Legendrian Submanifolds and Legendrian Knots

Alexander Uhlmann
auhlmann@ethz.ch

November 2021

Abstract

In this talk we will introduce legendrian submanifolds and give some important examples to motivate their importance. We will then discuss legendrian knots and the front projection and finally provide a proof of the C^0-approximation theorem.

1 Legendrian Submanifold

We begin our talk by introducing the notions of isotropic and legendrian submanifolds.

Definition 1.1 (Isotropic Submanifold)**.** Let (Y, ξ) be a contact manifold. A submanifold $K \subset Y$ is called an *isotropic* if $T_pK \subset \xi_p$ for all $p \in Y$.

We can see that the non-integrability condition on the contact structure puts a constraint on the dimension of such a submanifold:

Proposition 1.2. *Let* (Y, ξ) *be a contact manifold of dimension* $2n+1$ *and* $K \subset Y$ *an isotropic submanifold. Then* $\dim K \leq n$

Proof. Let $\imath_K : K \hookrightarrow Y$ be the inclusion map of K into Y, and α a one-form such that locally $\xi = \ker \alpha$. Since $T_pK \subset \xi_p$ for all $p \in Y$ we have $i_K^* \alpha = 0$. Because the exterior derivative is linear and commutes with the pullback this gives us $0 = d i_K^* \alpha = i_K^* d\alpha$. This is equivalent to T_pK being an *isotropic subspace* of the symplectic space ξ_p, as we saw last week [Aksb]. But now, since ξ_p has dimension $2n$ and T_pK is an isotropic subspace, we have $\dim T_pK \leq n$. □

As in the symplectic case, isotropic submanifolds of maximal dimension are singled out:

Definition 1.3 (Legendrian Submanifold)**.** Let (Y, ξ) be a contact manifold of dimension $2n+1$. An isotropic submanifold $K \subset Y$ is called an *legendrian* if $\dim K = n$.

1.1 Examples

We will now give a couple of examples to highlight the significance of legendrian submanifolds.

Remark 1.4 (Liouville form and the symplectic structure of T^*M). As we will be using it extensively we will briefly give an overview of these concepts here.
The *Liouville form* λ is the unique one-form on T^*M with the property that

$$\beta^*\lambda = \beta \text{ for any } \beta \in \Omega^1 M$$

At a point $m = (q, p) \in T^*M$ $\lambda_m : T_m T^*M \to \mathbb{R}$ can be written as

$$\lambda_m = \sum_{i=1}^{n} p_i dq^i$$

We can now use the Liouville form to endow T^*M with a symplectic structure by defining the *Poincaré two-form* as

$$\omega = -d\lambda = \sum_{i=1}^{n} dq^i \wedge dp_i$$

A more in-depth introduction of these concepts can be found in the talk introducing symplectic manifolds [Kau].

Example 1.5 (Jet Bundle)**.** Given a manifold K, we can define $J^1(K)$ the space of 1-jets of differentiable functions $f : K \to \mathbb{R}$. Let $\varepsilon(p)$ be the space of 1-germs at a point $p \in K$ defined as $\varepsilon(p) = C^1(K)/\sim_p$ where $C^1(K) = \{f : K \to \mathbb{R} | \, f \text{ differentiable}\}$ and $\sim_p$ is an equivalence relation on $C^1(K)$ defined as

$$\begin{aligned} f \sim_p g \iff f(p) &= g(p) \text{ and} \\ df_p &= dg_p \end{aligned}$$

The equivalence class $j^1_p f := [f]_{\sim_p} \in \varepsilon(p)$ is called the *1-jet* of f at p. We can now define J^1 as the bundle of the spaces of 1-germs:

$$J^1(K) = \bigcup_{p \in K} \{p\} \times \varepsilon(p)$$

Note that $J^1(K)$ can be given the structure of a $2n+1$-dimensional manifold such that

$$\begin{aligned} \Phi : J^1(K) &\to T^*K \times \mathbb{R} \\ j^1_p &\mapsto (df_p, f(p)) \end{aligned}$$

is a diffeomorphism. We can now endow $J^1(K)$ with a contact structure by observing that $\alpha = dz - \lambda$, where λ is the Liouville one-form, is a contact form on $T^*K \times \mathbb{R}$.

Every smooth function $f \in C^\infty(K)$ gives us a embedding

$$\begin{aligned} i_f : K &\hookrightarrow T^*K \times \mathbb{R} \\ p &\mapsto (df_p, f(p)) \end{aligned}$$

such that the image $i_f(K)$ is a legendrian submanifold of $T^*K \times \mathbb{R}$.
The fact that $i_f(K)$ is legendrian can be shown as follows:

Proof. We begin by showing that $i_f(K)$ is isotropic: For $p \in K$ and $f \in C^\infty(K)$ we have

$$\begin{aligned} i_f^*\alpha|_p &= i_f^* dz|_p - i_f^*\lambda|_p \\ &= di_f^* z|_p - df^*\lambda|_p \\ &= df|_p - df|_p \\ &= 0 \end{aligned}$$

This is equivalent to $T_p i_f(K) \subset \ker \alpha|_p$ for all $p \in i_f(K)$ which means $i_f(K)$ is an isotropic submanifold. Since $\dim K = n$ and $\dim T^*K \times \mathbb{R} = 2n+1$, $i_f(K)$ is a legendrian submanifold. □

Example 1.6 (Unit tangent bundle)**.** Let (M, g) be a riemannian manifold. The the *musical isomorphism*

$$\begin{aligned} \flat : TM &\to T^*M \\ X_p &\mapsto g(X_p, -) \end{aligned}$$

induces a metric g^* on $T * M$ by $g^* := (\flat^{-1})^* g$.
We then define the *unit tangent bundle STM* fiberwise as

$$ST_pM = \{X \in T_pM | g_p(X, X) = 1\}$$

The *unit cotangent bundle ST^*M* is defined equivalently using g^*. Note that STM and ST^*M are $2n-1$-dimensional submanifolds. Furthermore, by restricting the Liouville form λ we can gain a contact form $\alpha = \lambda|_{ST^*M}$ on ST^*M. Note that the flow of the Reeb vector field R_α on ST^*M is equal to the geodesic flow induced by g on STM. This was discussed more in-depth during the talk on reeb vector fields [Wan]. Now let

$$\begin{aligned} \pi : ST^*M &\to M \\ (q, \beta) &\mapsto q \end{aligned}$$

be the *footprint map*. Then the level sets $\pi^{-1}(\{q\})$ for $q \in M$ are legendrian submanifolds of ST^*M.

Proof. Let $q \in M$. We have that $q = const$ on $\pi^{-1}(\{q\})$ which implies that $dq^i = 0$ on $\pi^{-1}(\{q\})$. But since $\lambda = \sum_{i=1}^{n} p_i dq^i$, we have $\lambda|_{\pi^{-1}(\{q\})} = 0$ which implies that $\pi^{-1}(\{q\})$ is an isotropic submanifold.
Furthermore, since $\dim \pi^{-1}(\{q\}) = n - 1$ and $\dim ST^*M = 2n - 1$ we have that $\pi^{-1}(\{q\})$ is of maximal dimension and legendrian. □

1.2 Symplectization and Submanifolds

To further illustrate the connection between legendrian and lagrangian submanifolds we will consider the symplectization of a contact manifold.

Definition 1.7 (Symplectization)**.** Let (Y, ξ) be a dimension $2n-1$ contact manifold. We define its *symplectization* $S(Y, \xi) = \{(q, p) | q \in Y, p \in T_q^*Y, \ker p = \xi_q\}$. The restriction of the Poincaré two-form $\omega|_{S(Y,\xi)} = -d\lambda|_{S(Y,\xi)}$ gives the symplectic structure on $S(Y, \xi)$, which is then a $2n$-dimensional symplectic manifold.

The next proposition relates the legendrian submanifolds of (Y, ξ) to lagrangian submanifolds of $S(Y, \xi)$.

Proposition 1.8. *Let $K \subset (Y, \xi)$ be an isotropic submanifold. Then $L = \pi^{-1}(K) = \{(q, p) | q \in K, p \in T_q^*Y, \ker p = \xi_q\} \subset S(Y, \xi)$ is also an isotropic submanifold. Furthermore, if K is legendrian then L is lagrangian.*

Proof. First, let $K \subset (Y, \xi)$ be an isotropic submanifold. We will begin by showing that L is also isotropic. Note that since

$$\omega|_L = i_L^*\omega = -i_L^* d\lambda = -d i_L^*\lambda = -d\lambda|_L$$

is suffices to show that $\lambda|_L = 0$. Now let $(q, p) \in L$ then λ can be written on $T_{(q,p)}L$ as

$$\begin{aligned} \lambda_{(q,p)} : T_{(q,p)}L &\to R \\ (\partial q, \partial p) &\mapsto p(\partial q) \end{aligned}$$

But since $\partial q \in T_qK \subset \xi_q$ and $\ker p = \xi_q$ by definition of $S(Y, \xi)$ we have that $p(\partial q) = 0$ and $\lambda|_L = 0$. This concludes the first part of the proof. If we now assume that K is legendrian i.e. $\dim K = n - 1$, we get that L is lagrangian since $\dim L = \dim K + 1 = (n - 1) + 1$ and $\dim S(Y, \xi) = 2n$. □

2 Legendrian Knots

In this section we will give a brief introduction of knots in 3-manifolds and introduce the concepts of legendrian and transverse knots. We will then, in anticipation of the next talk, build up some basic machinery to better understand these structures.

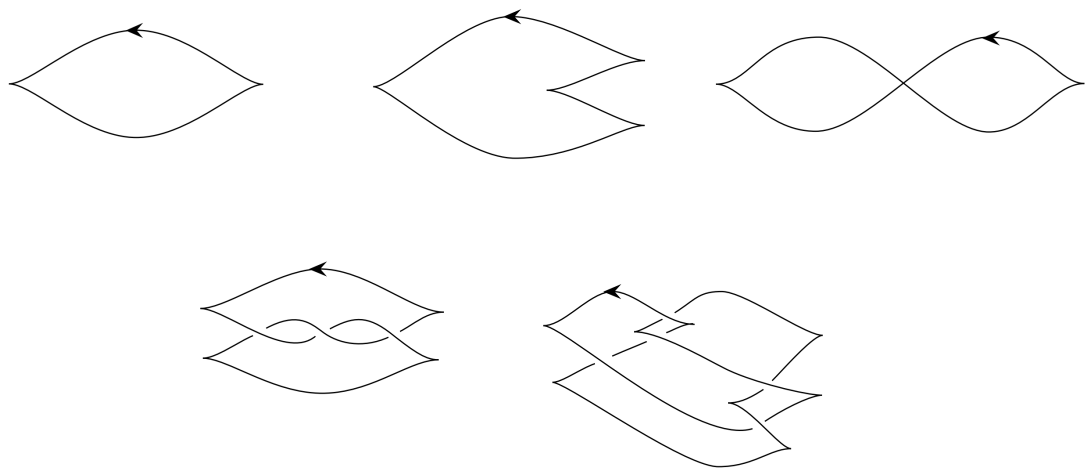

Figure 1: [Hon] Legendrian unknots, trefoil knot and 8-knot in the front projection.

Definition 2.1 (Knot)**.** For a 3-dimensional manifold M a *knot* is defined as an embedding $\gamma : S^1 \hookrightarrow M$. We will often also simply denote a knot by its image $K = \gamma(S^1)$.
We say that two knots γ_1, γ_2 are *isotopic* or *equivalent* if there exists an *ambient isotopy* between them. That is a $\Phi : M \times [0,1] \to M$ such that $\Phi_0 = id_M$, $\Phi_1 \circ \gamma_1 = \gamma_2$ and Φ_t a homeomorphism for all $t \in [0,1]$.
We say that a knot γ is *trivial* or an *unknot* if γ is isotopic to the boundary ∂D of the unit disk D.

For (Y, ξ) a 3-dimensional contact manifold we consider special types of knots:

Definition 2.2 (Legendrian and Transverse Knots)**.** A knot $\gamma \subset Y$ is called *legendrian* if $\gamma(S^1)$ is a legendrian submanifold of Y. Note that since $\dim S^1 = 1$ this is equivalent to $\gamma'(s) \in \xi_{\gamma(s)}$ or $\alpha(\gamma'(s)) = 0$ for all $s \in S^1$
A knot $\gamma \subset Y$ is called *transversal* if $\gamma'(s) \notin \xi_{\gamma(s)}$. If $\xi = \ker \alpha$ is cooriented we can distinguish between *positively oriented transversal knots* if $\alpha(\gamma'(s)) > 0$ and *negatively oriented transversal knots* if $\alpha(\gamma'(s)) < 0$ for all $s \in S^1$.

From now on we will only concern ourselves with knots in $\mathbb{R}^3$ with the standard contact structure $(\mathbb{R}^3, \xi_{st} = \ker \alpha_{st})$ for $\alpha_{st} = dz - xdy$.

2.1 Front Projection

In order to visualize knots on $(\mathbb{R}^3, \xi_{st})$, we draw their projection onto some subplane. The front projection is such a tool.

Definition 2.3 (Front Projection)**.** For any knot γ on $(\mathbb{R}^3, \xi_{st})$ with $\gamma(s) = (x(s), y(s), z(s))$, the *front projection* is $\lambda_F : S^1 \to \mathbb{R}^2$ given by

$$\gamma_F(s) = (y(s), z(s))$$

.

Remark 2.4. Note that for any legendrian knot γ on $(\mathbb{R}^3, \xi_{st})$ with $\gamma(s) = (x(s), y(s), z(s))$ the condition $\alpha_{st}(\gamma') = 0$ implies that

$$z' - xy' = 0$$

As a consequence, if $y'(s) = 0$, we have $z'(s) = 0$ and $x'(s) \neq 0$ since γ cannot have any singularities. But because the the front projection drops the x-coordinate, γ_F does indeed have a singularity at s. We will call such points *cusp points*. Examples of front-projected legendrian knots can be seen in figure 1.

Lemma 2.5. *Let $\gamma : S^1 \to (\mathbb{R}^3, \xi_{st})$ be a legendrian knot. Then its front projection γ_F does not have any vertical tangencies. Away from the cusp points, γ is recovered from its front projection via $x(s) = -\frac{z'(s)}{y'(s)}$ The knot γ is embedded if and only if γ_F has only transverse self-intersections.*

Proof. Suppose $y'(s) = 0$ for some $s \in S^1$, then by 2.4, $\gamma_F(s)$ is a cusp point and singular which means γ_F does not have a vertical tangency at s. The second fact also follows immediately from 2.4. And finally we note that γ self-intersects if and only if γ_F tangentially self-intersects. □

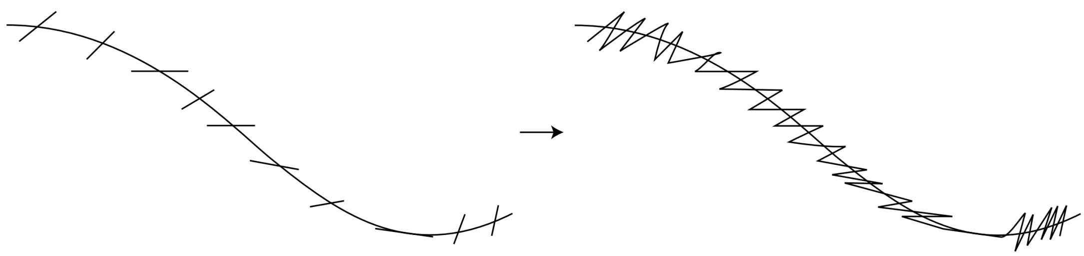

Figure 2: [Hon] Sketch of the zig-zag curves around γ_F used in the proof of theorem 2.6.

2.2 C^0-Approximation Theorem

We will end by briefly sketching the proof to an important approximation theorem which states that any knot can be approximated by an isotopic legendrian knot.

Theorem 2.6 (C^0-Approximation)**.** *Let γ be a knot in $(\mathbb{R}^3, \xi_{st})$. Then γ can be C^0–approximated by a legendrian knot isotopic to γ.*

Proof. The main idea is that we approximate the front projection of $\gamma_F(s) = (y(s), z(s))$ by defining $(y_0(s), z_0(s))$ to zig-zag sufficiently close around γ_F. We additionally make sure that $\frac{z_0'(s)}{y_0'(s)}$ is sufficiently close to $x(s)$ such that the equation in remark 2.4 approximately holds. Using lemma 2.5 to reconstruct x_0 $(x_0(s), y_0(s), z_0(s))$ then defines a legendrian knot which is C^0-close to γ. A sketch of the approximating curve $(y_0(s), z_0(s))$ can be found in figure 2. □

References

[Aksa] Metehan Aksay. "Lagrangian Submanifolds: Definitions, Examples". In: *Notes available at https://people.math.ethz.ch/~bacubulut/assets/week7-aksay-full.pdf* ().

[Aksb] Metehan Aksay. "Talk on Lagrangian Submanifolds". In: *Notes available at https://people.math.ethz.ch/~bacubulut/assets/week7-aksay-full.pdf* ().

[Gei08] Hansjörg Geiges. *An introduction to contact topology.* Vol. 109. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008, pp. xvi+440. ISBN: 978-0-521-86585-2. DOI: 10.1017/CBO9780511611438. URL: https://doi.org/10.1017/CBO9780511611438.

[Hon] Ko Honda. "Notes for math 599: contact geometry". In: *Lecture Notes available at http://www-bcf.usc.edu/~khonda/math599/notes.pdf* ().

[Kau] Reto Kaufmann. "Talk on Symplectic Manifolds". In: *Notes available at https://people.math.ethz.ch/~bacubulut/assets/week3_reto.pdf* ().

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Third. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017, pp. xi+623. ISBN: 978-0-19-879490-5; 978-0-19-879489-9. DOI: 10.1093/oso/9780198794899.001.0001. URL: https://doi.org/10.1093/oso/9780198794899.001.0001.

[Mur] aEmmy Murphy. "Lecture notes on (contact) geometry of manifolds". In: *Lecture Notes available at https://math.mit.edu/classes/18.966/2014SP/* ().

[Wan] Bangxin Wang. "Talk on Contact and Reeb Vector Fields". In: *Notes available at https://people.math.ethz.ch/~bacubulut/assets/week4_bangxin_full.pdf* ().