# Hamiltonian torus actions

Student seminar in symplectic vs. contact manifolds

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**Abstract**

We specialize the discussion of the previous talk about hamiltonian actions and moment maps to tori. We briefly introduce their Lie group structure and see how the definition of their hamiltonian actions simplifies, focusing on the case of the standard euclidean torus $\mathbb{T}^n$. Secondly, we look at a convexity result about the image of their associated moment maps, along with a few examples, most remarkably on complex projective spaces endowed with the Fubini–Study form.

## 1 Torus Lie groups

Let us start with the relevant Lie group theory.

**Definition 1.1.** A Lie group $\mathcal{T}$ is a **torus (Lie group)** if it is abelian (i.e. the smooth multiplication map is abelian), compact and connected. We denote its Lie algebra by $\mathfrak{t} = \mathrm{Lie}(\mathcal{T}) = T_e\mathcal{T}$ and observe it to be made of all translation invariant[1] vector fields on $\mathcal{T}$. Since the Lie bracket on $\mathfrak{t}$ is trivial,[2] the vector space $\mathfrak{t}$ itself is abelian.

Writing $\exp_{\mathcal{T}} : \mathfrak{t} \to \mathcal{T}$, $X \mapsto \gamma^X(1)$, $tX \mapsto \gamma^X(t)$ for the associated exponential map and denoting by $\mathfrak{t}_{\mathbb{Z}} \coloneqq \ker(\exp_{\mathcal{T}}) \subset \mathfrak{t}$ its **integral lattice** (a discrete additive subgroup), we can equivalently identify[3] $\mathcal{T}$ as the quotient Lie group $\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$.

[1]Recall that, for a general Lie group $G$ with Lie algebra $\mathfrak{g} = T_eG$, the evaluation map $\mathrm{eval}_e$ at the identity is a vector space isomorphism $\mathfrak{X}_l(G) \to \mathfrak{g}$, where $\mathfrak{X}_l(G) \coloneqq \{X \in \Gamma(TG) \mid Dl_a(b)[X(b)] = X(ab), \forall a, b \in G\}$ is the subset of vector fields on $G$ invariant under left-translation $l_a : b \in G \mapsto ab \in G$. If $G$ is abelian, we also get invariance under right-translation $r_a(b) = ba$.

[2]The argument is very simple: for $i : \mathcal{T} \to \mathcal{T}$ the smooth inversion, we have $i(ab) = b^{-1}a^{-1} = a^{-1}b^{-1} = i(a)i(b)$ for all $a, b \in \mathcal{T}$, so that $i$ is a Lie group homomorphism, thus its derivative $Di(e) : \mathfrak{t} \to \mathfrak{t}$, $Di(e)[X] = -X$ a Lie algebra isomorphism, whence $[X, Z] = [-X, -Z] = Di(e)[X, Z] = -[X, Z]$; this implies that $[\cdot, \cdot]$ is trivial on $\mathfrak{t}$ and $XZ = ZX$ for all $X, Z \in \mathfrak{t}$.

[3]This is a basic result in Lie group theory: for connected abelian Lie groups $G$, $\exp : \mathfrak{g} \to G$ is a surjective homomorphism with discrete kernel, hence the isomorphism theorem for Lie groups provides a Lie group isomorphism $\mathfrak{g}/\ker(\exp) \cong \mathrm{im}(\exp) = G$.

We call the dual vector space $\mathfrak{t}_{\mathbb{Z}}^* \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}_{\mathbb{Z}}, 2\pi\mathbb{Z})$ **weight lattice** of $\mathcal{T}$.

For most applications, here and in future, we will focus on the standard euclidean torus.

**Definition 1.2.** The **standard torus** of rank $n \geq 1$ is defined as $\mathbb{T}^n \coloneqq (\mathbb{S}^1)^n \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, its elements being of the form $(e^{i\theta_1}, ..., e^{i\theta_n}) \in \mathbb{T}^n$ for $\theta_1, ..., \theta_n \in \mathbb{R}$. Endowing $\mathbb{T}^n$ with the smooth multiplication map

$$(e^{i\theta_1}, ..., e^{i\theta_n}) \cdot (e^{i\vartheta_1}, ..., e^{i\vartheta_n}) = (e^{i(\theta_1+\vartheta_1)}, ..., e^{i(\theta_n+\vartheta_n)})$$

and smooth inversion

$$(e^{i\theta_1}, ..., e^{i\theta_n})^{-1} = (e^{-i\theta_1}, ..., e^{-i\theta_n}) \,,$$

we obtain the structure of an abelian compact connected Lie group. Hence, $\mathbb{T}^n$ is indeed a torus in the Lie group sense. Its Lie algebra is simply $\mathrm{Lie}(\mathbb{T}^n) = T_{\mathbb{1}}\mathbb{T}^n \cong \mathbb{R}^n$.

Due to the periodicity, we can also write

$$\mathbb{T}^n \cong (\mathbb{R}/2\pi\mathbb{Z})^n \cong \mathbb{R}^n/(2\pi\mathbb{Z})^n$$

(the quotient Lie group induced by $\exp_{\mathbb{T}^n}(\theta_1, ..., \theta_n) = (e^{i\theta_1}, ..., e^{i\theta_n})$, with kernel the integral lattice $(2\pi\mathbb{Z})^n$; here, the exponential map describes the flowing along each standard basis vector $X_i = (0, ..., 1, ..., 0)$ for time $\theta_i \in \mathbb{R}$). We then shorten $[\theta] = [\theta_1, ..., \theta_n] \coloneqq (e^{i\theta_1}, ..., e^{i\theta_n}) \in \mathbb{T}^n$, the identity element being $\mathbb{1} \coloneqq [0, ..., 0] \in \mathbb{T}^n$.

We can identify any abstract torus Lie group $\mathcal{T}^n$ with $\mathbb{T}^n$: choosing a basis $(X_i)_{i=1}^n$ for $\mathfrak{t} = \mathrm{Lie}(\mathcal{T})$ — which naturally sets an isomorphism of vector spaces $\mathcal{J} : \mathfrak{t} \to \mathbb{R}^n$ — so that $\mathfrak{t}_{\mathbb{Z}} \cong \mathrm{Lie}(\mathbb{T}^n)_{\mathbb{Z}} = (2\pi\mathbb{Z})^n$, we obtain a Lie group isomorphism[4]

$$\mathcal{T} \cong \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}} \xrightarrow{\mathcal{J}} \mathbb{R}^n/(2\pi\mathbb{Z})^n \cong \mathbb{T}^n \,,$$

where the identifications as quotient Lie groups come again from the respective exponential maps. We call $\mathcal{J}$ (modulo the exp's) a **splitting** of $\mathcal{T}$. Then the weight lattice $\mathfrak{t}_{\mathbb{Z}}^*$ gets mapped to $\mathrm{Hom}_{\mathbb{Z}}((2\pi\mathbb{Z})^n, 2\pi\mathbb{Z}) \cong \mathbb{Z}^n$.

**Remark 1.3.** By abelianity of $\mathbb{T}^n$, the inner automorphism $\varphi_a(b) = aba^{-1} = b$ is just the identity for all $a, b \in \mathbb{T}^n$, hence the adjoint action is trivial, $\mathrm{Ad}_a = D\varphi_a(\mathbb{1}) = \mathrm{id}_{\mathbb{T}^n}$, and so is its coadjoint: $\langle \mathrm{Ad}_a^*(X), Z\rangle = \langle X, \mathrm{Ad}_{a^{-1}}(Z)\rangle = \langle X, Z\rangle$ for all $X \in (\mathbb{R}^n)^*$ and $Z \in \mathbb{R}^n$, i.e. $\mathrm{Ad}_a^* = \mathrm{id}_{(\mathbb{R}^n)^*}$.

Observe that we preserved the dual notation for $(\mathbb{R}^n)^*$ so to avoid confusion regarding the associated pairing $\langle \cdot, \cdot \rangle$, which is just the standard scalar product.

Before proceeding, let us recall the definition of hamiltonian action and moment map given in the previous talk.

**Definition 1.4.** Let $(M, \omega)$ be a symplectic manifold, $G$ a Lie group with Lie algebra $\mathfrak{g}$. An action $\psi : G \to \mathrm{Diff}(M)$ of $G$ on $M$ is said to be **hamiltonian** if there exists a **moment map** $\mu : M \to \mathfrak{g}^*$, that is, a map such that:

[4] This is false on the topological level: $\mathcal{T}^n$ needs not be *homeomorphic* to $\mathbb{T}^n$! However, we can *define* torus Lie groups to be those isomorphic to $\mathbb{T}^n$ as Lie groups; this will automatically imply that they are abelian, compact and connected, lining up with our adopted definition.

1. For each $X \in \mathfrak{g}$, the associated smooth map $\mu^X \in \mathrm{C}^\infty(M)$, $\mu^X(p) := \langle \mu(p), X\rangle \equiv \mu(p)[X]$ and infinitesimal generator $X^\# \in \Gamma(TM)$, $X^\#(x) = X^\#_x := (\gamma^X_x)'(0) = \frac{d}{dt}|_{t=0}(x \exp(tX)) \in T_xM$ of the one-parameter subgroup $\gamma^X : t \in \mathbb{R} \mapsto \exp(tX) \in G$ are related as

   $$d\mu^X = -i_{X^\#}\omega \in \Omega^1(M)\,,$$

   or equivalently, $\langle d\mu_x(v), X\rangle = \omega_x(v, X^\#_x) \in \mathbb{R}$ for all $x \in M$, $v \in T_xM$.

2. For each $g \in G$, the diagram

   $$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{\mathrm{Ad}^*_g} & \mathfrak{g}^* \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu}\uparrow \\ M & \xrightarrow{\psi_g} & M \end{array}$$

   commutes, that is, $\mu(x)[\mathrm{Ad}_{g^{-1}}(Z)]\langle \mu(x), \mathrm{Ad}_{g^{-1}}(Z)\rangle \overset{\text{def}}{=} \langle \mathrm{Ad}^*_g(\mu(x)), Z\rangle \overset{!}{=} \langle \mu(\psi_g(x)), Z\rangle \equiv \mu(g \cdot x)[Z] \in \mathbb{R}$ for all $x \in M$, $Z \in \mathfrak{g}$.

If we take $G = \mathcal{T}$ in Definition 1.4 to be a torus, then we call the so obtained quadruple $(M, \omega, \mathcal{T}, \mu)$ a **hamiltonian torus space**.

Since, as observed above, any torus Lie group with a choice of Lie algebra basis is isomorphic to the standard euclidean torus, we are particularly motivated to unravel the definition of hamiltonian $\mathbb{T}^n$-space.

**Definition 1.5.** Let $(M, \omega)$ be a symplectic manifold. An action $\psi : \mathbb{T}^n \to \mathrm{Diff}(M), [\theta] \mapsto (\psi_{[\theta]} : x \mapsto [\theta] \cdot x)$ is a **hamiltonian torus action** if there exists a moment map $\mu : M \to (\mathbb{R}^n)^*$, that is, a map fulfilling:

1. For each standard basis vector $X_k \in \mathbb{R}^n$, the associated infinitesimal generator $X^\#_k \in \Gamma(TM)$ of $\gamma^{X_k} : \theta_k \in \mathbb{R} \mapsto \exp_{\mathbb{T}^n}(\theta_k X_k) = (0, ..., e^{i\theta_k}, ..., 0) \in \mathbb{T}^n$ is Hamiltonian with

   $$d\mu_k = -i_{X^\#_k}\omega \in \Omega^1(M)\,,$$

   where $\mu_k = x_k \circ \mu \in C^\infty(M)$ are the coordinate functions of $\mu$, $1 \le k \le n$.

2. $\mu_k([\theta] \cdot x) = \mu_k(x)$ for all $[\theta] \in \mathbb{T}^n$, $x \in M$, $1 \le k \le n$.

**Remark 1.6.**

- As for general Lie groups, when $n = 1$ condition 1. tells us that the moment map $\mu \in C^\infty(M)$ is a Hamiltonian function for each vector field $X^\# \in \Gamma(TM)$ associated to any $X \in \mathbb{R}$, then a Hamiltonian vector field. In higher dimension, along the $k$-th direction, this role is played by $\mu_k \in C^\infty(M)$ for $X^\#_k \in \Gamma(TM)$.
  In this sense, we could interpret the moment map as an "upgraded" version of Hamiltonian function.

- Compatibility with Definition 1.4 is straightforward: $\mu^{X_k} = \mu_k$ for each coordinate, so condition 1. holds for any $X \in \mathbb{R}^n$ (by linearity), while triviality of the coadjoint action (see Remark 1.3) simplifies the equivariance property 2. to the $\mathbb{T}^n$-invariance of each $\mu_k$.

- The moment map is not unique: assuming $\mu : M \to \mathbb{R}^n$ to be a moment map for a $\mathbb{T}^n$-action, so is $\mu + c$ for any $c \in \mathbb{R}^n$ (since $(\mu+c)_k = \mu_k + c$, and $d(\mu+c)_k = d\mu_k$). Moreover, any two moment maps $\mu^1, \mu^2$ for a given hamiltonian torus action on a connected manifold $M$ differ by just a constant (because $d(\mu^1 - \mu^2)_k = d\mu^1_k - d\mu^2_k = 0 \in \Omega^1(M)$, by condition 1., which implies $\mu^1 - \mu^2 = c \in \mathbb{R}^n$).

**Example 1.7.** Let us have a look at a few basic examples.

- Consider the standard symplectic manifold $(\mathbb{C}, \omega_0 = \frac{i}{2} dz \wedge d\bar{z})$ and the action of the 1-torus $\mathbb{S}^1$ by rotation,

  $$\psi : \mathbb{S}^1 \to \mathrm{Diff}(\mathbb{C}),\ \theta \equiv [\theta] \mapsto (\psi_\theta : z \mapsto e^{il\theta} z)\,,$$

  for a fixed $l \in \mathbb{Z}$. Then $\psi$ is hamiltonian, a valid moment map being

  $$\mu : \mathbb{C} \to \mathbb{R},\ z \mapsto \frac{1}{2} l |z|^2\,.$$

  A verification is best obtained in polar coordinates $(r, \varphi)$, with which $\omega_0 = rdr \wedge d\varphi$, $\psi_\theta(re^{i\varphi}) = re^{i(\varphi + l\theta)}$ and $\mu(re^{i\varphi}) = \frac{1}{2} lr^2$: then $X^\# = l\frac{\partial}{\partial\varphi} = i \cdot l(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$ clearly fulfills

  $$(-i_{X^\#}\omega_0)_{re^{i\varphi}} = (rdr \wedge d\varphi)(\cdot, l\frac{\partial}{\partial\varphi}) = rl \cdot dr = d\mu(re^{i\varphi})\,,$$

  and

  $$\mu(\psi_\theta(re^{i\varphi})) = \mu(re^{i(\varphi+l\theta)}) = \frac{1}{2} lr^2 = \mu(re^{i\varphi})\,,$$

  for all $\theta \in \mathbb{S}^1$, $re^{i\varphi} \in \mathbb{C}$.

- Similarly, in dimension $n$ and for fixed $l_1, ..., l_n \in \mathbb{Z}$, we have the hamiltonian torus action (*diagonal action*)

  $$\psi_0 : \mathbb{T}^n \to \mathrm{Diff}(\mathbb{C}^n),\ [\theta] \mapsto (\psi_{0,[\theta]} : (z_1, ..., z_n) \mapsto (e^{il_1\theta_1} z_1, ..., e^{il_n\theta_n} z_n)$$

  on $(\mathbb{C}^n, \omega_0 = \frac{i}{2}\sum_{i=1}^n dz_k \wedge d\bar{z}_k)$, with moment map

  $$\mu_0 : \mathbb{C}^n \to \mathbb{R}^n,\ (z_1, ..., z_n) \mapsto \frac{1}{2}(l_1|z_1|^2, ..., l_n|z_n|^2)\,.$$

  Here, $X^\#_k = l_k \frac{\partial}{\partial\theta_k} = i \cdot l_k(z_k\frac{\partial}{\partial z_k} - \bar{z}_k\frac{\partial}{\partial\bar{z}_k})$.
  For the choice $l_k = 1\ \forall 1 \le k \le n$, $(\mathbb{C}^n, \omega_0, \mathbb{T}^n, \mu_0)$ is called the **Darboux $\mathbb{T}^n$-model**.

- Consider the **complex projective space** $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where $z_1 \sim z_2$ iff $\exists \lambda \in \mathbb{C}^\times$ s.t. $z_1 = \lambda z_2$, then understood as the set[5] of all complex lines in $\mathbb{C}^{n+1}$ through the origin, which we may denote via homogeneous coordinates $[z] = [z_0 : z_1 : ... : z_n] = [\lambda z] \in \mathbb{C}\mathrm{P}^n$. To obtain the structure of a symplectic manifold, we look at

  $$\nu_{\mathrm{FS}} = \frac{i}{2(z_k\bar{z}^k)^2}\sum_{k=0}^n \sum_{k\neq j=0}^n (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k) \in \Omega^2(\mathbb{C}^{n+1} \setminus \{0\})\,;$$

[5] Alternatively, we can consider the complex projective space $\mathbb{C}\mathrm{P}^n$ as the homogeneous space $\mathbb{S}^{2n+1}/U(1)$ sitting inside $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, where $U(1) \equiv \mathbb{S}^1$ is the unitary circle group acting by multiplication on $\mathbb{S}^{2n+1}$. In particular, $\mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^3/U(1) \cong \mathbb{S}^2$, equipped with Fubini–Study form $\omega_{\mathrm{FS}} = \frac{1}{4} d\varphi \wedge dh$; see Example 2.4.

then the pullback $\omega_{\text{FS}} = \pi^*(\nu_{\text{FS}}) \in \Omega^2(\mathbb{C}\text{P}^n)$ under the mod$\sim$ projection $\pi$ of $\mathbb{C}^{n+1} \setminus \{0\}$ is a well-defined symplectic form, the **Fubini–Study form** on $\mathbb{C}\text{P}^n$. On the affine patch $U_j := \{[z] \in \mathbb{C}\text{P}^n \mid z_j \neq 0\} = \{[z_0 : ... : 1 : ... : z_n] \in \mathbb{C}\text{P}^n\}$, it can be written more succintly as

$$\omega_{\text{FS}} = \frac{i}{2}\partial\bar{\partial} f_j \qquad \text{for} \qquad f_j(z) := \ln\left(\frac{\bar{z}_k z^k}{\bar{z}_j z_j}\right) \in C^\infty(U_j)\,,$$

where $\partial := \sum_{i=0}^n \frac{\partial}{\partial z_i} dz_i$ and $\bar{\partial} := \sum_{i=0}^n \frac{\partial}{\partial \bar{z}_i} d\bar{z}_i$ (upon fixing $z_j = 1$ and considering the remaining affine coordinates $z_{\text{aff}} = (z_0, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \mathbb{C}^n$, we also have $f_j(z) = \ln(1 + |z_{\text{aff}}|^2)$). Then $(\mathbb{C}\text{P}^n, \omega_{FS})$ is a real $2n$-dimensional symplectic manifold.[6]
Then the standard action

$$\psi : \mathbb{T}^n \to \text{Diff}(\mathbb{C}\text{P}^n),\ [\theta] \mapsto (\psi_{[\theta]} : [z_0 : ... : z_n] \mapsto [z_0 : e^{i\theta_1} z_1 : ... : e^{i\theta_n} z_n])$$

is a hamiltonian torus action with (well-defined) moment map

$$\mu : \mathbb{C}\text{P}^n \to \mathbb{R}^n,\ [z] = [z_0 : ... : z_n] \mapsto \frac{1}{2}\left(\frac{|z_1|^2}{|z|^2}, ..., \frac{|z_n|^2}{|z|^2}\right).$$

## 2 Convexity of the moment map image

A peculiarity of hamiltonian torus actions is that the image of the associated moment maps is a convex polytope.[7]

**Theorem 2.1** (Atiyah, Guillemin–Sternberg)**.** *Let $(M, \omega)$ be a compact connected symplectic manifold with a hamiltonian action of $\mathbb{T}^n$ and associated moment map $\mu : M \to \mathbb{R}^n$. Then each $\mu^{-1}(c) \subset M$ for $c \in im(\mu)$ is a connected subset, and the image of $\mu$ is the convex hull spanned by the images of finitely many points $x_1, ..., x_N \in M$ fixed by the action:*[8]

$$\mu(M) = \left\{ \sum_{i=1}^N t_i \mu(x_i) \in \mathbb{R}^n \mid t_i \in [0,1] \ s.t. \ \sum_{i=1}^N t_i = 1 \right\} \subset \mathbb{R}^n$$

*We call $\mu(M)$ the **moment polytope**.*

**Remark 2.2.** Those of you who have knowledge of some basic results regarding locally convex vector spaces (such as $\mathbb{R}^n$) may have stumbled upon the following similarly-flavoured theorem[9] due to Krein–Milman:

> "A compact convex subset of a locally convex vector space is the closed convex hull of its extremal points",

[6] Actually, $\mathbb{C}\text{P}^n$ is something more: a Kähler manifold, that is, a manifold with mutually compatible complex, symplectic and Riemannian structure, all attached to the Kähler form $\omega_{\text{FS}}$.

[7] Two independently produced proofs of Theorem 2.1 can be found in M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14 (1982), 1–15, respectively V. Guillemin, S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. 67 (1982), 491–513.

[8] Still, note that there could be infinitely many fixed points!

[9] Consult for example M. Einsiedler, T. Ward, *Functional analysis, spectral theory, and applications*, Graduate Texts in Mathematics 276, Springer, (2017), ISBN: 978-3-319-58539-0.

where a point is called *extremal* if it cannot be written as a proper convex combination of any two points belonging to the subset. In fact, in the setting of Theorem 2.1, one can show that the image under $\mu$ of any point in $M$ fixed by the $\mathbb{T}^n$-action is an extremal point of $\mu(M)$.

**Remark 2.3.** (Sneak peek of symplectic toric manifold). In the setting of Theorem 2.1, if the hamiltonian torus action $\psi : \mathbb{T}^n \to \mathrm{Diff}(M)$ is *effective*, meaning that ($\psi_{[\theta]} = \mathrm{id}_M \implies [\theta] = \mathbb{1}$), then there are at least $n+1$ points of $M$ fixed by $\psi$ and, more importantly, $M$ must be of dimension at least $2n$.

Optimality regardining the dimension is reached in the case of **symplectic toric manifolds**: a symplectic toric manifold is any compact connected symplectic manifold $(M^{2n}, \omega)$ equipped with an effective hamiltonian torus action of $\mathbb{T}^n$ (so $\dim(M) = 2\dim(\mathbb{T}^n)$!) and a choice of moment map $\mu : M \to (\mathbb{R}^n)^*$.

We go no further: symplectic toric manifolds will be investigated in later talks.

**Example 2.4.**

- The moment map image for the Darboux $\mathbb{T}^n$-model discussed in Example 1.7 is the positive octant: $\mu_0(\mathbb{C}^n) = \mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n \mid x_k \geq 0 \ \forall 1 \leq k \leq n\}$ (hence the positive axis in dimension one).

- Consider the $\mathbb{S}^1$-action by rotation on the two-sphere $(\mathbb{S}^2, \omega = d\varphi \wedge dh)$, $e^{i\theta} \cdot (\varphi, h) = (\theta + \varphi, h)$, where $h : \mathbb{S}^2 \to \mathbb{R}$ is the height function. The associated moment map $\mu = -h$ has image $\mu(\mathbb{S}^2) = [-1, 1]$, exactly the convex hull spanned by $\mu(\varphi, \pm 1) = \mp 1$.
  Recall also from Example 1.7 that $\mathbb{S}^1$ acts on the homogeneous space $\mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^3/\mathbb{S}^1$ endowed with $\omega_{\mathrm{FS}} = \frac{1}{4} d\varphi \wedge dh$ as $e^{i\theta} \cdot [z_0 : z_1] = [z_0 : e^{i\theta} z_1]$, with $\mu([z_0 : z_1]) = \frac{|z_1|^2}{2(|z_0|^2 + |z_1|^2)}$. Then $\mu(\mathbb{C}\mathrm{P}^1) = [0, \frac{1}{2}]$.

- In general, the standard diagonal action of $\mathbb{T}^n$ on $\mathbb{C}\mathrm{P}^n$ with moment map as specified in Example 1.7 produces as moment polytope $\mu(\mathbb{C}\mathrm{P}^n) = \frac{1}{2}\Delta^n_{\mathrm{std}}$, "half" the standard $n$-dimensional simplex

  $$\Delta^n_{\mathrm{std}} \coloneqq \{x \in \mathbb{R}^n \mid 0 \leq x_1 \leq ... \leq x_n \leq 1\} \subset \mathbb{R}^n .$$

  This is best seen for $(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}})$: the hamiltonian $\mathbb{T}^2$-action

  $$\begin{aligned}(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] &= [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2] \\ &= [e^{-i\theta_1} z_0 : z_1 : e^{i(\theta_2 - \theta_1)} z_2] \\ &= [e^{-i\theta_2} z_0 : e^{i(\theta_1 - \theta_2)} z_1 : z_2]\end{aligned}$$

  clearly fixes the complex lines $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1] \in \mathbb{C}\mathrm{P}^2$, which are sent by

  $$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right)$$

  to respectively $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2}) \in \mathbb{R}^2$. The convex hull of the latters, $\{t(\frac{1}{2}, 0) + (1 - t)(0, \frac{1}{2}) \in \mathbb{R}^2 \mid t \in [0, 1]\}$, is precisely half the standard 2-simplex. Therefore, by Theorem 2.1, $\mu(\mathbb{C}\mathrm{P}^2) = \frac{1}{2}\Delta^2_{\mathrm{std}}$.

- We can also study the product symplectic manifold $((\mathbb{C}\mathrm{P}^1)^n, \omega_{\mathrm{FS}}^n)$ and the diagonal action of $\mathbb{T}^n$ on it (corresponding to the $\mathbb{S}^1$-action from above on each factor). By [CdS21][Exercise 1.4.7], the moment map is just the sum of the individual moment maps on each component and, accordingly, the moment polytope is the hypercube $[0, \frac{1}{2}]^n \subset \mathbb{R}^n$ (then a square for $n = 2$).

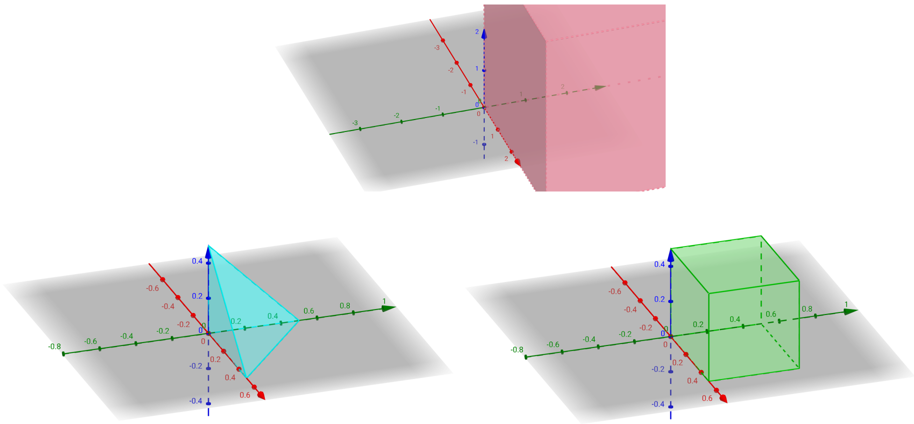

Figure 1: The convex polytopes $\mu_0(\mathbb{C}^3)$, $\mu(\mathbb{C}\mathrm{P}^3)$ and $\mu((\mathbb{C}\mathrm{P}^1)^3)$ under the respective moment maps discussed in Example 2.4. The 2-dimensional convex polytopes are simply obtained by projection onto the $xy$-plane.

**Remark 2.5.** The second-to-last bullet point above also lines up with what observed in Remark 2.2: for example, the image of the three fixed complex lines of $\mathbb{C}\mathrm{P}^2$ coincides with the extremal points of the standard 2-simplex, and, more generally, the complex lines of $\mathbb{C}\mathrm{P}^n$ fixed by the diagonal $\mathbb{T}^n$-action are seen to be mapped to the $(n+1)$-many vertices of $\frac{1}{2}\Delta^n_{\mathrm{std}}$.

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