# Motivating the importance of Legendrian submanifolds

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**Abstract**

In this talk we start by introducing the Weinstein neighbourhood Theorem that gives universal models for Lagrangian and Legendrian submanifolds. Then we discuss Legendrian and transverse knots: we define the classical invariants such as the Thurston-Bennequin, the rotation and the self-linking numbers and compute them from the front and Lagrangian projections. We also mention the Bennequin inequality and the Eliashberg-Fraser Theorem. We conclude by noticing how the classical invariants of Legendrian knots are related to Lagrangian cobordisms between them.

## 1 Weinstein Neighbourhood Theorem

We introduce some concepts useful for the following discussion.

**Definition 1.1.** A **symplectic homomorphism** $\Phi\colon (V_0,\omega_0) \to (V_1,\omega_1)$ between symplectic vector spaces $(V_i,\omega_i)$ is a homomorphism that preserves the symplectic form, i.e. $\omega_1 \circ (\Phi,\Phi) = \omega_0$. If $\Phi$ only preserves the conformal class of the symplectic form, i.e. $\omega_1 \circ (\Phi,\Phi) = \lambda\,\omega_0$ for some $\lambda \in \mathbb{R}_{>0}$, we call $\Phi$ a **conformal symplectic homomorphism**. If moreover $\Phi$ is an isomorphism, then we call $\Phi$ a **symplectic isomorphism**, or a **conformal symplectic isomorphism** respectively.

A **symplectic vector bundle homomorphism** $\Phi\colon (E_0,\omega_0) \to (E_1,\omega_1)$ **over** $\phi$ between symplectic vector bundles $(E_i,\omega_i) \to M_i$, where $\phi\colon M_0 \to M_1$ is a smooth map, is a vector bundle homomorphism over $\phi$ that preserves the symplectic form, i.e. $\Phi_p\colon (E_0|_p,\omega_0|_p) \to (E_1|_{\phi(p)},\omega_1|_{\phi(p)})$ is a symplectic homomorphism for every $p \in M_0$. If $\Phi$ preserves the symplectic form only conformally, i.e. $\omega_1|_{\phi(p)} \circ (\Phi_p,\Phi_p) = \lambda(p)\,\omega_0|_p$ for some smooth function $\lambda\colon M_0 \to \mathbb{R}_{>0}$, then we call $\Phi$ a **conformal symplectic vector bundle homomorphism over** $\phi$. If moreover $\Phi$ is a vector bundle isomorphism over $\phi$, we say that $\Phi$ is a **symplectic vector bundle isomorphism**, or a **conformal symplectic vector bundle isomorphism** respectively, **over** $\phi$. If $M_0 = M_1$ we simply call $\Phi$ a **(conformal) vector bundle homomorphism**, or **isomorphism** respectively.

**Definition 1.2.** A **complex structure** on a real vector space $V$ is an automorphism $J\colon V \to V$ satisfying $J^2 = -id_V$. If $(V,\omega)$ is a symplectic vector space, then a complex structure $J$ on $V$ is called **$\omega$-compatible** if it is a symplectic isomorphism and the bilinear form $\omega \circ (id_V, J)$ is positive definite.

A **complex structure** on a vector bundle $E \to M$ is a smooth family $J_p$ of complex structures on the fibres $E_p$, i.e. $J \in \Gamma(\mathrm{End}(E))$ and $J^2 = -id_E$. A complex structure on a symplectic vector bundle $(E,\omega)$ is called **$\omega$-compatible** if $J_p$ is $\omega_p$-compatible on $E_p$ for every $p \in M$. A complex structure on the tangent bundle of a manifold is called an **almost complex structure**. A symplectic manifold $(M, \omega)$ equipped with an $\omega$-compatible almost complex structure $J$ is called a **Kähler manifold**, in this case $M$ has also a natural Riemannian structure given by the metric $g := \omega \circ (id_{TM}, J)$.

Let $(Y, \xi = \ker \alpha)$ be a contact manifold with cooriented contact structure. A complex structure $J\colon \xi \to \xi$ on $\xi$ is called **$\xi$-compatible** if $J$ is $d\alpha|_\xi$-compatible.

*Remark* 1.3. The space of $\omega$-compatible complex structures on a symplectic vector space $(V, \omega)$, equipped with the subspace topology inherited from $\mathrm{End}(V)$, is non-empty and contractible, see [Gei08, Proposition 1.3.10]. This directly translates to the same statement for $\omega$-compatible complex structures on a symplectic vector bundle $(E, \omega) \to M$, see [Gei08, Proposition 2.4.5].

Let $(Y, \xi = \ker \alpha)$ be a contact manifold with cooriented contact structure. Then $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle over $Y$. Note that if $\alpha$ is replaced by $\lambda\alpha$ for some smooth function $\lambda\colon Y \to \mathbb{R}_{>0}$, then $d(\lambda\alpha)|_\xi = \lambda d\alpha|_\xi$, that is, the **conformal class** of the symplectic bundle structure depends only on the cooriented contact structure $\xi$. This allows us to make the following definition.

**Definition 1.4.** Let $K \subset (Y, \xi = \ker \alpha)$ be an isotropic submanifold in a contact manifold with cooriented contact structure. Write $(TK)^\perp \subset \xi|_K$ for the subbundle of $\xi|_K$ that is symplectically orthogonal to $TK$ with respect to the symplectic structure $d\alpha|_\xi$. Since the conformal class of this symplectic structure only depends on the contact structure $\xi$, not on the choice of contact form $\alpha$ defining $\xi$, the bundle $(TK)^\perp$ is determined by $\xi$. Note that the fact that $K$ is isotropic implies that $TK \subset (TK)^\perp$. The quotient bundle

$$CSN_Y(K) := (TK)^\perp / TK$$

equipped with the conformal symplectic structure induced by $d\alpha$ is called the **conformal symplectic normal bundle** of $K$ in $Y$. So the normal bundle $\mathrm{Nor}\, K \cong (TY|_K)/TK$ of $K$ in $Y$ splits as

$$\mathrm{Nor}\, K \cong (TY|_K)/(\xi|_K) \oplus (\xi|_K)/(TK)^\perp \oplus CSN_Y(K). \tag{1}$$

Analogously, let $L \subset (M, \omega)$ be an isotropic submanifold in a symplectic manifold. Write $(TL)^\perp \subset TM|_L$ for the subbundle of $TM|_L$ that is symplectically orthogonal to $TL$ with respect to the symplectic structure $\omega$. Note that in this setting the whole conformality argument can be dropped. We then define the **symplectic normal bundle** of $L$ in $M$ as

$$SN_M(L) := (TL)^\perp / TL$$

equipped with the symplectic structure induced by $\omega$. In this case the normal bundle of $L$ in $M$ splits as

$$\mathrm{Nor}\, L \cong (TM|_L)/(TL)^\perp \oplus SN_M(L). \tag{2}$$

*Remark* 1.5. In the contact case, observe that if $\dim Y = 2n+1$ and $\dim K = k \leq n$, then the rank of $(TK)^\perp$ is $2n-k$, so the ranks of the three summands in the above splitting are

1, $k$ and $2(n-k)$, respectively. In the symplectic case, if $\dim M = 2n$ and $\dim L = l \leq n$ then the ranks of the two summands in the above splitting are $l$ and $2(n-l)$. Our aim in this section is to show that a neighbourhood of $K$ in $Y$ is determined, up to contactomorphism, by the conformal symplectic isomorphism class of $CSN_Y(K)$ and analogously a neighbourhood of $L$ in $M$ is determined up to symplectomorphism by the symplectic isomorphism class of $SN_M(L)$. As a first step, we show that the the topological bundle type of the normal bundle $\operatorname{Nor} K$ is completely determined by the conformal symplectic normal bundle. The bundle $(TY|_K)/(\xi|_K)$ is a trivial line bundle because $\xi$ is cooriented. The Reeb vector field $R_\alpha$ induces a nowhere vanishing section of this quotient bundle, so it is convenient to identify $(TY|_K)/(\xi|_K)$ with the line subbundle $\langle R_\alpha \rangle \subset TY$ spanned by $R_\alpha$. It is immediate that the second summand in Eq. (1) is determined by the topological type of $K$ alone, since $(\xi|_K)/(TK)^\perp \cong T^*K$ via the vector bundle isomorphism $\Psi\colon (\xi|_K)/(TK)^\perp \to T^*K$, $[X] \mapsto i_X d\alpha|_{TK}$. The same map also yields $(TM|_L)/(TL)^\perp \cong T^*L$ for the symplectic case. Additionally, denoting by $J$ a $\xi$-compatible complex structure on $\xi$ we have a vector bundle isomorphism $(\xi|_K)/(TK)^\perp \cong J(TK)$ and a conformal symplectic vector bundle isomorphism $CSN_Y(K) \cong (TK \oplus J(TK))^\perp$, see [Gei08, Proposition 2.5.5]. In the symplectic case, an $\omega$-compatible complex structure on $TM$ yields a vector bundle isomorphism $(TM|_L)/(TL)^\perp \cong J(TL)$ and a symplectic vector bundle isomorphism $SN_M(L) \cong (TL \oplus J(TL))^\perp$ in the same way. Thus Eq. (1) also reads

$$\operatorname{Nor} K \cong \langle R_\alpha \rangle \oplus J(TK) \oplus (TK \oplus J(TK))^\perp \tag{3}$$

and analogously Eq. (2) reads

$$\operatorname{Nor} L \cong J(TL) \oplus (TL \oplus J(TL))^\perp. \tag{4}$$

We can use Eq. (3) (or Eq. (4) respectively) to extend a conformal symplectic vector bundle isomorphism at the level of conformal symplectic normal bundles (or a symplectic vector bundle isomorphism at the level of symplectic normal bundles) to a vector bundle isomorphism at the level of normal bundles. Applying Gray's stability Theorem (or Moser's isotopy Theorem) to a suitable family of contact structures (symplectic structures) we then obtain the following results, see [Gei08, Theorem 2.5.8] for the contact case.

**Theorem 1.6.** *Let $(Y_i, \xi_i)$ for $i = 0, 1$ be contact manifolds with closed isotropic submanifolds $K_i$. If there is a conformal symplectic vector bundle isomorphism*

$$\Phi\colon CSN_{Y_0}(K_0) \to CSN_{Y_1}(K_1)$$

*over a diffeomorphism $\phi\colon K_0 \to K_1$, then $\phi$ extends to a contactomorphism*

$$\psi\colon \mathcal{N}(K_0) \to \mathcal{N}(K_1)$$

*of suitable neighbourhoods $\mathcal{N}(K_i)$ of $K_i$ in $Y_i$, such that the maps $D\psi|_{CSN_{Y_0}(K_0)}$ and $\Phi$ are bundle homotopic as conformal symplectic vector bundle isomorphisms.*

*Analogously, let $(M_i, \omega_i)$ for $i = 0, 1$ be symplectic manifolds with closed isotropic submanifolds $L_i$. If there is a symplectic vector bundle isomorphism*

$$\Phi\colon SN_{M_0}(L_0) \to SN_{M_1}(L_1)$$

*over a diffeomorphism $\phi\colon L_0 \to L_1$, then $\phi$ extends to a symplectomorphism*

$$\psi\colon \mathcal{N}(L_0) \to \mathcal{N}(L_1)$$

*of suitable neighbourhoods $\mathcal{N}(L_i)$ of $L_i$ in $M_i$, such that the maps $D\psi|_{SN_{M_0}(L_0)}$ and $\Phi$ are bundle homotopic as symplectic vector bundle isomorphisms.*

*Proof.* Let us start with the contact case. Choose contact forms $\alpha_i$ of $\xi_i$ for $i = 0, 1$, scaled in such a way that $\Phi$ is actually a symplectic vector bundle isomorphism with respect to the symplectic structures on $CSN_{Y_i}(K_i)$ given by $d\alpha_i$. We regard $CSN_{Y_i}(K_i)$ and

$$\operatorname{Nor} K_i \cong \langle R_{\alpha_i} \rangle \oplus J_i(TK_i) \oplus (TK_i \oplus J_i(TK_i))^{\perp}$$

as subbundles of $TY_i|_{K_i}$. Let $\Phi_R\colon \langle R_{\alpha_1} \rangle \to \langle R_{\alpha_2} \rangle$ be the vector bundle isomorphism over $\phi$ obtained by the requirement $\Phi_R(R_{\alpha_0}(p)) := R_{\alpha_1}(\phi(p))$. Let $\Psi_i\colon J_i(TK_i) \to T^*K_i$ be the vector bundle isomorphism defined by taking the interior product with $d\alpha_i$ as in Remark 1.5. Note that on a smooth manifold $K$ there is a canonical symplectic structure $\Omega_K$ on $TK \oplus T^*K$ given by

$$\Omega_K(X \oplus \eta, X' \oplus \eta') := \eta(X') - \eta'(X).$$

Moreover, one can easily show, see [Gei08, Lemma 2.5.7], that the map

$$id_{TK_i} \oplus \Psi_i\colon (TK_i \oplus J_i(TK_i), d\alpha_i) \to (TK_i \oplus T^*K_i, \Omega_{K_i})$$

is a symplectic vector bundle isomorphism. Now notice that

$$D\phi \oplus (D\phi^{-1})^*\colon (TK_0 \oplus T^*K_0, \Omega_{K_0}) \to (TK_1 \oplus T^*K_1, \Omega_{K_1})$$

is a symplectic vector bundle isomorphism, where $(D\phi^{-1})^*\colon T^*K_0 \to T^*K_1$ denotes the adjoint of $D\phi^{-1}$. It follows that

$$D\phi \oplus \Psi_1^{-1} \circ (D\phi^{-1})^* \circ \Psi_0\colon (TK_0 \oplus J(TK_0), d\alpha_0) \to (TK_1 \oplus J(TK_1), d\alpha_1)$$

is also a symplectic vector bundle isomorphism over $\phi$. We define

$$\widetilde{\Phi} := \Phi_R \oplus D\phi \oplus \Psi_1^{-1} \circ (D\phi^{-1})^* \circ \Psi_0 \oplus \Phi\colon \operatorname{Nor} K_0 \to \operatorname{Nor} K_1$$

which is a symplectic vector bundle isomorphism over $\phi$. Let $\tau_i\colon \operatorname{Nor} K_i \to Y_i$ be tubular maps, i.e. $\tau_i$ are embeddings such that under the identification of $K_i$ with the zero section of $\operatorname{Nor} K_i$, the map $\tau|_{K_i}$ is the inclusion $K_i \subset Y_i$, and with respect to the splitting $T(\operatorname{Nor} K_i)|_{K_i} \cong TK_i \oplus \operatorname{Nor} K_i \cong$

$TY_i|_{K_i}$,the derivative $D\tau_i$ induces the identity on $\operatorname{Nor} K_i$ along $K_i$. Then

$$\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1} \colon \mathcal{N}(K_0) \to \mathcal{N}(K_1)$$

is a diffeomorphism of suitable neighbourhoods $\mathcal{N}(K_i)$ of $K_i$ inducing the vector bundle isomorphism $D\phi \oplus \widetilde{\Phi} \colon TY_0|_{K_0} \to TY_1|_{K_1}$ along $\phi$, which by construction pulls $\alpha_1$ back to $\alpha_0$ and $d\alpha_1$ back to $d\alpha_0$. Hence, $\alpha_0$ and $(\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1})^* \alpha_1$ are contact forms on $\mathcal{N}(K_0)$ that coincide on $TY_0|_{K_0}$, and so do their differentials. Now consider the family of 1-forms given by

$$\beta_t = (1-t)\alpha_0 + t(\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1})^* \alpha_1 \quad \text{for} \quad t \in [0,1].$$

On $TY_0|_{K_0}$ we have that $\beta_t = \alpha_0$ and $d\beta_t = d\alpha_0$. Since the contact condition $\alpha \wedge (d\alpha)^n = 0$ is an open condition, we may assume, by shrinking $\mathcal{N}(K_0)$ if necessary, that $\beta_t$ is a contact form on $\mathcal{N}(K_0)$ for all $t \in [0,1]$. By Gray's stability Theorem there exists an isotopy $\psi_t$ of $\mathcal{N}(K_0)$ fixing $K_0$ and such that $\psi_t^* \beta_t = \lambda_t \alpha_0$ for some smooth family of smooth functions $\lambda_t \colon \mathcal{N}(K_0) \to \mathbb{R}_{>0}$, for $t \in [0,1]$. It follows that the composition $\psi := \tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1} \circ \psi_1$ is the desired contactomorphism.

The proof of the symplectic case is obtained in the same way using Eq. (4); in particular the factor relative to the Reeb field is absent. All the vector bundle isomorphisms work in the same way, up to replacing the conformal symplectic vector bundle $(\xi_i, d\alpha_i)$ with $(TM_i, \omega_i)$, that means that all the terms "conformal" can be dropped. We then use the map $\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1}$ to pull $\omega_1$ back on $\mathcal{N}(L_0)$ in order to apply Moser's isotopy Theorem, which concludes the proof in the analogous way. □

**Corollary 1.7.** Let $K_i \subset (Y_i, \xi_i)$ for $i = 0,1$ be closed Legendrian submanifolds. If they are diffeomorphic, then they have contactomorphic neighbourhood.

Analogously, let $L_i \subset (M_i, \omega_i)$ for $i = 0,1$ be closed Lagrangian submanifolds. If they are diffeomorphic, then they have symplectomorphic neighbourhoods.

*Proof.* The conformal symplectic normal bundle of a Legendrian submanifold has rank 0, i.e. it is the zero vector bundle, thus we can apply Theorem 1.6. The same is true for the symplectic normal bundle of a Lagrangian submanifold. □

**Example 1.8.** Let $K \subset (Y^3, \xi)$ be a Legendrian knot in a contact 3-manifold. We identify a neighbourhood of $K$ with $S^1 \times \mathbb{R}^2$, where $K \cong S^1 \times \{0\}$. Then, with $S^1$-coordinate $\theta$ and Cartesian coordinates $(x,y)$ on $\mathbb{R}^2$, the contact structure

$$\cos(\theta)dx - \sin(\theta)dy = 0$$

provides a model for a neighbourhood of $K$.

**Example 1.9.** Let $K$ be a smooth manifold and denote by $\mathcal{E}_p$ be the space of germs at $p \in K$ of smooth $\mathbb{R}$-valued functions on $K$. We define an equivalence relation on $\mathcal{E}_p$ by

$$f_1 \sim f_2 :\Longleftrightarrow f_1(p) = f_2(p) \quad \text{and} \quad df_1(p) = df_2(p).$$

The equivalence class $j^1_p f$ of a germ $f \in \mathcal{E}_p$ is called a 1-jet at $p$. We define the 1-**jet bundle** of $K$ as

$$J^1K := \bigsqcup_{p \in K} \mathcal{E}_p / \sim .$$

There exists a unique smooth structure on $J^1K$ such that the bijection $j^1_p f \mapsto (f(p), df_p)$ is a diffeomorphism, then $J^1K$ can be identified with $\mathbb{R} \times T^*K$.

Let $\lambda$ be the Liouville form on $T^*K$ given in local coordinates $(q^i, p^i)$ on $T^*K$ by $\lambda = \sum_i p^i dq^i$. Writing $z$ for the $\mathbb{R}$-coordinate on $\mathbb{R} \times T^*K$, we have a natural contact structure on $\mathbb{R} \times T^*K$ given by $\xi = \ker(dz - \lambda)$ which is inherited by $J^1K$ thanks to the diffeomorphism as above. We denote the inherited contact structure on $J^1K$ by $\xi_{jet}$; thus $(J^1K, \xi_{jet})$ is canonically a contact manifold.

Every smooth function $f \in C^\infty(K)$ determines a Legendrian embedding of $K$ into $(J^1K, \xi_{jet})$ given by

$$j^1(f) \colon K \to \mathbb{R} \times T^*K, p \mapsto (f(p), df_p).$$

In particular, the identically zero function corresponds to the zero section $j^1(0) \cong K \subset T^*K \subset \mathbb{R} \times T^*K$. If $K \subset (Y, \xi)$ is a closed Legendrian submanifold, by Corollary 1.7 a neighbourhood of $K$ in $Y$ is contactomorphic to a neighbourhood of $j^1(0)$ in $J^1K$; in this sense, for every Legendrian submanifold the zero section of its 1-jet bundle provides a model.

An analogous discussion shows that the zero section of cotangent bundles provide universal models for Lagrangian submanifolds. We summarize these special cases which are both known as the Weinstein neighbourhood Theorem, see [MS17, Theorem 3.4.13] for a direct proof of the symplectic case.

**Theorem 1.10** (Weinstein). *Let $K \subset (Y, \xi)$ be a closed Legendrian submanifold of a contact manifold. Then a neighbourhood of $K$ in $Y$ is contactomorphic to a neighbourhood of $j^1(0)$ in $J^1K$.*

*Analogously, let $L \subset (M, \omega)$ be a closed Lagrangian submanifold of a symplectic manifold. Then a neighbourhood of $L$ in $M$ is contactomorphic to a neighbourhood of the zero section in $T^*L$.*

# 2 Legendrian Knots

The study of knots in 3-manifolds is a venerable subject in its own right, the main question being the classification of knots in 3-space up to isotopy. However, knots also play an important supporting role in the geometric topology of 3-manifolds and when we turn to contact 3-manifolds the situation is analogous. Here there are two distinguished classes of knots: the Legendrian and the transverse ones. Again, these are interesting objects in their own right, and much can be said about the classification of these knots up to isotopy through knots of the given type.

We first discuss how Legendrian and transverse knots can be visualised by two kinds of 2-dimensional projections adapted to the contact structure: the front and the Lagrangian projection. Then we introduce the classical invariants: the Thurston-Bennequin invariant and rotation number of Legendrian knots, and the self-linking number of transverse knots. We discuss relations between these invariants and compute them from the knot projections.

**Definition 2.1.** A **knot** in a 3-manifold $Y$ is an embedding of $S^1$ into $Y$. We say that a knot is **topologically trivial**, or an **unknot**, if it is isotopic to the boundary of an embedded disk in $Y$ through an ambient isotopy $\phi_t\colon Y \to Y$. We say that a knot is **homologically trivial** if its integral homology class as a 1-simplex in $Y$ is trivial.

Let now $(Y, \xi)$ be a 3-dimensional contact manifold. A **Legendrian knot** in $Y$ is an embedding $\gamma\colon S^1 \to Y$ such that $\dot\gamma \subset \xi$, or equivalently $\alpha(\dot\gamma) = 0$ for every local contact form $\alpha$. We denote by $K$ the image of $\gamma$, which is a Legendrian submanifold of $Y$.

On the other hand, a **transverse knot** in $Y$ is an embedding $\gamma\colon S^1 \to Y$ such that $\dot\gamma \oplus \xi = TY$. If $\xi = \ker\alpha$ is cooriented, we say that the knot is **positively** or **negatively transverse** depending on whether $\alpha(\dot\gamma) > 0$ or $\alpha(\dot\gamma) < 0$.

We shall frequently have to associate with some given Legendrian knot a transverse or a second Legendrian knot, where this associated knot ought to be canonically defined up to transverse or Legendrian isotopy, respectively. This happens, for instance, when we want to prove the analogue for transverse knots of a result already proved for Legendrian knots.

**Definition 2.2.** Let $\gamma\colon S^1 \to K \subset Y$ be a Legendrian knot. By Example 1.8, a neighbourhood of $K$ in $(Y, \xi)$ is contactomorphic to a neighbourhood of the Legendrian knot

$$\theta \mapsto (\theta, x = 0, y = 0) \in S^1 \times \mathbb{R}^2, \ \theta \in S^1,$$

with contact structure on $S^1 \times \mathbb{R}^2$ given by the contact form

$$\alpha = \cos(\theta)dx - \sin(\theta)dy.$$

We identify $\gamma$ with this model Legendrian $S^1$. Observe that the radial vector field $X := x\partial_x + y\partial_y$ is a contact vector field, since $\mathcal{L}_X\alpha = \alpha$. On the boundary torus $S := \{x^2 + y^2 = \delta\}$ for $\delta > 0$ of a thin cylinder around $K$, there are two distinguished curves along which $X$ is tangent to $\xi$, namely

$$\gamma_\pm := (\theta, \pm\delta\sin(\theta), \pm\delta\cos(\theta)).$$

Observe that $\gamma_\pm$ is obtained from $\gamma$ by pushing it in the direction of the vector field $\pm(\sin(\theta)\partial_x + \cos(\theta)\partial_y)$ tangent to $\xi$ but transverse to $\dot\gamma$. Moreover, we have that $\alpha(\dot\gamma_\pm) = \pm\delta$. Different choices of $\delta > 0$ yield knots that are isotopic as transverse knots. We therefore call $\gamma_\pm$ the **positive** or **negative**, respectively, **transverse push-off** of $\gamma$.

The torus $S$ is foliated by the Legendrian curves $\theta \to (\theta, x_0, y_0)$, with $x_0^2 + y_0^2 = \delta$. There are two further distinguished Legendrian curves on $S$, along which the contact planes are tangent to $S$. These are given by $\gamma_l(\theta) := (\theta, \pm\delta\cos(\theta), \mp\delta\sin(\theta))$. Notice that these Legendrian curves lie in the same homotopy class of $\gamma_\pm$ as curves on $S$. We call either of them a **Legendrian push-off** of $K$. By varying the parameter $\delta$ we see that these Legendrian push-offs are Legendrian isotopic to $K$.

Finally, given a transverse knot in a contact 3-manifold, we may think of it as the knot $\theta \mapsto (\theta, x = 0, y = 0), \theta \in S^1$ in $S^1 \times \mathbb{R}^2$ with contact structure given by $d\theta + xdy - ydx = 0$. This has Legendrian push-offs, but in contrast with the previous cases there is no canonical choice. For instance, for any $k \in \mathbb{Z} \setminus \{0\}$ we can form the Legendrian push-off

$$\theta \mapsto (\theta, \frac{1}{k}\cos(k^2\theta), \frac{1}{k}\sin(k^2\theta)).$$

The invariants that we introduce later are supposed to distinguish Legendrian knots up to Legendrian isotopy.

**Definition 2.3.** Two Legendrian knots $K_0$ and $K_1$ in a contact manifold $Y$ are **Legendrian isotopic** if there is a smooth map $\Phi\colon S^1 \times [0,1] \to Y$ such that $\Phi_0(S^1) = K_0$, $\Phi_1(S^1) = K_1$ and for every $t \in [0,1]$ we have that $K_t := \Phi_t(S^1) := \Phi(S^1, t)$ is a Legendrian knot.

Analogously, two transverse knots $K_0$ and $K_1$ in a contact manifold $Y$ are **transverse isotopic** if there is a smooth map $\Phi\colon S^1 \times [0,1] \to Y$ such that $\Phi_0(S^1) = K_0$, $\Phi_1(S^1) = K_1$ and for every $t \in [0,1]$ we have that $K_t := \Phi_t(S^1) := \Phi(S^1, t)$ is a transverse knot.

The following Proposition follows from Gray's stability Theorem and it shows that Legendrian isotopies are equivalent to contact isotopies, see [Gei08, Theorem 2.6.2].

**Proposition 2.4.** *Two Legendrian knots $K_0$, $K_1$ are Legendrian isotopic if and only if there exists a one-parameter family of contact diffeomorphisms $\phi_t\colon Y \to Y$ such that $\phi_0 = id$ and $\phi_1(K_0) = K_1$.*

For questions such as surgery descriptions of contact 3-manifolds, the Legendrian or transverse knots of primary interest are those in $S^3$ with its standard contact structure. Given any knot in $S^3$, we may assume that it misses a given point and regard it as a knot in $\mathbb{R}^3$. Since $S^3$ without a point and equipped with the standard contact structure is contactomorphic to $\mathbb{R}^3$ with its standard contact structure, any Legendrian knot in $S^3$ may be regarded as a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ where $\xi_{st} = \ker \alpha_{st}$ and $\alpha_{st} = dz + xdy$.

**Definition 2.5.** The **front projection** of a parametrised curve $\gamma(s) = (x(s), y(s), z(s))$ in $(\mathbb{R}^3, \xi_{st})$ is the curve

$$\gamma_F(s) := (y(s), z(s));$$

its **Lagrangian projection** is the curve

$$\gamma_L(s) := (x(s), y(s)).$$

Figure 1: ([MS17]) Front projections of four different Legendrian knots in $\mathbb{R}^3$.

**Example 2.6.**

**Lemma 2.7.** *Let $\gamma\colon (a,b) \to (\mathbb{R}^3, \xi_{st})$ be a Legendrian immersion. Then its front projection $\gamma_F$ does not have any vertical tangencies. Away from the cusp points, $\gamma$ is recovered from its front projection via $x(s) = -\frac{z'(s)}{y'(s)} = -\frac{dz}{dy}$, i.e. $x(s)$ is the negative slope of the front projection. The curve $\gamma$ is embedded if and only if $\gamma_F$ has only transverse self-intersections.*

**Lemma 2.8.** *Let $\gamma\colon (a,b) \to (\mathbb{R}^3, \xi_{st})$ be a Legendrian immersion. Then its Lagrangian projection $\gamma_L(s) = (x(s), y(s))$ is also an immersed curve. The curve $\gamma$ is recovered from $\gamma_L$ via*

$$z(s_1) = z(s_0) - \int_{s_0}^{s_1} x(s)y'(s)ds.$$

*A Legendrian immersion $\gamma\colon S^1 \to (\mathbb{R}^3, \xi_{st})$ has a Lagrangian projection that encloses zero area. Moreover, $\gamma$ is embedded if and only if every loop in $\gamma_L$ (except, in the closed case, the full loop $\gamma_L$) encloses a non-zero oriented area. Any curve immersed in the $(x,y)$-plane is the Lagrangian projection of a Legendrian curve in $(\mathbb{R}^3, \xi_{st})$, unique up to translation in the $z$-direction. A closed immersed curve $\gamma_L$ in the $(x,y)$-plane lifts to a Legendrian immersion of $S^1$ in $(\mathbb{R}^3, \xi_{st})$ if and only if $\int_{\gamma_L} xdy = 0$.*

**Definition 2.9.** A **knot diagram** $K_P$ is the image of one or two disjoint knots in $\mathbb{R}^3$ under a generic projection onto a 2-plane $P$ in $\mathbb{R}^3$ such that the projected knots are immersed except for transverse double points. A knot diagram is called **oriented** if every knot is oriented; in this case the images of the knots inherit the orientation from the respective knots.

**Definition 2.10.** The **writhe** of an oriented knot diagram $K_P$ is the signed number of self-crossings of the diagram, where the sign of the crossing is given as shown in the following image.

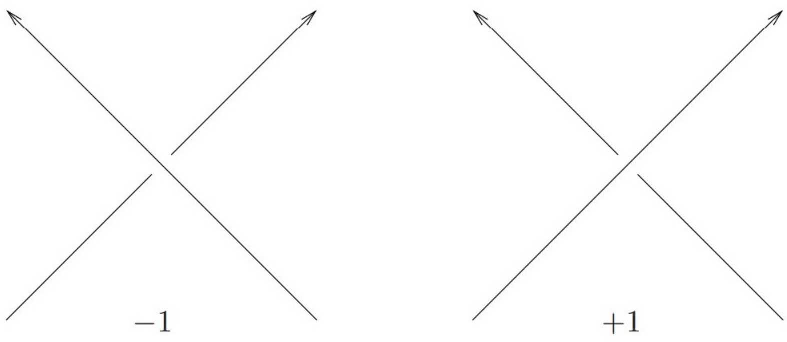


Figure 2: ([Gei08]) Signs of crossings in a knot diagram: Right-handed is positive, left-handed is negative.

We denote this number by writhe($K_P$). We denote the number of points where $K_P$ fails to be an immersion by cusps($K_P$).

*Remark* 2.11. If $K$ and $K'$ are isotopic, then we choose an orientation on $K$ and the orientation on $K'$ will be the one such that $K$ and $K'$ are isotopic as oriented knots. Then the writhe of a knot diagram obtained from $K$ and $K'$ is independent of the chosen orientation of $K$, since if the orientation is reversed, both strands at each crossing reverse their direction, which leaves the sign of the crossing unchanged.

# 3 Topological preliminaries

**Proposition 3.1** ([Gei08])**.** *Let $Y$ be a closed, oriented 3-manifold. Every homology class $c \in H_1(Y, \mathbb{Z})$ is represented by a knot $K_c$ in $Y$.*

**Proposition 3.2** ([Gei08])**.** *Let $Y$ be a closed, oriented 3-manifold. Every homology class $c \in H_2(Y, \mathbb{Z})$ is represented by a smoothly embedded closed, oriented surface $\Sigma_c \subset Y$.*

**Proposition 3.3** ([Gei08])**.** *Let $K$ be a nullhomologous oriented knot in a closed, oriented 3-manifold $Y$. Let $c \in H_2(Y, K)$ be a relative class that maps to $\partial_* c = [K]$, the generator of $H_1(K)$, under the boundary homomorphism $\partial_* \colon H_2(Y, K) \to H_1(K)$; such a class exists by the homology long exact sequence of the pair $(Y, K)$. Then there is an embedded compact, oriented surface $\Sigma_c$ in $Y$ with $\partial \Sigma_c = K$ as oriented manifolds and $[\Sigma_c]_{(Y,K)} = c$.*

**Definition 3.4.** Let $K$ be a nullhomologous knot in an oriented closed 3-manifold $Y$. An embedded connected, compact, orientable surface $\Sigma$ with $\partial\Sigma = K$ is called a **Seifert surface** for $K$. Under these assumptions such a surface exists by Proposition 3.3. If $K$ is orientated, we say that $\Sigma$ is **compatible** if $\Sigma$ is oriented in such a way that the induced orientation on $\partial\Sigma = K$ coincides with the orientation on $K$.

**Definition 3.5.** Let $Y$ be an oriented 3-manifold with $H_1(Y, \mathbb{Z}) \cong 0$; in the following we are interested to the cases where $Y$ is $\mathbb{R}^3$ or $S^3$. Let $K$ be a knot in $Y$ and denote by $\mathcal{N}$ a closed tubular neighbourhood of $K$ in $Y$. Such a neighbourhood is diffeomorphic to the solid torus

$S^1 \times D^2$, since it is the only orientable $D^2$-bundle over $S^1$. Let $\mathcal{C}$ be the closure of $Y \setminus \mathcal{N}$ in $Y$, then by the Mayer-Vietoris sequence for $Y = \mathcal{N} \cup \mathcal{C}$ with $\mathcal{N} \cap \mathcal{C} \cong T^2$ we have

$$H_2(Y) \cong 0 \to H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(\mathcal{N}) \oplus H_1(\mathcal{C}) \cong \mathbb{Z} \oplus H_1(\mathcal{C}) \to H_1(Y) \cong 0.$$

It follows that $H_1(\mathcal{C}) \cong \mathbb{Z}$ and that on $T^2 \cong \partial\mathcal{N}$ there are two distinct curves, unique up to isotopy.

1. The **meridian** $\mu$, defined as a simple closed curve whose homotopy class generates the kernel of the homomorphism $H_1(T^2) \to H_1(\mathcal{N})$.

2. The **preferred longitude** $\lambda$, a simple closed curve whose homotopy class generates the kernel of the homomorphism $H_1(T^2) \to H_1(\mathcal{C})$.

We give $T^2 = \partial\mathcal{N}$ the induced orientation. We also assume $K$ to be oriented. Then $\lambda$ can be oriented by requiring it to be isotopic to $K$ in $\mathcal{N}$ as oriented curve; the orientation we choose for $\mu$ is the one that turns $\mu$, $\lambda$ into a positive basis for that homology group. With an appropriate choice of generator for $H_1(\mathcal{C}) \cong \mathbb{Z}$, the homomorphism $H_1(T^2) \to H_1(\mathcal{N}) \oplus H_1(\mathcal{C})$ is then characterised by $[\mu] \to (0,1)$ and $[\lambda] \to (1,0)$. By excision and homotopy equivalence, we have that

$$H_2(Y, Y \setminus K) \cong H_2(\mathcal{N}, \partial\mathcal{N}),$$

generated by a meridional disc; this follows from considering the homology long exact sequence of the pair $(S^1 \times D^2, T^2)$. The homology exact sequence of the pair $(Y, Y \setminus K)$ then shows that the kernel of the homomorphism

$$H_1(Y \setminus K) \to H_1(Y)$$

induced by inclusion is isomorphic to $\mathbb{Z}$, generated by the homology class $[\mu]$.

**Definition 3.6.** Let $K'$ be a second oriented knot disjoint from $K$. Then the homology class $[K']_{Y\setminus K}$ lies in the kernel of the above homomorphism, hence

$$[K']_{Y\setminus K} = n[\mu]$$

for a uniquely determined $n \in \mathbb{Z}$. This number, which is clearly an isotopy invariant, is called the **linking number** $\mathsf{lk}(K, K')$ of $K$ and $K'$.

*Remark* 3.7. Observe that the linking number is additive in the following sense. Given two disjoint oriented knots $K_1', K_2'$ in the complement of $K$, the homology class $[K_1' + K_2']_{Y\setminus K}$ represented by their connected sum $K_1' + K_2'$ is the sum of the homology classes represented by the two knots, hence

$$\mathsf{lk}(K, K_1' + K_2') = \mathsf{lk}(K, K_1') + \mathsf{lk}(K, K_2').$$

Moreover, it is obvious that $\mathsf{lk}(K, \mu) = 1$. In particular, adding a right-handed twist to $K'$ with respect to $K$, i.e. passing to the connected sum $K' + \mu$, increases the linking number by 1. Finally, notice that $\mathsf{lk}(K, K')$ changes sign if the orientation of either $K$ or $K'$ is reversed.

**Proposition 3.8.** *Let $K, K' \subset \mathbb{R}^3$ be two disjoint oriented knots given by an oriented knot diagram $K_P$. Then $\mathsf{lk}(K, K')$ equals the number of the crossings where $K'$ crosses under $K$, counted with sign as in Definition 2.10.*

*Proof.* A positive crossing of $K'$ under $K$ can be turned into an overcrossing by replacing $K'$ with the connected sum $K' - \mu$; analogously, a negative undercrossing can be turned into an overcrossing by passing to $K' + \mu$. Hence, if $n \in \mathbb{Z}$ denotes the total number of times the knot $K'$ crosses under $K$, then $K' - n\mu$ crosses over $K$ at all crossings. This implies $\mathsf{lk}(K, K' - n\mu) = 0$, and hence $\mathsf{lk}(K, K') = n$. □

# 4 The classical invariants

For this whole section, let $(Y, \xi)$ be a **closed orientable** contact 3-manifold with cooriented contact structure and assume that $Y$ is oriented in such a way that $\alpha \wedge d\alpha$ is a positive volume form on $Y$ for every local contact form $\alpha$. Let $K \subset Y$ be an **nullhomologous** oriented knot in $Y$ with compatible Seifert surface $\Sigma$. Note that the case $(Y, \xi) = (\mathbb{R}^3, \xi_{st})$ is included in the whole discussion as one can pass to the one-point compactification $S^3$; moreover, both $K$ and $\Sigma$ can always be assumed to miss the compactification point. Note that every knot in $\mathbb{R}^3$ or $S^3$ is clearely nullhomologous.

**Definition 4.1.** Let $K$ be Legendrian. Since $TK \subset \xi$, the normal bundle $\operatorname{Nor} K \cong (TY|_K)/TK$ of $K$ in $Y$ splits into

$$\operatorname{Nor} K \cong (TY|_K)/(\xi|_K) \oplus (\xi|_K)/TK.$$

A **contact framing** of $K$ is a trivialisation of $\operatorname{Nor} K$ defined by a choice of this splitting.

*Remark* 4.2. The geometrical interpretation of a contact framing of $K$ is the following. In order to define a framing of $K$, we need to specify a parallel curve to $K$, which we take as a longitude $\lambda$ on the boundary of a tubular neighbourhood $\mathcal{N}$ of $K$ in $Y$. A contact framing can be the one corresponding to the parallel curve we get by pushing $K$ in a direction transverse to $\xi$. In the definition above, this would correspond to choosing a section of the factor $(TY|_K)/(\xi|_K)$. Note that a canonical choice for such a section is the Reeb field. Alternatively, a contact framing can be the one obtained by pushing $K$ in a direction tangent to $\xi$ but transverse to $K$, i.e. by a transverse push-off. This would correspond to choosing a section of $(\xi|_K)/TK$, and yields the same overall trivialisation of $\operatorname{Nor} K$.

**Definition 4.3.** A **surface (or Seifert) framing** of $K$ is the trivialisation of the normal bundle $\operatorname{Nor} K$ corresponding to the parallel curve obtained by pushing $K$ along $\Sigma$.

*Remark* 4.4. An equivalent characterisation of a surface framing is that it is determined by a parallel curve that has linking number 0 with $K$. In particular, the definition of a surface framing is independent of the choice of Seifert surface.

**Definition 4.5.** Let $K$ be Legendrian. The **Thurston-Bennequin invariant** of $K$, denoted by $\mathsf{tb}(K)$, is the twisting of a contact framing relative to a surface framing of $K$, with right-handed twists being counted positively. In other words, if we choose a vector field along $K$ transverse to $\xi$, such as the Reeb field, and define a parallel knot $K'$ by pushing $K$ along this vector field, then

$$\mathsf{tb}(K) = \mathsf{lk}(K, K'),$$

where $K'$ is given the orientation that makes $K$ and $K'$ isotopic as oriented knots.

*Remark* 4.6. It follows from Proposition 2.4 that $\mathsf{tb}(K)$ is indeed invariant under Legendrian isotopies of $K$. Moreover, observe that $\mathsf{tb}(K)$ does not depend on the choice of orientation on $K$.

**Example 4.7.** Consider $S^3 \subset \mathbb{R}^4$ with its standard contact structure $\xi_{st} = \ker \alpha$, where

$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2,$$

consider the Legendrian unknot

$$K = \{x_1^2 + x_2^2 = 1,\ y_1 = y_2 = 0\}.$$

This circle bounds the 2-disc

$$\Sigma := \{x_1^2 + x_2^2 + y_1^2 = 1,\ y_1 \geq 0,\ y_2 = 0\},$$

so the surface framing of $K$ is given by the vector field $\partial_{y_1}$ (or equivalently $\partial_{y_2}$). In order to compute $\mathsf{tb}(K)$, we choose an orientation of $K$, say the one defined by the unit tangent vector field $T := x_1 \partial_{x_2} - x_2 \partial_{x_1}$. Along $K$, the volume form $\alpha \wedge d\alpha$ on $S^3$ takes the form

$$(\alpha \wedge d\alpha)|_K = 2x_1 dy_1 \wedge dx_2 \wedge dy_2 + 2x_x dy_2 \wedge dx_1 \wedge dy_1.$$

The interior product of this volume form with $T$ is $-2dy_1 \wedge dy_2$. So with respect to the chosen orientation of $K$, the oriented surface framing is given by the ordered basis $(\partial_{y_2}, \partial_{y_1})$. The vector field $N := x_1 \partial_{y_2} - x_2 \partial_{y_1}$ along $K$ lies in $\xi_{st}$, and together with $T$ spans the contact distribution. Therefore, $\mathsf{tb}(K)$ is given by the number of rotations of $N$ relative to the ordered basis $(\partial_{y_2}, \partial_{y_1})$ as we traverse $K$ once in the chosen positive direction, which is easily seen to be $\mathsf{tb}(K) = -1$.

**Example 4.8.** In the front projection picture for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$, consider the "shark" $K$, see Figure 3. Since the vector field $\partial_z$ is everywhere transverse to the standard contact structure $\xi_{st} = \ker(dz + x dy)$, a parallel Legendrian knot $K'$ corresponding to the contact framing is simply given by pushing the front projection of the knot in the $z$-direction. In this particular example we find that $\mathsf{tb}(K) = \mathsf{lk}(K, K') = -2$.

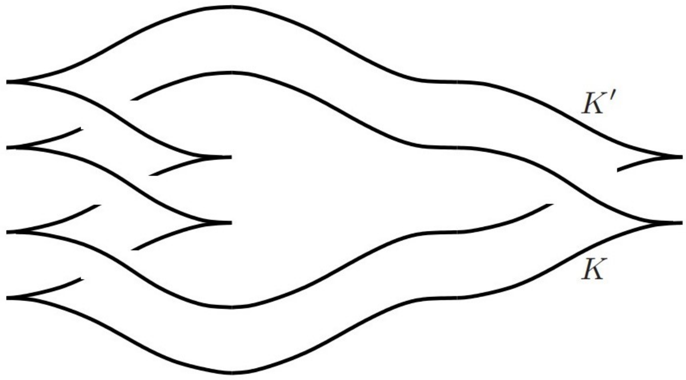


Figure 3: ([Gei08]) Front projection of a topological unknot $K$ with $\mathsf{tb}(K) = -2$.

We want to generalise this example and give a recipe for computing the Thurston-Bennequin invariant of a Legendrian knot in standard $\mathbb{R}^3$ from its front or Lagrangian projection.

**Proposition 4.9.** *Let $K \subset (\mathbb{R}^3, \xi_{st})$ be Legendrian and denote by $K_F$ the knot diagram of $K$ obtained by the front projection. Then*

$$\mathsf{tb}(K) = \mathsf{writhe}(K_F) - \frac{1}{2}\mathsf{cusps}(K_F).$$

*Proof.* We compute $\mathsf{tb}(K)$ as in Example 4.8, i.e. we define a parallel copy $K'$ of $K$, with the induced orientation, by pushing $K$ in the $z$-direction, and then compute $\mathsf{tb}(K)$ as linking number $\mathsf{lk}(K, K')$. That linking number can be computed by counting the crossings of $K'$ under $K$ with sign. It is easy to see that a self-crossing of $K$ will contribute a crossing of $K'$ under $K$ of the same sign, a cusp on the right will give a negative crossing of $K'$ under $K$, and a cusp on the left will give a crossing of $K'$ over $K$. Since there are as many cusps on the left as cusps on the right, the claimed formula follows. □

**Proposition 4.10.** *Let $K \subset (\mathbb{R}^3, \xi_{st})$ be Legendrian and denote by $K_L$ the knot diagram of $K$ obtained by the Lagrangian projection. Then*

$$\mathsf{tb}(K) = \mathsf{writhe}(K_L).$$

*Proof.* The parallel copy $K'$ of $K$ may be obtained by pushing $K$ in a direction tangent to $\xi_{st}$ and transverse to $K$. Under the Lagrangian projection, $\xi_{st}$ projects isomorphically onto the $(x, y)$-plane. This implies that the image of $K'$ under the projection can be represented as parallel to the image of $K$ in the $(x, y)$-plane. The result now follows from Proposition 3.8. □

Let $[\Sigma] \in H_2(Y, K)$ for the relative homology class represented by $\Sigma$. Notice that not all relative homology classes can be represented in this way, since by definition $\Sigma$ is assumed to be connected, and its orientation is fixed by that of $K$. As a surface with boundary, $\Sigma$ retracts to its 1-skeleton, which by the classification of surfaces is a bouquet of $2g$ copies of $S^1$. Since the only orientable vector bundle of rank 2 over $S^1$ is the trivial one, we can find a trivialisation of $\xi|_\Sigma$ which is compatible with the given orientation of $\xi$. Let $\gamma\colon S^1 \to K \subset Y$ be an orientation preserving parametrisation of $K$. If $K$ is Legendrian, given an oriented trivialisation $\xi|_\Sigma = \Sigma \times \mathbb{R}^2$ there is an induced map $\gamma' \colon S^1 \to \mathbb{R}^2 \setminus \{0\}$.

**Definition 4.11.** Let $K$ be Legendrian. The **rotation number** relative to $\Sigma$ is defined as

$$\mathsf{rot}(K, [\Sigma]) := \deg(\gamma'),$$

where deg denotes the degree. In other words, $\mathsf{rot}(K, [\Sigma])$ counts the number of rotations of the positive tangent vector to $K$ relative to the trivialisation of $\xi|_\Sigma$ as we go once around $K$.

*Remark* 4.12. By Proposition 2.4 we have that $\mathsf{rot}(K, [\Sigma])$ is invariant under Legendrian isotopies. However, in contrast with the Thurston-Bennequin invariant, the rotation number does depend on the choice of orientation of $K$. If $\bar{K}$ denotes $K$ with reversed orientation, we have $\mathsf{rot}(K, [\Sigma]) = -\mathsf{rot}(\bar{K}, -[\Sigma])$. Also, the sign of the rotation number depends on the choice of orientation for $\xi$.

*Remark* 4.13. It turns out that the rotation number $\mathsf{rot}(K, [\Sigma])$ does not depend on the choice of trivialisation of $\xi|_\Sigma$, see [Gei08, Lemma 3.5.14]. Moreover, if $\xi$ has zero Euler class (which is true if $\xi$ is a trivial bundle) then $\mathsf{rot}(K, [\Sigma])$ is independent on the class $[\Sigma] \in H_2(Y, K)$ representing $\Sigma$, thus we simply write $\mathsf{rot}(K)$, see [Gei08, Proposition 3.5.15]. This discussion applies in particular to $(\mathbb{R}^3, \xi_{st})$ and $(S^3, \xi_{st})$.

**Example 4.14.** Consider the standard Legendrian unknot $K$ in $(S^3, \xi_{st})$ as in Example 4.7. The vector field

$$x_1\partial_{x_2} - x_2\partial_{x_1} + y_2\partial_{y_1} - y_1\partial_{y_2}$$

is a nowhere vanishing section of $\xi_{st}$ that coincides along $K$ with the tangent vector field $T$ to $K$. This shows that $\mathsf{rot}(K) = 0$.

The next proposition gives a formula for computing the rotation number of a knot from its front projection. Notice that the orientation of $K$ allows us to speak of cusps being oriented upwards or downwards.

**Proposition 4.15.** *Let $K$ be Legendrian in $(\mathbb{R}^3, \xi_{st})$. Then*

$$\mathsf{rot}(K) = \frac{1}{2}(\mathsf{cusps}_-(K) - \mathsf{cusps}_+(K)),$$

*where $\mathsf{cusps}_\pm(K)$ denotes the total number of upwards/ downwards oriented cusps in the front projection of $K$.*

*Proof.* Note that $e_1 := \partial_x$ and $e_2 := \partial_y - x\partial_z$ define a positively oriented trivialisation of $\xi_{st}$. Thus $\mathsf{rot}(K)$ can be computed by counting with sign how often the positive tangent vector to $K$ crosses $e_1$ as we travel once along $K$. Since $x$ equals the negative slope of the front projection, points of $K$ where the positive tangent vector equals $e_1$ are exactly the left cusps oriented downwards and the right cusps oriented upwards. At a left cusp oriented downwards, the tangent vector to $K$, expressed in terms of $e_1, e_2$, changes from having a negative component in the $e_2$-direction to a positive one, i.e. such a cusp yields a positive contribution to $\mathsf{rot}(K)$. Analogously, one sees that a right cusp oriented upwards gives a negative contribution to the rotation number, thus the claim follows. ☐

**Proposition 4.16.** *Let $K$ be Legendrian, parametrised by $\gamma\colon S^1 \to K \subset \mathbb{R}^3$ in $(\mathbb{R}^3, \xi_{st})$ and $\gamma_L$ its Lagrangian projection. Then*

$$\mathsf{rot}(K) = \deg(\gamma_L').$$

*Proof.* Since a global trivialisation of $\xi$ is given by the vector fields $\partial_x$ and $\partial_y - x\partial_z$, $\mathsf{rot}(\gamma_L)$ can be computed by counting with sign how often $\gamma_L'$ crosses $e_1$. ☐

We now introduce some relations between the two classical invariants of Legendrian knots that we introduced. In particular, the Thurston-Bennequin invariant and the rotation number of a homologically trivial Legendrian knot in a tight contact 3-manifold give a lower bound to the Euler characteristic of a Seifert surface for the knot, see [Gei08, Theorem 6.6.36].

**Theorem 4.17** (Legendrian Bennequin Inequality)**.** *Let $K$ be Legendrian in a tight contact 3-manifold with compatible Seifert surface $\Sigma$. Then*

$$\mathsf{tb}(K) + |\mathsf{rot}(K)| \leq -\chi(\Sigma),$$

*where $\chi(\Sigma)$ denotes the Euler characteristic of $\Sigma$.*

**Example 4.18.** When $K$ is the unknot, we may take the Seifert surface $\Sigma$ to be a disc. So the Bennequin inequality reads

$$\mathsf{tb}(K) + |\mathsf{rot}(K)| \leq -1.$$

This yields the restriction $\mathsf{tb}(K) \leq -1$.

The following result states that in a tight contact 3-manifold, the Thurston-Bennequin invariant and the rotation number completely characterise topologically trivial Legendrian knots, see [EF98].

**Theorem 4.19** (Eliashberg-Fraser)**.** *Let $K, K'$ be two topologically trivial Legendrian knots in a tight contact 3-manifold. Then, $K$ and $K'$ are Legendrian isotopic if and only if $\mathsf{tb}(K) = \mathsf{tb}(K')$ and $\mathsf{rot}(K) = \mathsf{rot}(K')$.*

*Remark* 4.20. This result has been extended to some topologically non-trivial knot types by Etnyre and Honda [EH01]. In general, however, the classical invariants do not suffice for classifying Legendrian knots in a given knot type.

**Definition 4.21.** Let $K$ be transverse. We choose a non-vanishing section $X$ of $\xi|_\Sigma$ and push $K$ in the direction of $X$ to obtain a parallel copy $K'$ of $K$. We orient $K'$ with the corresponding orientation on $K$. Then the **self-linking number** of $K$ relative to $\Sigma$ is defined as

$$\mathsf{sl}(K, \Sigma) := \mathsf{lk}(K, K').$$

*Remark* 4.22. The number $\mathsf{sl}(K, \Sigma)$ is invariant under transverse isotopies, see [Gei08, Theorem 2.6.12]. It turns out that that $\mathsf{sl}(K, \Sigma)$ is independent of the choice of $X$, see [Gei08, Remark 3.5.29]. As for the rotation number of Legendrian knots, if $\xi$ has zero Euler class, then $\mathsf{sl}(K, \Sigma)$ is independent on $\Sigma$, in this case we simply write $\mathsf{sl}(K)$, see [Gei08, Proposition 3.5.30].

**Proposition 4.23.** *Let $K \subset (\mathbb{R}^3, \xi_{st})$ be transverse and denote by $K_F$ the knot diagram given by its front projection. Then*

$$\mathsf{sl}(K) = \mathsf{writhe}(K_F).$$

*Proof.* This argument is completely analogous to that used for proving Proposition 4.9. We can take $X = \partial_x$ in the definition of $\mathsf{sl}(K)$. This means that $K'$ is obtained from $K$ by pushing it vertically with respect to the front projection. Hence, by a small isotopy we may assume that the front projection of $K'$ is a parallel curve to the front projection of $K$. We then observe that each crossing of the front projection of $K$ contributes a crossing of $K'$ underneath $K$ of the corresponding sign. The result follows by Proposition 3.8. □

*Remark* 4.24. The analogue of Theorem 4.19 for transverse knots has been proved by Eliashberg [Eli93]: topologically trivial transverse knots in a tight contact 3-manifold with the same self-linking number are transverse isotopic.

The following result gives a useful relation between the three invariants we have seen, see [Gei08, Proposition 3.5.36].

**Proposition 4.25.** *Let $K$ be Legendrian in a contact 3-manifold. Let $K_\pm$ be the positive/ negative transverse push-off of $K$. We may regard $\Sigma$ also as a Seifert surface of $K_\pm$ since $K$ and $K_\pm$ are topologically isotopic. Then*

$$\mathsf{sl}(K_\pm, \Sigma) = \mathsf{tb}(K) \mp \mathsf{rot}(K, [\Sigma]).$$

The following result is the analogous of Theorem 4.17 for transverse knots, see [Gei08, Theorem 4.6.34].

**Theorem 4.26** (Transverse Bennequin Inequality)**.** *Let $K$ be transverse in a tight contact 3-manifold. Then*

$$\mathsf{sl}(K) \leq -\chi(\Sigma).$$

# 5 Lagrangian Cobordism between Legendrian Knots

We now shift our attention to Lagrangian cobordisms between Legendrian knots in contact 3-manifolds. We see that the classical invariants of Legendrian knots give obstructions to the existence of a Lagrangian cobordism in $(\mathbb{R}^3, \xi_{st})$.

**Definition 5.1.** Let $Y$ be a closed oriented 3-manifold with positive contact structure $\xi$, i.e. $\alpha \wedge d\alpha$ is a positive volume form on $Y$ for every contact form $\alpha$. Let $(M, \omega) := (\mathbb{R} \times Y, d(e^s\alpha))$ be the simplectisation of $Y$. Let $\Sigma$ be a compact orientable surface with two distinct points $p^-$ and $p^+$ removed. Let $(s^-, \theta) \in (-\infty, -T) \times S^1$ and $(s^+, \theta) \in (T, +\infty) \times S^1$ for some fixed $T > 0$ be cylindrical coordinates around $p^-$ and $p^+$ respectively. Let $\gamma^-, \gamma^+ \colon S^1 \to Y$ be homologically trivial Legendrian knots in $Y$. Then

- $\gamma^-$ is **Lagrangian cobordant** to $\gamma^+$, denoted by $\gamma^- \prec_\Sigma \gamma^+$, if there exists a Lagrangian embedding $L \colon \Sigma \to \mathbb{R} \times Y$ such that

1. $\forall (s^-,\theta) \in (-\infty,-T) \times S^1 : L(s^-,\theta) = (s^-, \gamma^-(\theta))$ and
2. $\forall (s^+,\theta) \in (T,+\infty) \times S^1 : \quad L(s^+,\theta) = (s^+, \gamma^+(\theta))$.

- $\gamma^-$ is **Lagrangian concordant** to $\gamma^+$ if $\gamma^- \prec_\Sigma \gamma^+$ and $\Sigma$ is homeomorphic to a cylinder; we denote this particular case by $\gamma^- \prec \gamma^+$.

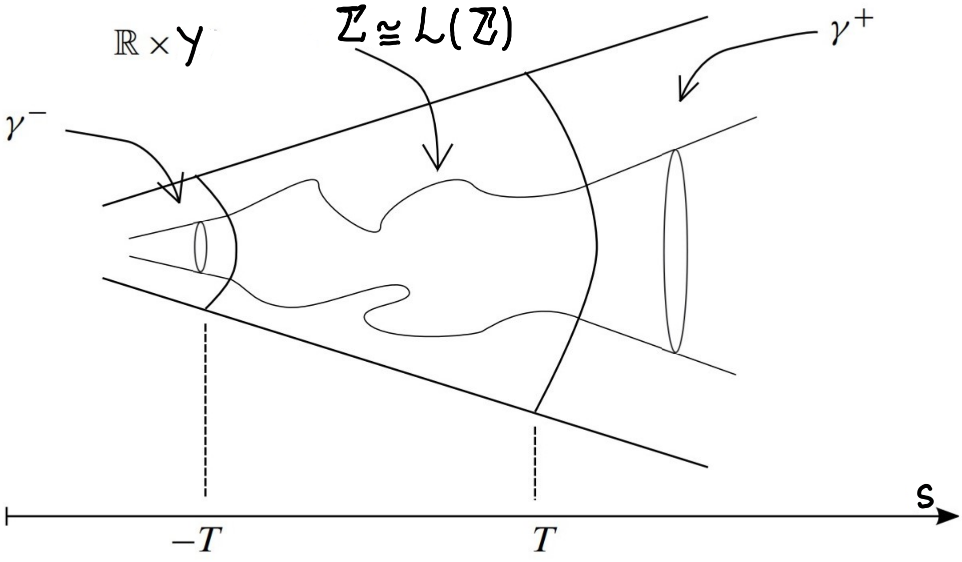


Figure 4: ([Cha10]) Lagrangian concordance: $\gamma^- \prec \gamma^+$.

*Remark* 5.2. The previous definition is motivated by the following: every $\mathbb{R}$-invariant Lagrangian submanifold in $M$ projects to a Legendrian submanifold of $Y$, and any Legendrian submanifold lifts to an $\mathbb{R}$-invariant Lagrangian submanifold. In particular any Legendrian submanifold is Lagrangian concordant to itself.

However, for the theory of Lagrangian concordance to be more intimately related to Legendrian knot theory, we want a relation up to Legendrian isotopy rather than a relation on the Legendrian submanifolds themselves. Indeed, it turns out that that any Legendrian isotopy in the contact manifold $Y$ gives rise to a Lagrangian cylinder in the symplectisation $M$, see [Cha10, Theorem 1.1].

The following result can be found in [Cha10, Theorem 1.2].

**Theorem 5.3** (Chantraine)**.** *Let $K^-, K^+ \subset (\mathbb{R}^3, \xi_{st})$ be Legendrian knots and assume that $K^- \prec_\Sigma K^+$. Then the following holds:*

1. $\mathsf{rot}(K^-) = \mathsf{rot}(K^+)$,
2. $\mathsf{tb}(K^+) - \mathsf{tb}(K^-) = -\chi(\Sigma) = 2g(\Sigma)$, *where $g(\Sigma)$ denotes the genus of $\Sigma$.*

**Example 5.4.** In particular, there does not exist a Lagrangian cobordism between the Legendrian knots in the following picture (where the knots are represented by their front projections), since the two knots have different rotation numbers.

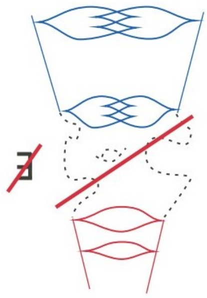


Figure 5: ([Tra12]) $\mathsf{rot}(K^+) = 1$ and $\mathsf{rot}(K^-) = 0$.

On the other side, Theorem 5.3 does not rule out the possibility of a Lagrangian cobordism between the Legendrian knots in the following picture.

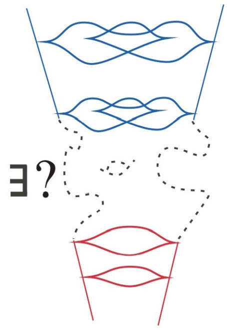


Figure 6: ([Tra12]) $\mathsf{tb}(K^+) = 1, \mathsf{rot}(K^+) = 0$ and $\mathsf{tb}(K^-) = -1, \mathsf{rot}(K^-) = 0$.

However, if such Lagrangian cobordism exists, it must have genus equal to 1. This is in contrast to the smooth situation, where one can increase the genus by merely adding handles to a cobordism. This indicates a rigidity/efficiency of Lagrangian cobordisms.

# References


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