lagrangian subspaces of symplectic vector spaces.

Proposition 1. *Let $Y \subseteq V$ be a linear subspace of the symplectic real vector space (V, Ω) of dimension $2n$. Then the following are equivalent:*

- *Y is Lagrangian, that is $Y^\Omega = Y$*
- *Y is isotropic and coisotropic*
- *Y is isotropic and* $\dim Y = n$

We will close this section with a few examples and some remarks about Lagrangian subspaces.

Example. Let (V, Ω) be a symplectic vector space with a canonical symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Then, the span of $\{e_1, \ldots, e_n\}$ is a Lagrangian subspace of V.

In fact, given a Lagrangian subspace of V with a basis $\{e_1, \ldots, e_n\}$, one can complete this set to a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for (V, Ω) by picking $f_i \in span\{e_1, \ldots, \hat{e_i}, \ldots, e_n\}^\Omega$ such that $\Omega(e_i, f_i) = 1$.

Moreover, if we are given a Lagrangian subspace Y of V, and W is *any* subspace complementary to Y, then one can build a Lagrangian complement to Y *canonically* from W.

Example. Given a vector space Y of dimension n, the direct sum $V = Y \oplus Y^*$ can be given a symplectic structure with the symplectic form Ω_0 defined as $\Omega_0(u \oplus f, v \oplus g) = g(u) - f(v)$. In this case, $Y = \{u \oplus 0\} \subseteq V$ is a Lagrangian subspace of V since $\dim Y = \dim Y/2$ and $\Omega_0(u \oplus 0, v \oplus 0) = 0$.

Conversely, if Y is a given Lagrangian subspace of (V, Ω), then (V, Ω) is symplectomorphic to $(Y \oplus Y^*, \Omega_0)$ where Ω_0 is defined as before.

2 Lagrangian Submanifolds

Let X be a submanifold of M with the proper injective immersion $i : X \to M$. In this case, we regard X as a subset of M and T_pX as a subspace of T_pM through the identifications $p = i(p) \subseteq M$ and $v = i_*(v) \subseteq T_pM$.

Definition 2. Let (M, ω) be a symplectic manifold. A submanifold $L \subseteq M$ is a ***Lagrangian submanifold*** if at each point $p \in L$, the subspace $T_pL \subseteq T_pM$ is a Lagrangian subspace of (T_pM, ω_p).

Equivalently, a submanifold $L \subseteq M$ is a ***Lagrangian submanifold*** if $\dim L = \dim M/2$ and $i^*\omega = 0$ where $i : L \to M$ is the inclusion.

Symplectic, isotropic, coisotropic submanifolds are defined analogously.

Example. Let $(M_1, \omega_1), (M_1, \omega_1)$ be symplectic manifolds of dimension $2n$. Then define the two form $\tilde{\omega}$ on $M_1 \times M_2$ as

$$\tilde{\omega} = pr_1^*\omega_1 - pr_2^*\omega_2$$

where $pr_i : M_1 \times M_2 \to M_i$ are the projections. Then $(M_1 \times M_2, \tilde{\omega})$ is a symplectic manifold of dimension 4n. Indeed,

$$d\tilde{\omega} = pr_1^*d\omega_1 - pr_2^*d\omega_2 = 0$$

and

$$\tilde{\omega}^{2n} = \pm\binom{2n}{n} pr_1^*(\omega_1)^n \wedge pr_2^*(\omega_2)^n$$

is nonvanishing

Remark. One can choose varying nonzero real coefficients λ_1, λ_2 to get different symplectic forms on $M_1 \times M_2$ which are similarly defined as

$$\tilde{\omega}_{\lambda_1,\lambda_2} = \lambda_1 pr_1^*\omega_1 + \lambda_2 pr_2^*\omega_2$$

Non degeneracy and closedness of the two form $\tilde{\omega}_{\lambda_1,\lambda_2}$ follows from a similar argument. In this case, the symplectic form we defined above is simply $\tilde{\omega}_{1,-1}$

Proposition 2. *Using the above notation, a diffeomorphism $f : M_1 \to M_2$ is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$*

Proof. Let $\gamma : M_1 \to M_1 \times M_2$ defined as $\gamma(p) = (p, f(p))$, mapping M_1 to the graph of f in $M_1 \times M_2$. Then we have

$$\begin{aligned}\gamma^*\tilde{\omega} &= \gamma^*pr_1^*\omega_1 - \gamma^*pr_2^*\omega_2 \\ &= (pr_1 \circ \gamma)^*\omega_1 - (pr_2 \circ \gamma)^*\omega_2 \\ &= \omega_1 - f^*\omega_2\end{aligned}$$

Thus, $\omega_1 = f^*\omega_2$ if and only if $\gamma^*\tilde{\omega} = 0$, that is if and only if the image of γ is a Lagrangian submanifold. $\square$

In particular, if we take $(M, \omega) = (M_1, \omega_1) = (M_2, \omega_2)$, then the product $(M \times M, \tilde{\omega} = pr_1^*\omega - pr_2^*\omega)$ is a symplectic manifold. Consider the image L of the diagonal embedding $\Delta : M \to M \times M$. We have

$$\begin{aligned} \Delta^*\tilde{\omega} &= \Delta^* pr_1^*\omega - \Delta^* pr_2^*\omega \\ &= (pr_1 \circ \Delta)^*\omega - (pr_2 \circ \Delta)^*\omega \\ &= (id_L)^*\omega - (id_L)^*\omega = 0 \end{aligned}$$

Therefore, the diagonal embedding of M as $L \subset M \times M$ is a Lagrangian submanifold of $(M \times M, \tilde{\omega} = pr_1^*\omega - pr_2^*\omega)$

Example. (Graphs of closed 1-forms are Lagrangian submanifolds of the cotangent bundle) Let us recall the symplectic structure on the cotangent bundle $M^{2n} = T^*X$ of a smooth manifold X^n.

The ***tautological 1-form*** λ is defined pointwise for $p = (x, \xi) \in M$ as

$$\lambda_p = (d\pi_p)^*\xi$$

where $(d\pi_p)^*$ is the dual map of the derivative of the projection $\pi : M \to X$. Equivalently, if $v \in T_pM = T_p(T^*X)$, then

$$\lambda_p(v) = \xi(d\pi_p(v))$$

The ***canonical symplectic 2-form*** ω on $M = T^*X$ is then defined as

$$\omega = -d\lambda$$

If (U, x^i) is a local coordinate chart on X and (T^*U, x^i, ξ_i) is the corresponding cotangent coordinates on M, then locally we have

$$\lambda = \sum \xi_i dx^i \text{ and } \omega = \sum dx^i \wedge d\xi_i$$

If $\mu \in \Gamma(T^*X)$ is a 1-form, we will denote it by $s_\mu : X \to M$ when we consider it just as a smooth map. From the coordinate representations and the fact that $\pi \circ s_\mu = id_X$, the map $s_\mu : X \to M$ is a proper injective immersion. That is, the image $X_\mu = s_\mu(X)$ is an n-dimensional submanifold of the cotangent bundle M^{2n}

Lemma 2. With the above notation, $s_\mu^*\lambda = \mu$

Proof. Let $v \in T_xX$, then

$$(s_\mu^*\lambda)_x(v) = \lambda_{s_\mu(x)}(d(s_\mu)_x(v)) = \mu_x(d\pi_{(x,\mu_x)}d(s_\mu)_x(v)) = \mu_x(v)$$

□

Proposition 3. *$X_\mu = s_\mu(X)$ is a Lagrangian submanifold of M^{2n} if and only if μ is closed.*

Proof. We already have that the dimension of X_μ is half of the dimension of M. Let $i : X_\mu \to M$ be the inclusion. Then

$$\begin{aligned} i^*\omega = 0 &\Leftrightarrow \tau^*i^*(\omega) = 0 \\ &\Leftrightarrow (i \circ \tau)^*(-d\lambda) = 0 \\ &\Leftrightarrow (s_\mu)^*(-d\lambda) = 0 \\ &\Leftrightarrow -d(s_\mu)^*\lambda = 0 \\ &\Leftrightarrow -d\mu = 0 \\ &\Leftrightarrow d\mu = 0 \end{aligned}$$

where $\tau : X \to X_\mu$ is the diffeomorphism defined as $\tau(x) = (x, \mu_x)$, such that $s_\mu = i \circ \tau$. □

Remark. Every smooth function f on X generates a Lagrangian submanifold as the image of $\mu = df$.

If X is simply connected, then every closed 1-form can be written as $\mu = df$ for some smooth function f. This f is called the ***generating function*** of the Lagrangian submanifold X_μ.

Example. As our last example we will discuss the conormal bundle $N^*S \subseteq T^*X$ of a submanifold $S^k \subseteq X^n$. The ***conormal space*** at $x \in S$ is

$$N_x^*S = \{\xi \in T_x^*X : \xi(v) = 0 \text{ for all } v \in T_xS\}$$

The ***conormal bundle*** is defined as

$$N^*S = \{(x, \xi) \in T^*X : x \in S, \xi \in N_x^*S\}$$

If (U, x^i) is a local coordinate chart on X such that S is defined by vanishing of $x^{k+1}, \dots, x^n$ and (T^*U, x^i, ξ_i) is the corresponding cotangent coordinates on T^*X, then N^*S is defined by the vanishing of $x^{k+1}, \dots, x^n, \xi_1, \dots, \xi_k$. Therefore, N^*S is an n-dimensional submanifold of the 2n-dimensional manifold T^*X. Moreover, since locally $\lambda = \sum \xi_i dx^i$, we have at $p \in N^*S$

$$i^*\lambda_p = \lambda_p|_{T_pN^*S} = \sum_{i>k} \xi_i(p) dx_p^i|_{span_{i\leq k}\{\partial_i|_p\}} = 0$$

where $i : N^*S \to T^*X$ is the inclusion. Consequently, $i^*\omega = -di^*\lambda = 0$ and the conormal bundle is a Lagrangian submanifold of the cotangent bundle.

References

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