A review of Lie Group Actions and the Notion of Moment Map

Full Version

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Abstract

In this talk we will consider symplectic and hamiltonian actions of Lie groups, which are in some sense a generalization of symplectic and hamiltonian vector fields.

1 A refresher on Lie Groups

This section will briefly recall some basic definitions and properties of Lie groups. For a more in-depth explanation with proofs, see the lecture note by Merry [Mer21].

Definition 1. A Lie group G is a (smooth) manifold with a group structure such that multiplication and inversion are smooth maps.

A (real) Lie algebra $\mathfrak{g}$ is a vector space with a bilinear pairing $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* such that $[v,w] = -[w,v]$ and

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

For example, $\Gamma(TG)$, i.e. the vector fields on G, form a Lie algebra with

$$[X,Y] := \mathcal{L}_X Y$$

given by the Lie derivative. For $g \in G$, consider the left multiplication map

$$\begin{aligned} l_g : G &\to G \\ h &\mapsto gh \end{aligned}$$

We say that a vector field X is left-invariant if $(l_g)_* X = X$.

Lemma 2. *If X, Y are left-invariant, then so is $[X, Y]$.*

Based on this lemma, will associate to every Lie group G the Lie algebra $\mathfrak{g}$ consisting of left-invariant vector fields on G. We may identify $\mathfrak{g}$ with the tangent space at the identity element:

Lemma 3. *Let $e \in G$ be the identity element. For every $v \in T_eG$, there is a unique left-invariant vector field $X \in \Gamma(TG)$ with $X_e = v$. In particular, we may identity $\mathfrak{g}$ with T_eG.*

Let G act on itself via conjugation, i.e. for $g \in G$, we consider

$$\psi_g : h \mapsto g \cdot h \cdot g^{-1}$$

We have $\psi_g(e) = e$, so we may differentiate there to get a linear map

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

where we use the identification $\mathfrak{g} = T_eG$. This gives us the *adjoint representation*

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g \end{aligned}$$

Dualizing this, we get the *coadjoint representation*

$$\begin{aligned} \mathrm{Ad}^* : G &\to \mathrm{GL}(\mathfrak{g}^*) \\ g &\mapsto \mathrm{Ad}^*_g \end{aligned}$$

which for $\xi \in \mathfrak{g}^*$ is defined via

$$\mathrm{Ad}^*_g(\xi)(X) = \xi(\mathrm{Ad}_{g^{-1}}(X)) \qquad \forall X \in \mathfrak{g}$$

(We use g^{-1} to ensure that $\mathrm{Ad}^*_{gh} = \mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h$, so we indeed get an action.)

Remark. If G is abelian, then Ad and Ad^* are both the trivial action. In most of our applications, this will be the case.

2 Hamiltonian Actions

Definition 4 (Action)**.** A *(smooth left) action* of a Lie group G on a manifold M is a group morphism

$$\psi : G \mapsto \mathrm{Diff}(M)$$

such that the evaluation map

$$\begin{aligned} \mathrm{ev}_\psi : G \times M &\to M \\ (g,x) &\mapsto g \cdot x := \psi_g(x) \end{aligned}$$

is smooth.

Example. Actions of $\mathbb{R}$ on M are given by the flows of complete vector fields

- Let X be a complete vector field, then the flow ρ_t defines a morphism $\mathbb{R} \to \mathrm{Diff}(M)$ as $\rho_{s+t} = \rho_s \circ \rho_t$.
- Let ψ be an action of $\mathbb{R}$ on M, then

  $$X_p = \frac{\mathrm{d}\psi_t(p)}{\mathrm{d}t}\Big|_{t=0}$$

  defines a complete vector field.

Any action of a Lie group G on M induces a Lie algebra morphism

$$\begin{aligned} \mathfrak{g} &\to \Gamma(TM) \\ X &\mapsto X^\sharp \end{aligned}$$

as follows: By restricting the action to the one-parameter subgroup of G defined by the flow $\{\rho_t(e) \big| t \in \mathbb{R}\}$, we get an action of $\mathbb{R}$ on M. By the previous example, this action corresponds to a complete vector field $X^\sharp$. This is often called the "infinitesimal action".

Definition 5. An action ψ is *symplectic* if ψ_g is a symplectomorphism for every $g \in G$.

In my first talk, I have shown that the flow of a complete symplectic vector field is a one-parameter family of symplectomorphisms. Hence, similar to the previous example, we have

$$\{\text{symplectic } \mathbb{R}\text{-actions on } M\} \leftrightarrow \{\text{symplectic vector fields on } M\}$$

Hamiltonian actions How should we define hamiltonian actions? Based on the previous examples, we would want that

$$\{\text{hamiltonian } \mathbb{R}\text{-actions on } M\} \leftrightarrow \{\text{hamiltonian vector fields on } M\}$$

Recall that hamiltonian vector fields X satisfies

$$\iota_X \omega = -dH$$

for some hamiltonian function $H \in C^\infty(M)$. This $G = \mathbb{R}$ case is a useful example to keep in mind when looking at the general definition.

Definition 6. An action ψ is *hamiltonian* if there is a map

$$\mu : M \to \mathfrak{g}^*$$

with the following properties:

- For every $X \in \mathfrak{g}$, let

  $$\begin{aligned} \mu^X : M &\to \mathbb{R} \\ p &\mapsto \mu(p)(X) \end{aligned}$$

  Then $X^\sharp$ is a hamiltonian vector field with hamiltonian function μ^X. In other words,

  $$\iota_{X^\sharp}\, \omega = -\,\mathrm{d}\mu^X$$

- Let Ad^* be the coadjoint action of G on $\mathfrak{g}^*$. Then

  $$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu \quad \forall g \in G$$

  (If G is abelian, then Ad^* is trivial, so this condition is just $\mu \circ \psi_g = \mu$. This will be the case once we consider tori.)

(M, ω, G, μ) is then called a *hamiltonian G-space* and μ is called a *moment map.*

Example. Let $G = \mathbb{R}$ or $G = S^1$, then $\mathfrak{g} \cong \mathbb{R}$ and $\mathfrak{g}^* \cong \mathbb{R}$. Let $\mu : M \to \mathbb{R}$ be a moment map. Let $X = 1 \in \mathfrak{g}$, then $\mu^X = \mu$. As X generates $\mathfrak{g}$, the first condition is just

$$\iota_{X^\sharp}\, \omega = -d\mu$$

i.e. the action has to be the flow of a hamiltonian vector field. The second condition is

$$\mu = \underbrace{\mu \circ \psi_t}_{=\psi_t^* \mu} \qquad \forall t \in \mathbb{R}$$

The first condition implies the second one, as the flow of a hamiltonian vector field preserves the hamiltonian function (see my first talk).

Here is a special case of this:

Example. Let $G = S^1$ with coordinate σ and $M = S^2$ with coordinates (θ, h). Consider the action by rotation, i.e. $\sigma \cdot (\theta, h) = (\theta + \sigma, h)$. This action preserves the surface area, so it is symplectic. This action moreover preserves $\mu = -h$, so it is hamiltonian.

Here is a more involved example where we get to see the coadjoint action.

Example. Let $M = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates (q_i, p_i) standard symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i = \mathrm{d}\lambda$$

where

$$\lambda = \sum_{i=1}^{n} p_i \, \mathrm{d}q_i$$

or $\lambda = (\mathrm{d}q)^T \cdot p$ if we define the vectors

$$\mathrm{d}q = \begin{pmatrix} \mathrm{d}q_1 \\ \vdots \\ \mathrm{d}q_n \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

Let $G = \mathrm{SO}_n(\mathbb{R})$, then

$$\mathfrak{g} = \left\{ B \in M_n(\mathbb{R}) \middle| B + B^T = 0 \right\}$$

Let G act on M via

$$\psi_A(q, p) = (A \cdot q, \ A \cdot p)$$

then for $X \in \mathfrak{g}$, the infinitesimal action is

$$X^\sharp_{(q,p)} = \frac{\partial}{\partial t}\bigg|_{t=0} \Big((I_n + tX) \cdot p, \ (I_n + tX) \cdot q \Big) = \left(X \cdot \frac{\partial}{\partial p} + X \cdot \frac{\partial}{\partial q} \right)$$

where we have column vectors

$$\frac{\partial}{\partial p} = \begin{pmatrix} p_1 \cdot \frac{\partial}{\partial p_1} \\ \vdots \\ p_n \cdot \frac{\partial}{\partial p_n} \end{pmatrix}, \quad \frac{\partial}{\partial q} = \begin{pmatrix} q_1 \cdot \frac{\partial}{\partial q_1} \\ \vdots \\ q_n \cdot \frac{\partial}{\partial q_n} \end{pmatrix}$$

The action ψ is symplectic:

$$\psi_A^* \lambda = (\mathrm{d}(q \circ A))^T \cdot (Ap) = (A \cdot \mathrm{d}q)^T \cdot (A \cdot p) = \mathrm{d}q^T \cdot \underbrace{A^T \cdot A}_{=I_n} \cdot p$$

hence also

$$\psi_A^*\omega = \psi_A^* \,\mathrm{d}\lambda = \mathrm{d}(\psi_A^*\lambda) = \mathrm{d}\lambda = \omega$$

(see also the section in my first talk on exact symplectic manifolds).

Moreover, the action is hamiltonian with moment function

$$\begin{aligned} \mu : M &\to \mathfrak{g} \\ (q,p) &\mapsto \left[B \mapsto p^T \cdot B \cdot q\right] \end{aligned}$$

(where $q^T \cdot B \cdot p = p^T \cdot B^T \cdot q = -p^T \cdot B \cdot q$ by the description of $\mathfrak{g}$). This is indeed a moment map:

- We first need to check that $\iota_{X^\sharp}\omega = -d\mu^X$. Fix $X \in \mathfrak{g}$, then

  $$\mu^X(p,q) = p^T X q$$

  hence

  $$\mathrm{d}\mu^X\Big|_{p,q} = p^T X \cdot \mathrm{d}q + q^T X^T \cdot \mathrm{d}p = p^T X \cdot \mathrm{d}q - q^T X \cdot \mathrm{d}p$$

  as $X^T = -X$. On the other hand, we have

  $$\iota_{X^\sharp}\omega = q^T X \cdot \mathrm{d}p - p^T X \cdot \mathrm{d}q$$

- Next, let's check equivariance. We have

  $$\mathrm{Ad}_A B = ABA^{-1} \qquad \forall A \in \mathrm{SO}_n(\mathbb{R}), B + B^T = 0$$

  therefore

  $$\begin{aligned} (\mu \circ \psi_A)(p,q)(B) = \mu(Ap, Aq)(B) &= p^T \underbrace{A^T}_{A^{-1}} BAq \\ &= \mu(p,q)(A^{-1}BA) = \mu(p,q)(\mathrm{Ad}_{A^{-1}}(B)) \end{aligned}$$

  so we indeed get the coadjoint action.

Example. If we want to do the same example with $G = \mathrm{GL}_n(\mathbb{R})$, then we need to define the action as

$$\psi_A(p,q) = (A \cdot p, (A^{-1})^T \cdot q)$$

to get a symplectic (and also hamiltonian) action.

This example generalizes to arbitrary cotangent bundles:

Example. Let ψ be any action of a Lie group G on any manifold N. This induces an action $\overline{\psi}$ on the cotangent bundle $(T^*N, -\,\mathrm{d}\lambda)$ given by

$$\overline{\psi}_g(x,p) := (\psi_g(x), (\psi_{g^{-1}})^* p)$$

This is action is the lift of a diffeomorphisms and hence symplectic (see notes of my first talk). It is moreover hamiltonian with moment map

$$\mu(X) = \iota_{X^\sharp}\lambda = \lambda(X^\sharp)$$

This lifting of actions to hamiltonian actions in some sense generalizes the lifting of vector fields to hamiltonian vector fields.

3 Products

Hamiltonian actions behave well with respect to products. For proofs of these lemmas, see Section 4 of [Wie21].

Lemma 7. *Suppose we are given hamiltonian actions of a Lie group G on two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with moment maps $\mu_1 : M_1 \to \mathfrak{g}^*$ and $\mu_2 : M_2 \to \mathfrak{g}^*$. Equip $M_1 \times M_2$ with the product symplectic structure $\pi_1^*\omega_1 + \pi_2^*\omega_2$. Then the diagonal action of G on $M_1 \times M_2$ is hamiltonian with $\mu = \mu_1 \circ \pi_1 + \mu_2 \circ \pi_2$.*

Lemma 8. *Let G_1 and G_2 be two Lie groups, each with a hamiltonian actions ψ_1 and ψ_2 on the same manifold M, with moment maps $\mu_1 : M \to \mathfrak{g}_1^*$ and $\mu_2 : M \to \mathfrak{g}_2^*$. Suppose in addition that μ_1 is ψ_2-invariant and vice-versa. Then the product action defined by*

$$\psi_{(g_1,g_2)}(x) = (\psi_1)_{g_1}((\psi_2)_{g_2}(p))$$

is hamiltonian with moment map $\mu_1 \oplus \mu_2$.

For us, the most important case of this is $G = \mathbb{T}^n = S^1 \times \cdots \times S^1$.

4 Differential of Moment Maps

Let (M, ω, G, μ) be a hamiltonian G-space. Fix a point $p \in M$. We will use the following notation:

$\mathcal{O}$ orbit of p.

G_p stabilizer of p.

$\mathfrak{g}_p$ Lie algebra of G_p.

For a vector subspace $W \subseteq V$, recall the annihilator

$$W^0 = \left\{\xi \in V^* \middle| \xi\big|_W \equiv 0\right\} \subseteq V^*$$

For a subspace $W \subseteq V$ of a symplectic space V, recall the symplectic complement

$$W^\omega = \left\{v \in V \middle| \omega(v, w) = 0 \ \forall w \in W\right\}$$

Lemma 9. *Let $\mu : M \to g^*$ be a moment map and fix $p \in M$. Then the differential*

$$\mathrm{d}\mu_p : T_pM \to T_{\mu(p)}\, \mathfrak{g}^* = \mathfrak{g}^*$$

satisfies

$$\begin{aligned}\ker(\mathrm{d}\mu_p) &= (T_pO)^\omega \\ \mathrm{im}(\mathrm{d}\mu_p) &= (\mathfrak{g}_p)^0 = \left\{\xi \in \mathfrak{g}^* \middle| \xi(\mathfrak{g}_p) = \{0\}\right\}\end{aligned}$$

Proof. **Kernel** By definition of moment map, we have

$$-(\mathrm{d}\mu_p(v))(X) = \omega_p(X_p^\sharp, v) \qquad \forall X \in \mathfrak{g}, v \in T_pM$$

Then

$$\begin{aligned}\mathrm{d}\mu_p(v) = 0 &\Leftrightarrow (\mathrm{d}\mu_p(v))(X) = 0 && \forall X \in \mathfrak{g} \\ &\Leftrightarrow \omega_p(X_p^\sharp, v) = 0 && \forall X \in \mathfrak{g} \\ &\Leftrightarrow v \in \mathrm{Span}\left\{X_p^\sharp \middle| X \in \mathfrak{g}\right\}^\omega \\ &\Leftrightarrow v \in (T_p\mathcal{O})^\omega\end{aligned}$$

Dimension In particular, we have

$$\begin{aligned}\dim(\ker \mathrm{d}\mu_p) &= \dim M - \dim G + \dim G_p \\ \dim(\mathrm{im}\, \mathrm{d}\mu_p) &= \dim G - \dim G_p = \dim(\mathfrak{g}_p)^0\end{aligned}$$

by rank-nullity. Thus, it suffices to show that

$$\mathrm{im}(\mathrm{d}\mu_p) \subseteq (\mathfrak{g}_p)^0$$

Image We will now show that

$$(\mathrm{d}\mu_p(v))(X) = 0 \qquad \forall X \in \mathfrak{g}_p, v \in T_pM$$

Let $X \in \mathfrak{g}_p$, then the action of $\{\exp tX, t \in \mathbb{R}\}$ stabilizes p, thus $X_p^\sharp = 0$. Thus, for any $v \in T_pM$, we have

$$(\mathrm{d}\mu_p(v))(X) = -\omega_p(\underbrace{X_p^\sharp}_{=0}, v) = 0$$

□

Corollary 10. *In the above setting,*

- $\mathrm{d}\mu_p$ *is* injective *if and only if the orbit* $\mathcal{O}_p$ *is open.*
- $\mathrm{d}\mu_p$ *is* surjective *if and only if the stabilizer of* p *is discrete.*

References

[Mer21] Will J. Merry. Lecture notes for a two-semester course on Differential Geometry. https://www.merry.io/courses/differential-geometry/, 2021. [Online; accessed 30-September-2021].

[Wie21] Manuel Wiedmer. Moment Maps From Lie Groups, Semester Paper. https://people.math.ethz.ch/~acannas/Student_Papers/Semester_Papers/2021_manuel_wiedmer_sp_moment_maps_from_lie_groups.pdf, 2021. [Online; accessed 15-November-2021].