Lagrangian submanifolds and Arnold's conjecture

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November 4, 2021

Abstract

Following the introduction to Lagrangian submanifolds by Metehan Aksay, in this 30min talk I will try to motivate the importance of Lagrangians in symplectic topology, focusing on proving a special case of Arnold's conjecture. Each result can be found in McDuff & Salamon [1], although they have been streamlined to prove the main theorem as fast as possible, and proofs have been added or altered for additional clarity.

0.1 Reminder of concepts

Most of these were mentioned in previous talks, I include here for ease of understanding.

Definition 0.1. A *linear symplectomorphism* $\varphi : V \to V$ is an isomorphism of symplectic vector space (V, ω) such that the symplectic form is preserved under pullback:

$$\varphi^*\omega(v, w) := \omega(\varphi(v), \varphi(w)) = \omega(v, w).$$

Proposition 0.2. *A linear map $\varphi : V \to V$ is a linear symplectomorphism if and only if the graph $\Gamma_\varphi := \{(v, \varphi(v)) \mid v \in V\}$ is a Lagrangian subspace of $(V \times V, (-\omega) \times \omega)$.*

Proof. This is an easy exercise from the definitions, given in the previous talk, of symplectic complement $W^\omega := \{w \mid \omega(v, w) = 0 \, \forall v \in W\}$ and Lagrangian subspaces $\{W \subset V \mid W^\omega = W\}$. □

Definition 0.3.

1. A *symplectomorphism* on a symplectic manifold is a diffeomorphism such that it preserves the symplectic form:

$$\varphi^*\omega = \omega.$$

2. A *symplectic isotopy* is a smooth map $\varphi : [0, 1] \times M \to M$ such that φ_t is a symplectomorphism for every t and $\varphi_0 = id$, the identity.

3. A *Hamiltonian isotopy* is a symplectic isotopy where the generating family of vector fields X_t defined by $\partial_t \varphi_t =: X_t \cdot \varphi_t$ is such that $\iota(X_t)\omega = dH_t$ for a family of smooth functions H_t called the time-dependent Hamiltonian.

Remark. If the manifold is simply connected then $H^1_{dR}(M) = 0$, all closed 1-forms are exact, in particular every symplectic isotopy is Hamiltonian.

For general symplectic manifolds (M, ω), there's a significant proposition.

Proposition 0.4. *Let (M, ω) be a symplectic manifold and $\psi : M \to M$ a diffeomorphism, then ψ is a symplectomorphism if and only if the graph of ψ, Γ_ψ, is a Lagrangian submanifold of $(M \times M, (-\omega) \times \omega)$.*

Proof. Note we have the identification $T_{(p,\psi(p))}(M \times M) \cong T_pM \times T_{\psi(p)}M$ and under this identification $T(\Gamma_\psi) = \mathrm{graph}(D\psi)$.

$$\begin{aligned}
&\Gamma_\psi \text{ is Lagrangian} \\
&\Updownarrow \\
(-\omega) \times \omega = 0 &\text{ on } T(\Gamma_\psi) = \mathrm{graph}(D\psi) \\
&\Updownarrow \\
((-\omega) \times \omega)(\xi, D\psi\xi)(\eta, D\psi\eta) &\equiv -\omega(\xi, \eta) + \psi^*\omega(\xi, \eta) \\
&= 0 \;\forall \xi, \eta \in T_pM \;\forall p \in M \\
&\Updownarrow \\
&\psi \text{ is a symplectomorphism.}
\end{aligned}$$

□

This highlights the use of Lagrangians for investigating symplectomorphisms, which we explore later.

1 Graphs of 1-forms on cotangent bundles

In the case of the canonical cotangent bundle (T^*Q, ω_{can}) which, as shown in previous talks, is a symplectic manifold with a canonical 1-form $\lambda_{can} \in \Omega^1(T^*Q)$ and $\omega_{can} := -d\lambda_{can}$ is called the canonical symplectic form.

Remark. This is *the* 2-form, up to multiplication by a non-zero constant, such that any diffeomorphism of Q lifts to a symplectomorphism of T^*Q, i.e. you have functoriality.

Proposition 1.1. *(Tautological Property) The canonical 1-form λ_{can} is uniquely characterised by the tautological property: $\sigma^*\lambda_{can} = \sigma$ for every 1-form $\sigma : Q \to T^*Q$.*

Proof. This is a local property so we reduce to working in coordinates (x^i, y^i) of T^*Q restricted to the coordinate neighbourhood. Recall $\lambda_{can} := \sum y_i dx^i$. Also let $\sigma = a^i dx^i$ with $\hat{\sigma}$ the coordinate representation

$$\begin{aligned} d\sigma\left(\frac{\partial}{\partial x^j}|_p\right) &= \sum_{i=1}^n \frac{\partial \hat{\sigma}^i}{\partial x^j}\frac{\partial}{\partial x^i}|_{\sigma(p)} + \sum_{i=1}^n \frac{\partial \hat{\sigma}^{n+i}}{\partial x^j}\frac{\partial}{\partial y^i}|_{\sigma(p)} \\ &= \frac{\partial}{\partial x^j}|_{\sigma(p)} + \sum_{i=1}^n \frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial y^i}|_{\sigma(p)} \end{aligned}$$

so

$$\begin{aligned} \sigma^*\lambda_{can}\left(\frac{\partial}{\partial x^j}|_p\right) &= \lambda_{can} d\sigma\left(\frac{\partial}{\partial x^j}|_p\right) \\ &= \lambda\left(\frac{\partial}{\partial x^j}|_{\sigma(p)} + \sum_{i=1}^n \frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial y^i}|_{\sigma(p)}\right) \\ &= \sum_{i=1}^n a^i(p) dx^i\left(\frac{\partial}{\partial x^j}|_{\sigma(p)}\right) \\ &= a^j(p) \\ &= \sigma\left(\frac{\partial}{\partial x^j}|_p\right) \end{aligned}$$

as required. □

Proposition 1.2. *The graph $\Gamma_\sigma \subset T^*Q$ of a 1-form on $(T^*Q, -d\lambda_{can})$ is Lagrangian if and only if σ is closed.*

Proof. Since it is a section, $\sigma : Q \to T^*Q$ is an embedding and the submanifold is half the dimension of T^*Q. Then the canonical symplectic form $-d\lambda_{can}$ vanishes on Γ_σ if and only if $0 = \sigma^* d\lambda_{can} = d(\sigma^*\lambda_{can}) = d\sigma$ on Q. □

Theorem 1.3. *(Weinstein's Lagrangian Neighbourhood Theorem) Let (M, ω) be a symplectic manifold and $L \subset M$ a compact Lagrangian submanifold. Denote T^*L_0 the zero section of T^*L. Then there exists*

- *a diffeomorphism of neighbourhoods $\Phi : \mathcal{N}(L) \to \mathcal{N}(T^*L_0)$,*
- *such that $\Phi(p) = 0_p \;\; \forall p \in L$,*
- *and $\Phi^*\omega_{can} = \omega$.*

Proof. The proof relies on definitions and results not covered yet and is not illuminating to the topic, so we only give a sketch. For a full proof, the reference is Thm 3.32 in McDuff, Salamon [1].

The similarity to Moser's isotopy is not accidental. One can use a 'complex structure' J to define an isomorphism between the normal bundle of L and TL, this induces a metric g_J on M. The resulting exponential map defines Φ with the required pullback on the zero section. 'Moser's trick' as discussed in previous talks, finishes the argument. $\square$

Remark. Note if (M, ω) is compact, then the diagonal Δ is a compact Lagrangian of $(M \times M, (-\omega) \times \omega)$ so we get a diffeomorphism $\Phi : \mathcal{N}(\Delta) \to \mathcal{N}(T^*M_0)$ such that $\Phi^*(\omega_{can}) = (-\omega) \times \omega$.

Proposition 1.4. *Let $(M, -d\lambda)$ be any exact symplectic manifold. Suppose ψ_t is a Hamiltonian isotopy, then $\psi_t^*\lambda - \lambda = dF_t$ for a smooth family of functions $F_t : M \to \mathbb{R}$.*

Proof. For generating vector field X_t of isotopy ψ_t, we shift the derivative to zero and use *Cartan's Magic Formula*

$$\frac{d}{dt}\psi_t^*\lambda = \psi_t^*(\mathcal{L}_{X_t}\lambda) = \psi_t^*(\iota(X_t)d\lambda + d\iota(X_t)\lambda) = \psi_t^* d(\iota(X_t)\lambda - H_t)$$

then we can integrate the expression inside the exterior derivative, $F_t = \int_0^t (\iota(X_s)\lambda - H_s) \circ \psi_s ds$ to get $dF_t = \psi_t^*\lambda - \lambda$ by the Fundamental Theorem of Calculus. $\square$

Definition 1.5. For a map $\psi : M \to M$, the map $gr_\psi : M \to M \times M$ is defined as $gr_\psi(p) := (p, \psi(p))$.

Corollary 1.6. *Let $\alpha \in \Omega^1(M \times M)$ be such that $-d\alpha = (-\omega) \times \omega$ and $\alpha|\Delta = 0$, in the sense $\iota^*\alpha \equiv 0$. Then for Hamiltonian symplectomorphisms $\psi : M \to M$, the form $gr_\psi^*\alpha \in \Omega(M)$ is exact.*

Proof. The symplectomorphism $id \times \psi : M \times M \to M \times M$ is Hamiltonian and $gr_\psi = (id \times \psi) \circ \iota$ where ι is the diagonal embedding. By the previous proposition, evaluating at $t = 1$, there's an $F : M \times M \to \mathbb{R}$ such that $dF = (id \times \psi)^*\alpha - \alpha$. Note $\iota^*\alpha = 0$ and so

$$gr_\psi^*\alpha = \iota^*(id \times \psi)^*\alpha = \iota^*(\alpha + dF) = d(F \circ \iota).$$

$\square$

2 Arnold's conjecture

Definition 2.1. A critical point $p \in M$ of a real valued function $F : M \to \mathbb{R}$ is such that $dF_p : T_pM \to T_{F(p)}\mathbb{R}$ is zero. Equivalently $dF : M \to T^*M$ maps p to the zero cotangent 0_p.

Conjecture 2.2. *(Arnold's conjecture – 1960s) Let $\psi : M \to M$ be a Hamiltonian symplectomorphism of a compact symplectic manifold (M, ω). Then ψ must have at least as many fixed points as a function on M must have critical points.*

Note this puts a topological constraint on the symplectic structure of (M, ω) through Morse theory [2] which states that the number of critical points of a *Morse function* is bounded below by the sum of the Betti numbers of the manifold. We can now prove Arnold's conjecture in the special case when the symplectomorphism is near enough the identity.

Theorem 2.3. *The Arnold conjecture is satisfied for every Hamiltonian symplectomorphism ψ, of a compact symplectic manifold, if it is sufficiently close to the identity in the C^1-topology.*

Proof. Recall $graph(\psi)$ is a Lagrangian submanifold of $(M \times M, (-\omega) \times \omega)$, and it is close to the diagonal. A neighbourhood of the diagonal can be identified with a neighbourhood of the zero section of $(T^*M, -d\lambda_{can})$, this is by *Weinstein's Lagrangian Neighbourhood Theorem* using that the diagonal is a compact Lagrangian itself. Denote $\Phi : \mathcal{N}(\Delta) \to \mathcal{N}(T^*M_0)$ the diffeomorphism.

Using this diffeomorphism, proposition 1.2 and that ψ is C^1-close enough to id_M, the Lagrangian submanifold (in T^*M) is a graph of a closed 1-form $\sigma : M \to T^*M$, i.e.

$$\Phi(graph(\psi)) = graph(\sigma).$$

Controlling the derivatives ensures $graph(\psi)$ can be considered as a graph over the diagonal, and therefore σ must exist.

We claim ψ being Hamiltonian and close to the identity implies it must be the graph of an *exact* form,

$$\Phi(graph(\psi)) = graph(\sigma) = graph(dF).$$

Indeed, $\Phi^*\lambda_{can}$ is as required in corollary 1.6, so $gr_\psi^*\Phi^*\lambda_{can}$ is exact, then by the tautological property $dF = gr_\psi^*\Phi^*\lambda_{can} = (\Phi \circ gr_\psi)^*\lambda_{can} = \Phi \circ gr_\psi = \sigma$.

The fixed points of ψ correspond to critical points of F

$$\begin{aligned} Fix(\psi) &= \Delta \cap graph(\psi) \\ &= \Delta \cap \Phi^{-1}(graph(dF)) \\ &= \Phi^{-1}(T^*M_0 \cap graph(dF)) \\ &= \Phi^{-1}(Crit(F)) \end{aligned}$$

and the theorem follows. □

References

[1] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*, volume 27. Oxford University Press, 2017.

[2] John Milnor. *Morse theory (AM-51)*, volume 51. Princeton University Press, 2016.